UNIT – II

Strain Energy due to axial, bending and torsional loads

Castigliano’s theorems

Maxwell's and Betis Reciprocal theorem

UNIT I load method

Application to beams, trusses, frames, rings, etc.

ENERGY METHODS

Strain Energy:

The strain energy of a member will be defined as the increase in energy associated with the deformation of the member. The strain energy is equal to the work done by a slowly increasing load applied to the member.

2. Define Strain energy density.

The strain-energy density of a material will be defined as the strain energy per unit volume.

3. Define Modulus of toughness.

The area under the entire stress-strain diagram was defined as the modulus of toughness and is a measure of the total energy that can be acquired by the material.

4. Define Modulus of resilience.

The area under the stress-strain curve from zero strain to the strain $\in_y$ at yield is referred to as the modulus of resilience of the material and represents the energy per unit volume that the material can absorb without yielding. We wrote

$$u_y = \frac{\sigma_y^2}{2E}$$

5. Write the expression for strain energy under axial load.

If the rod is of uniform cross section of area A, the strain energy is

$$U = \int_0^L \frac{p^2}{2AE} dx$$

6. Write the expression for strain energy due to bending.

For a beam subjected to transverse loads the strain energy associated with the normal stresses is

$$U = \int_0^L \frac{M^2}{2EI} dx$$

where M is the bending moment and EI the flexural rigidity of the beam.

7. Write the expression for strain energy due to shearing stresses.

The strain energy associated with shearing stresses, the strain-energy density for a material in pure shear is

$$u = \frac{\tau_{xy}^2}{2G}$$

where t_{xy} is the shearing stress and G the modulus of rigidity of the material.

8. Write the expression for strain energy due to torsion.

For a shaft of length Land uniform cross section subjected at its ends to couples of magnitude T the strain energy was found to be

$$U = \frac{T^2 L}{2GJ}$$

Where J is the polar moment of inertia of the cross-sectional area of the shaft.

9. Explain strain energy for a general state of stress.

The strain energy of an elastic isotropic material under a general state of stress and expressed the strain energy density at a given point in terms of the principal stresses σ_a, σ_b and σ_c at that point:

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2v\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right]$$

The strain-energy density at a given point was divided into two parts: u_υ, associated with a change in volume of the material at that point, and u_d, associated with a distortion of the material at the same point. We wrote $u = u_\upsilon + u_d$, where

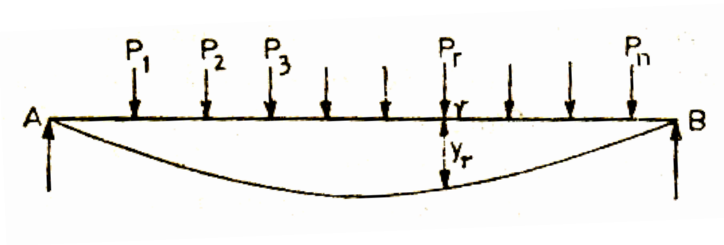
$$u_\upsilon = \frac{1-2v}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2$$

and

$$u_d = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right]$$

10. Define Castigliano's theorem.

In any beam or truss subjected to any load system, the deflection at any point r is given by the partial differential coefficient of the total strain energy stored with respect to a force P_r acting at the point r in the direction in which the deflection is desired.



Figure

Figure shows a structure AB carrying a load system P_1, P_2, P_3 ….P_r, ….P_n.

Let the deflection at the point r be y_r.
Let W_e = External work done be the given load system
W_i = Corresponding strain energy stored.

$$\therefore W_e = W_i$$

$$y_r = \lim_{\Delta P_r \to 0} \cdot \frac{\Delta W_i}{\Delta P_r}$$

$$y_r = \frac{\partial W_i}{\partial P_r}$$

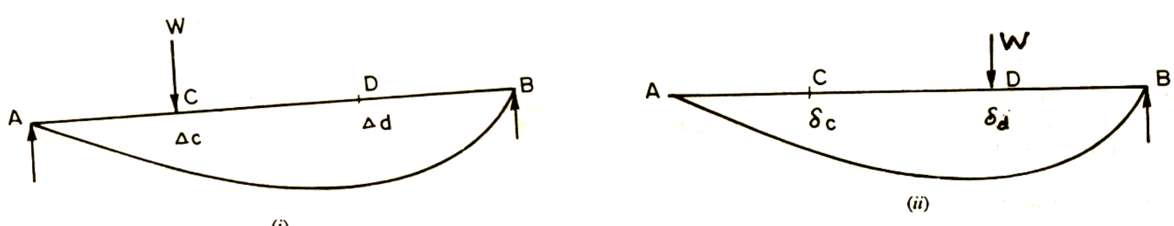
=Partial differential coefficient of the total strain energy stored with respect to P_r.

11. Define Maxwell's reciprocal theorem.

In any beam of truss the deflection at any point D due to a load W at any other point C is the same as the deflection at C due to the same load W applied at D.

Figure (i) shows a structure AB carrying a load W applied at any point C. Let the deflection at C be Δc. Let the deflection at another point D be Δd.

Figure (ii) shows the same structure AB carrying the same load W at D. Let the deflections at C and D be δ_c and δ_d respectively.



Figure

12. Give the relation between number of joints and the number of members in a perfect frame.

Let there be n members and j joints in a perfect frame, Fig. (a)

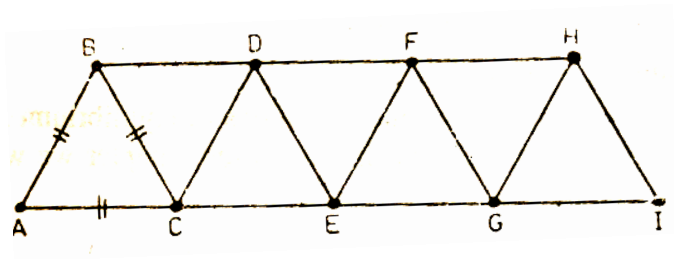


Fig. (a)

Suppose we remove three members AB, BC and CA and the three joints A, B and C. We are now left with (n – 3) members and (j – 3) joints.

Studying this remaining part of the frame (Fig. (b)), we find that the number of members in such that, for each joint, there are two members.

Hence for the (j – 3) joints we have 2(j – 3) members.

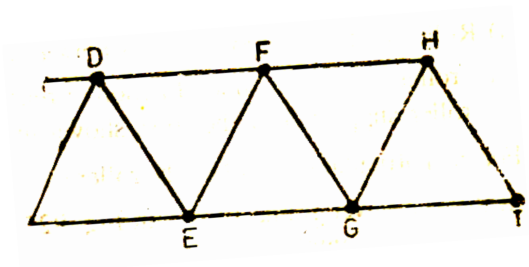


Fig. (b)

$$\therefore \quad n - 3 = 2(j - 3)$$

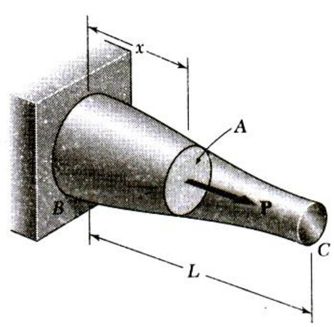
$$\therefore \quad n = 2j - 3$$

Hence for a stable frame the minimum number of members required = twice the number of joints minus three.

13. Derive the expression for Strain Energy under Axial Loading.

Strain Energy under Axial Loading

When a rod is subjected to centric axial loading, the normal stresses σ_x can be assumed uniformly distributed in any given transverse section. Denoting by A the area of the section located at a distance x from the end B of the rod and by P the internal force, we write $\sigma_x = P/A$.



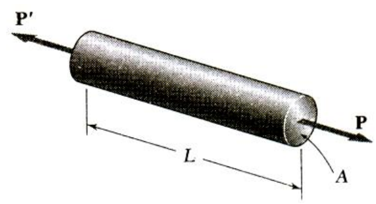
Figure

$$U = \int \frac{P^2}{2EA^2} dV$$

or, setting dV = A dx,

$$U = \int_0^L \frac{P^2}{2AE} dx$$

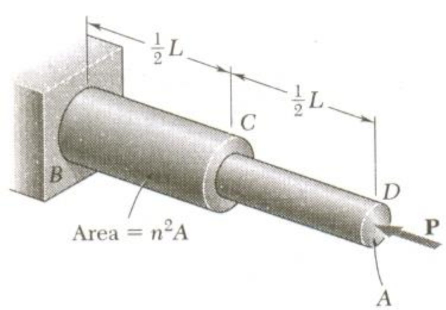
In the case of a rod of uniform cross section subjected at its ends to equal and opposite forces of magnitude P.



Figure

$$U = \frac{P^2 L}{2AE}$$

14. A rod consists of two portions BC and CD of the same material and same length, but of different cross sections. Determine the strain energy of the rod when it is subjected to a centric axial load P, expressing the result in terms of P, L, E, the cross-sectional area A of portion CD, and the ratio n of the two diameters.



Figure

$$U_n = \frac{P^2\left(\frac{1}{2}L\right)}{2AE} + \frac{P^2\left(\frac{1}{2}L\right)}{2\left(n^2 A\right)E} = \frac{P^2 L}{4AE}\left(1 + \frac{1}{n^2}\right)$$

or

$$U_n = \frac{1+n^2}{2n^2}\frac{P^2 L}{2AE}$$

We check that, for n = 1, we have

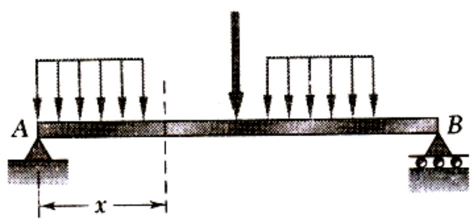
$$U_1 = \frac{P^2L}{2AE}$$

which is the expression given in equation for a rod of length L and uniform cross section of area A. We also note that, for n > 1, we have $U_n < U_1$; for example, when n = 2, we have $U_2 = \left(\frac{5}{8}\right)$ U_1. Since the maximum stress occurs in portion CD of the rod and is equal to $\sigma_{max} = P/A$, it follows that, for a given allowable stress, increasing the diameter of portion BC of the rod results in a decrease of the overall energy-absorbing capacity of the rod. Unnecessary changes in cross-sectional area should therefore be avoided in the design of members that may be subjected to loadings, such as impact loadings, where the energy-absorbing capacity of the member is critical.

15. Derive the expression for strain energy in bending.

Strain Energy in Bending

Consider a beam AB subjected to a given loading and let M be the bending moment at a distance x from end A. Neglecting for the time being the effect of shear, and taking into account only the normal stresses $\sigma_x = My/I$,



Figure

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2y^2}{2EI^2} dV$$

Setting dV = dA dx, where dA represents an element of the cross-sectional area, and recalling that $M^2/2EI^2$ is a function of x alone, we have

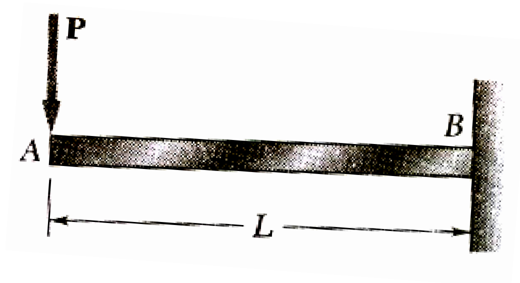
$$U = \int_0^L \frac{M^2}{2EI^2} \left(\int y^2 dA\right) dx$$

Recalling that the integral within the parentheses represents the moment of inertia I of the cross section about its neutral axis, we write

$$U = \int_0^L \frac{M^2}{2EI} dx$$

16. Determine the strain energy of the prismatic cantilever beam AB taking into account

only the effect of the normal stresses.



Figure

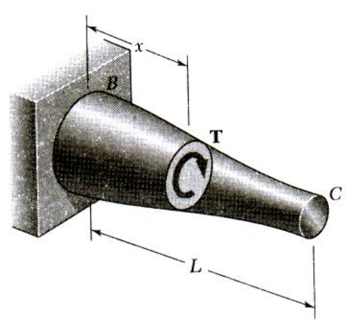
The bending moment at a distance x from end A is M = - P_x. Substituting this expression we write

$$U = \int_0^L \frac{P^2x^2}{2EI}dx = \frac{P^2L^3}{6EI}$$

17. Derive expression for strain energy due to torsion.

Strain Energy in Torsion

Consider a shaft BC of length L subjected to one or several twisting couples. Denoting by J the polar moment of inertia of the cross section located at a distance x from B and by T the internal torque in that section, we recall that the shearing stresses in the section are τ_{xy} = Tρ/J. Substituting for τ_{xy} we have



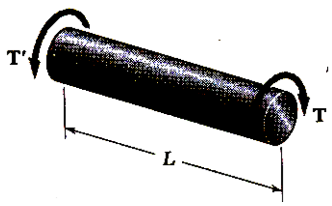
$$U = \int \frac{\tau_{xy}^2}{2G}dV = \int \frac{T^2\rho^2}{2GJ^2}dV$$

Setting dV = dA dx, where dA represents an element of the cross-sectional area, and observing that $T^2/2GJ^2$ is a function of x alone, we write

$$U = \int_0^L \frac{T^2}{2GJ^2}\left(\int \rho^2 dA\right)dx$$

Recalling that the integral within the parentheses represents the polar moment of inertia J of the cross section, we have

$$U = \int_0^L \frac{T^2}{2GJ} dx$$

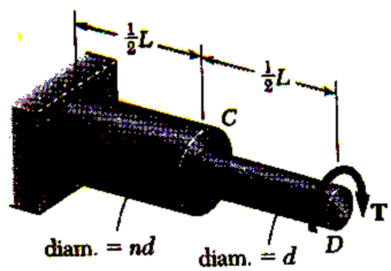


Figure

In the case of a shaft of uniform cross section subjected at its ends to equal and opposite couples of magnitude T yields.

$$U = \frac{T^2 L}{2GJ}$$

18. A circular shaft consists of two portions BC and CD of the same material and same length, but of different cross sections. Determine the strain energy of the shaft when it is subjected to a twisting couple T at end D, expressing the result in terms of T, L, G, the polar moment of inertia J of the smaller cross section, and the ratio n of the two diameters.



Figure

$$U_n = \frac{T^2\left(\frac{1}{2}L\right)}{2GJ} + \frac{T^2\left(\frac{1}{2}L\right)}{2G\left(n^4 J\right)} = \frac{T^2 L}{4GJ}\left(1 + \frac{1}{n^4}\right)$$

or

$$U_n = \frac{1+n^4}{2n^4}\frac{T^2 L}{2GJ}$$

We check that, for n = 1, we have

$$U_1 = \frac{T^2 L}{2GJ}$$

Which is the expression given in equation for a shaft of length L and uniform cross section. We also note that, for n > 1, we have $U_n < U_1$; for example, when n = 2, we have $U_2 = \left(\frac{17}{32}\right) U_1$.

Since the maximum shearing stress occurs in the portion CD of the shaft and is proportional to the torque T, we note as we did earlier in the case of the axial loading of a rod that, for a given allowable stress, increasing the diameter of portion BC of the shaft results in a decrease of the overall energy-absorbing capacity of the shaft.

19. Find the deflection at the free end of a cantilever carrying a concentrated load at the free end. Assume uniform flexural rigidity.

Solution:-

Figure shows a cantilever carrying a point load P at the free end A. The bending moment at any section distant x from the free end is given by

$$M = -Px$$

∴ Strain energy stored by the cantilever

$$W_i = \int \frac{M^2 dx}{2EI} = \int_0^l \frac{P^2 x^2 dx}{2EI} = \frac{P^2}{2EI} \cdot \frac{l^3}{3}$$

$$\therefore \quad W_i = \frac{p^2\ l^3}{6EI}$$

∴ By the first theorem of Castiglione, the deflection in the line of action of the forceP,

$$= \delta = \frac{\partial W_i}{\partial P} = \frac{(2P)l^3}{6EI} = \frac{Pl^3}{3EI}$$

20. Find the central deflection of a simply supported beam carrying a concentrated load at mid span. Assume uniform flexural rigidity.

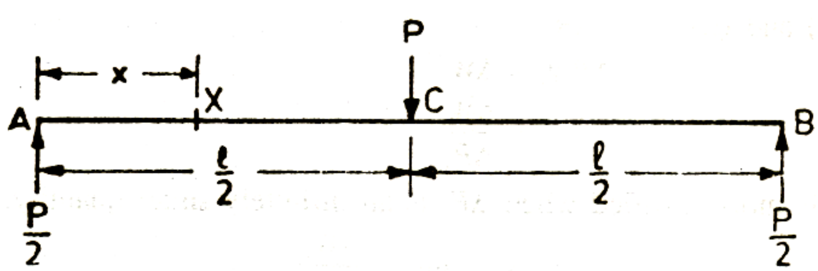
Solution:-

Figure shows a beam AB simply supported at A and B and carrying a central load P. Each reaction

$$= \frac{P}{2}$$

The bending moment at any section in AC, distant x from the end A is given by,

$$M = \frac{P}{2}x$$



Figure

∴ Strain energy stored by the beam

$$W_i = \int \frac{M^2 dx}{2EI} = 2 \int_0^{1/2} \frac{p^2 x^2}{4} \frac{dx}{2EI}$$

$$= \frac{p^2}{4EI} \cdot \frac{1}{3} \cdot \frac{l^3}{8} = \frac{p^2 l^3}{96EI}$$

∴ $$W_i = \frac{p^2 l^3}{96EI}$$

∴ The deflection in the line of action P is given by

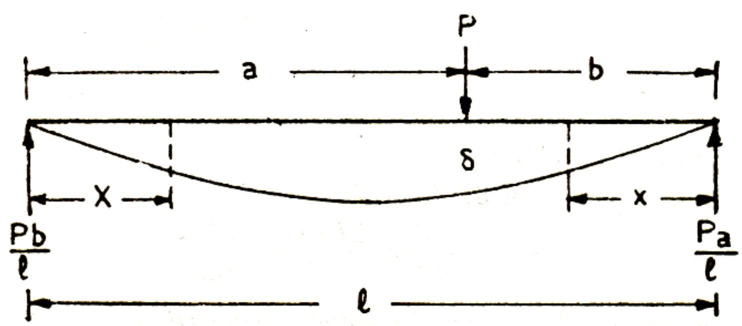
$$\delta = \frac{\partial W_i}{\partial P} = \frac{2pl^3}{96EI}$$

∴ $$\delta = \frac{pl^3}{48EI}$$

21. A simply supported beam carries a point load P eccentrically on the span. Find the deflection under the load. Assume uniform flexural rigidity.

Solution:-

Figure shows a beam AB of span l which carries a load P at C.



Figure

Let AC = a and BC = b.

Reaction at A= $\frac{Pb}{l}$

Reaction at B = $\frac{.Pa}{l}$

The strain energy stored by the beam AB

W_i = strain energy stored by AC
+ strain energy stored by BC

$$= \int_0^a \left(\frac{Pb}{l}x\right)^2 \frac{dx}{2EI} + \int_0^b \left(\frac{Pa}{l}x\right)^2 \frac{dx}{2EI}$$

$$= \frac{p^2b^2a^3}{6EIl^2} + \frac{p^2a^2b^3}{6EIl^2} = \frac{p^2a^2b^2}{6EIl^2}(a+b)$$

Since $a + b = l$

$$W_i = \frac{p^2a^2b^2}{6EIl}$$

$\therefore$ Deflection under load P is given by

$$\delta = \frac{\partial W_i}{\partial P} = \frac{.(2P)a^2b^2}{6EIl} = \frac{Pa^2b^2}{3EIl}$$

22. Define: Strain Energy

When an elastic body is under the action of external forces the body deforms and work is done by these forces. If a strained, perfectly elastic body is allowed to recover slowly to its unstrained state. It is capable of giving back all the work done by these external forces. This work done in straining such a body may be regarded as energy stored in a body and is called strain energy or resilience.

23. Define: Proof Resilience.

The maximum energy stored in the body within the elastic limit is called Proof Resilience.

24. Write the formula to calculate the strain energy due to axial loads (tension).

$$U = \int \frac{P^2}{2AE} dx \quad \text{limit 0 to L}$$

Where,

P = Applied tensile load.
L = Length of the member
A = Area of the members
E = Young's modulus.

25. Write the formula to calculate the strain energy due to bending.

$$U = \int \frac{M^2}{2EI} dx \quad \text{limit 0 to L}$$

Where,

M = Bending moment due to applied loads.
E = Young's modulus
I = Moment of inertia

26. Write the formula to calculate the strain energy due to torsion

$$U = \int \frac{T^2}{2GJ} dx \qquad \text{limit 0 to L}$$

Where, T = Applied Torsion
G = Shear modulus or Modulus of rigidity
J = Polar moment of inertia

27. Write the formula to calculate the strain energy due to pure shear

$$U = K \int \frac{T^2}{2GA} dx \qquad \text{limit 0 to L}$$

Where, V = Shear load
G = Shear modulus or Modulus of rigidity
A = Area of cross section
K = Constant depends upon shape of cross section.

28. Write the down the formula to calculate the strain energy due to pure shear, if shear stress is given.

$$U = \frac{\tau^2 V}{2G}$$

Where, τ = Shear stress
G = Shear modulus or Modulus of rigidity
V = Volume of the material.

29. Write the down the formula to calculate the strain energy, if the moment value is given

$$U = \frac{M^2 L}{2EI}$$

Where, M = Bending moment
L = Length of the beam
E = Young's modulus
I = Moment of inertia

30. Write the down the formula to calculate the strain energy, if the torsion moment value

is given.

$$U = \frac{T^2 L}{2GJ}$$

Where, T = Applied Torsion
L = Length of the beam
G = Shear modulus or Modulus of rigidity
J = Polar moment of inertia

31. Write down the formula to calculate the strain energy, if the applied tension load is given.

$$U = \frac{P^2 L}{2AE}$$

Where, P = Applied tensile load.
L = Length of the member.
A = Area of the member
E = Young's modulus.

32. Write the Castigliano's first theorem.

In any beam or truss subjected to any load system, the deflection at any point is given by the partial differential coefficient of the total strain energy stored with respect to force acting at a point.

$$\delta = \frac{\partial U}{\partial P}$$

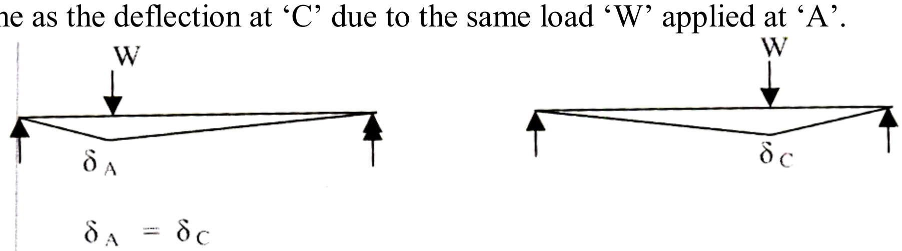
Where, δ = Deflection
U = Strain Energy stored
P = Load

33. What are the uses of Castigliano's first theorem?

1. To determine the deflection of complicated structure.
2. To determine the deflection of curved beams, springs.

34. Define: Maxwell Reciprocal Theorem.

In any beam of truss the deflection at any point 'A' due to a load 'W' at any other point 'C' is the same as the deflection at 'C' due to the same load 'W' applied at 'A'.



35. Define: Unit load method.

The external load is removed and the unit load is applied at the point, where the deflection or rotation is to found.

36. Give the procedure for unit load method.

1. Find the forces P1, P2, ………. in all the members due to external loads.
2. Remove the external loads and apply the unit vertical point load at the joint if the vertical deflection is required and find the stress.
3. Apply the equation for vertical and horizontal deflection.

37. Compare the unit load method and Castigliano's first theorem.

In the unit load method, one has to analyze the frame twice to find the load and deflection. While in the latter method, only one analysis is needed.

38. Find the strain energy per unit volume, the shear stress for a material is given as 50 N/mm^2. Take G = 80000 N/mm^2.

$U = \frac{\tau^2}{2G}$ per unit volume.

= 50^2 / (2 x 80000).

= 0.015625 N / mm^2. per unit volume.

39. Find the strain energy per unit volume, the tensile stress for a material is given as 150 N/mm^2. Take E = 2 x 10 N/mm^2.

$U = \frac{f^2}{2E}$ Per unit volume

= $(150)^2$ / (2 × (2x 10^2)

= 0.05625 N/mm ^{2} per unit volume.

40. Define: Modulus of resilience.

The proof resilience of a body per unit volume. (ie) The maximum energy stored in the body within the elastic limit per unit volume.

41. Define Trussed Beam?

A beam strengthened by providing ties and struts is known as Trusted Beams.