MATRICES AND CALCULUS

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CONTENTS

UNIT – 1

MATRICES

1.1 INTRODUCTION

Matrices play an important role in Engineering and Mathematics. Areas like linear programming, game theory, Markov models and some statistical models have matrix algebra as the base. It has application in electrical circuits, quantum mechanics, Nuclear Physics and other applied science subjects.

Definition:

A matrix is an rectangular arrangement of elements in rows and columns. If a matrix has m rows and n columns it is called an $m \times n$ matrix. We use capital letters such as A, B, C.. etc to denote matrices.

$$\begin{array}{cl} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} & \begin{array}{l} 1^{st} \text{ row } (R_1) \\ 2^{nd} \text{ row } (R_1) \\ 3^{rd} \text{ row } (R_1) \end{array} \\ \begin{array}{ccc} 1^{st} & 2^{nd} & 3^{rd} \\ \text{column} & \text{column} & \text{column} \\ C_1 & C_2 & C_3 \end{array} & \end{array}$$

Examples:

1. $\begin{pmatrix} 2 & 1 & 4 \\ 5 & 6 & 3 \end{pmatrix}$ is a 2×3 matrix.

2. $\begin{pmatrix} 5 & 6 & 7 \\ 1 & 4 & 3 \\ -2 & 0 & 9 \end{pmatrix}$ is a 3×3 matrix.

3. $\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$ is a 3×1 matrix.

TYPES OF MATRICES

1. Row Matrix:

A matrix in which there is only one row is called a row matrix.

Example: $[3 \quad 4 \quad 5]_{1 \times 3}$

2. Column Matrix:

A matrix in which there is only one column is called a column matrix.

Example: $\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}_{3 \times 1}$

3. Square matrix:

A matrix in which the number of rows and columns are equal is called a square matrix.

Example: $A = \begin{bmatrix} 1 & 2 \\ 7 & 9 \end{bmatrix}_{2 \times 2} \quad B = \begin{bmatrix} 7 & 1 & 6 \\ 8 & 2 & 3 \\ 4 & 9 & J \end{bmatrix}_{3 \times 3}$

Determinant of a matrix

If A is a square matrix, $|A|$ is called the determinant of the matrix A and

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Transpose of a Matrix:

The Transpose of a matrix is obtained by interchanging rows into columns and columns into rows.

$$\text{If } A = \begin{bmatrix} 4 & -3 \\ 2 & 5 \\ 1 & 8 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 5 & 8 \end{bmatrix}$$

4. Null matrix or Zero Matrix:

A matrix in which all the entries are zero is called a Null matrix or Zero matrix.

Example: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a null matrix of order 3×3

5. Triangular matrices:

1. A square matrix is said to be an upper triangular matrix if all the elements below the leading diagonal are zero.

Example: $\begin{pmatrix} 1 & 3 & 5 \\ 0 & 9 & -1 \\ 0 & 0 & -2 \end{pmatrix}$ is an upper triangular matrix.

2. A square matrix is said to be a lower triangular matrix if all the elements above the leading diagonal are zero.

Example: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & -5 \end{pmatrix}$ is a lower triangular matrix.

6. Diagonal matrix: (D)

A square matrix in which all the entries except the main diagonal entries are zero is known as diagonal matrix.

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a diagonal matrix of order 3.

7. Scalar matrix:

A diagonal matrix in which all the elements are same is called a scalar matrix.

Example: $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a scalar matrix.

8. Non Singular and Singular Matrix

For any square matrix, A if $|A| \neq 0$, then A is called a non singular matrix otherwise it is called a singular matrix.

9. Unit matrix or identity matrix (I)

A scalar matrix in which the diagonal entries are unity is called a unit matrix or identity matrix.

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix of order 3.

10. Symmetric and non-symmetric matrix

A symmetric matrix is a matrix that is equal to its transpose otherwise it is called non-symmetric matrix.

Example: $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$ Symmetric $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 4 & 5 \\ 7 & 1 & 3 \end{bmatrix}$

non symmetric

1.2 CHARACTERISTIC EQUATION

Given a square matrix A, the determinant of $(A - \lambda I)$ matrix equated to zero is known as the characteristic equation of the matrix A.

i.e., $|A - \lambda I| = 0$ ... (1)

If A is of order n, the characteristic equation is a nth degree polynomial in λ. The roots of the characteristic equation are known as characteristic roots (or) latent roots (or) eigen values of A.

Expanding this determinant shown in (1) we get a nth degree polynomial equation in λ. Solving this equation we get n values for λ. These values are known as eigen values and the equation (1) is known as the characteristic equation.

The vectors corresponding to each of these n values of λ are known as eigen vectors.

The eigen vectors of the matrix are given by the homogeneous equations

$$(A - \lambda I)\, X = 0$$

i.e. $\left(a_{11} - \lambda\right)x_1 + a_{12}\,x_2 + ... + a_{1n}\,x_n = 0$

$$a_{21}\,x_1 + \left(a_{22} - \lambda\right)x_2 + \ldots + a_{2n}\,x_n = 0$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$a_{n1}\,x_1 + a_{n2}\,x_2 + ... + \left(a_{nn} - \lambda\right)x_n = 0$$

Method to find the characteristic equation:

Case (i): For 2×2 matrix

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the characteristic equation of A

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where $S_1 =$ Sum of main diagonal elements

$S_2 = |A| =$ determinant of A

Case (ii): For 3×3 matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1 =$ Sum of the main diagonal elements

$S_2 =$ Sum of the minors of the main diagonal elements

$S_3 = |A| =$ determinant of A.

WORKED EXAMPLES

Example 1: Find the characteristic equation of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$

Solution:

The characteristic equation is

$$\lambda^2 - S_1\lambda + S_2 = 0 \qquad \text{... (1)}$$

$$\text{Let, } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

S_1 = Sum of the main diagonal elements

$= 1 + 4 = 5$

$$\boxed{\boldsymbol{S_1 = 5}}$$

$S_2 = |A|$

$= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$= 4 - 6 \Rightarrow = -2$

$$\boxed{\boldsymbol{S_2 = -2}}$$

Put $S_1 = 5$ and $S_2 = -2$ in (1),

Hence the characteristic equation is

$$\boxed{\boldsymbol{\lambda^2 - 5\lambda + 2 = 0}}$$

Example 2: Find the characteristic equation of

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

Solution:

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \quad \text{... (1)}$$

Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

$S_1 =$ Sum of the main diagonal elements

$S_1 = 1 + 1 + 1 = 3$

$$\boxed{\mathbf{S_1 = 3}}$$

$S_2 =$ Sum of the minors of main diagonal elements

$$S_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$= (1 - 1) + (1 - 1) + (1 - 1)$

$$\boxed{\mathbf{S_2 = 0}}$$

$S_3 = |A|$

$$|A| = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$= 1\,(1 - 1) + 1\,(-1 - 1) - 1\,(1 + 1)$

$= 1\,(0) + 1\,(-2) - 1\,(2)$

$= 0 - 2 - 2 \Rightarrow -4$

$$\boxed{\mathbf{S_3 = -4}}$$

Put $S_1 = 3$, $S_2 = 0$ and $S_3 = -4$ in (1)

Hence the characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 3\lambda^2 + 0\lambda + 4 = 0}}$$

Example 3: Find the characteristic equation of the following matrices

(i) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution:

(i) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of main diagonal elements

$$\boxed{S_1 = 6}$$

S_2 = Sum of the minors of main diagonal elements

$$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$= (2 - 0) + (6 - 2) + (3 + 2)$

$= 2 + 4 + 5$

$$\boxed{S_2 = 11}$$

$$S_3 = |A|$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1\,(6-2) - 1\,(2-4)$$

$$= 1\,(4) - 1\,(-2)$$

$$= 4 + 2$$

$$\boxed{\mathbf{S_3 = 6}}$$

Put S_1, S_2 and S_3 in (1)

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}}$$

(ii) $\boldsymbol{A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}}$

The characteristic equation is

$$\lambda^3 - S_1\,\lambda^2 + S_2\,\lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of main diagonal elements

$= 4 + 3 + 1$

$$\boxed{\mathbf{S_1 = 8}}$$

$$S_2 = \begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix}$$

$= (12 + 10) + (3 - 8) + (4 - 4)$

$= 22 - 5 + 0$

$$\boxed{\mathbf{S_2 = 17}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{vmatrix}$$

$= 4\,(3 - 8) - 2\,(-5 + 4) - 2\,(-20 + 6)$

$= 4\,(-5) - 2\,(-1) - 2\,(-14)$

$= -20 + 2 + 28$

$$\boxed{\mathbf{S_3 = 10}}$$

Put S_1, S_2 and S_3 in (1)

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0}}$$

(iii) $\boldsymbol{A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}}$

The characteristic equation is

$$\lambda^3 - S_1\,\lambda^2 + S_2\,\lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$= 11 - 2 - 6$

$$\boxed{\mathbf{S_1 = 3}}$$

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix} + \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix}$$

$= (-22 + 28) + (12 - 20) + (-66 + 70)$

$= 6 - 8 + 4$

$$\boxed{\mathbf{S_2 = 2}}$$

$$S_3 = |A| = \begin{vmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{vmatrix}$$

$$= 11\,(12 - 20) + 4\,(-42 + 50) - 7\,(-28 + 20)$$

$$= 11\,(-8) + 4\,(8) - 7\,(-8)$$

$$= -88 + 32 + 56$$

$$= -88 + 88$$

$$\boxed{\mathbf{S_3 = 0}}$$

Substitute S_1, S_2, S_3 in (1)

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 3\lambda^2 + 2\lambda - 0 = 0}}$$

(iv) $\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{-7} & \mathbf{2} & \mathbf{-3} \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - S_1\,\lambda^2 + S_2\,\lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$$= 2 + 1 - 3$$

$$\boxed{\mathbf{S_1 = 0}}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & -3 \end{vmatrix}$$

$$= (2 - 4) + (-3 - 2) + (-6 + 0)$$

$$= -2 - 5 - 6$$

$$\boxed{\mathbf{S_2 = -13}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{vmatrix}$$

$= 2(-3-2) - 2(-6+7) + 0$

$= 2(-5) - 2(1)$

$= -10 - 2$

$$\boxed{S_3 = -12}$$

Substitute S_1, S_2 and S_3 in (1),

The characteristic equation is

$$\boxed{\lambda^3 - 0\lambda^2 - 13\lambda + 12 = 0}$$

(v) $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$= 2 + 1 - 1$

$$\boxed{S_1 = 2}$$

$S_2 = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}$

$= (2+2) + (-1-3) + (-2-3)$

$= 4 - 4 - 5$

$$\boxed{S_2 = -5}$$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$= 2(-1-3) + 2(-1-1) + 3(3-1)$

$= 2(-4) + 2(-2) + 3(2)$

$= -8 - 4 + 6$

$$\boxed{S_3 = -6}$$

The characteristic equation is

$$\boxed{\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0}$$

1.3 EIGEN VALUES AND EIGEN VECTORS

A linear transformation is defined by the equations

$$y_1 = a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n$$

$$y_2 = a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n$$

....

....

$$y_n = a_{n1} x_1 + a_{n2} x_2 + ... + a_{nn} x_n$$

which can be written as

$$Y = AX \text{ where } Y = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

The linear transformation of the form $Y = AX = \lambda X$ where λ is a scalar assumes much importance in the engineering applications. Let us find the values for λ for which this type of linear transformation occur.

$AX = \lambda X$, where X is **not a zero vector**

i.e. $AX - \lambda X = 0$

i.e. $AX - \lambda IX = 0$

i.e. $(A - \lambda I)X = 0$

which is a homogeneous system of equations. For this system a non trivial solution exists if $|A - \lambda I| = 0$.

$$\text{i.e.} \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad .... \ (1)$$

Note: Eigen values is also called latent roots, characteristic roots and proper values

Eigen vector is also called latent vector, characteristic vector and proper vector.

1.4 PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS

1. If all the eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$ of a matrix A are distinct, then the corresponding eigen vectors $X_1, X_2, \ldots X_n$ are all linearly independent.
2. If two or more eigenvalues of a matrix are equal, then the eigenvectors may be linearly independent or linearly dependent.
3. A square matrix A and its transport A' have the same characteristic values.

Taking $A = (a_{ij}), i, j = 1, 2, \dots n$, we have

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} - \lambda & \dots a_{2n} \\ & \dots\dots & \\ a_{n1} & a_{n2} & \dots a_{nn} - \lambda \end{vmatrix}$$

$$|A' - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{21} & \dots a_{n1} \\ a_{12} & a_{22} - \lambda & \dots a_{n2} \\ & \dots\dots & \\ a_{1n} & a_{n1} & \dots a_{nn} - \lambda \end{vmatrix}$$

$$|A - \lambda I| = |A' - \lambda I|$$

By changing rows into columns in the determinant, we get

A and A' have the same characteristic function and hence possess the same characteristic roots.

4. The product of the eigenvalues of the square matrix A is $|A|$.

   If $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigenvalues of A, then

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$$

   Setting, $\lambda = 0$, $\quad |A| = (-1)^n (-1)^n \lambda_1 \lambda_2 \dots \lambda_n = \lambda_1 \lambda_2 \dots \lambda_n$

Note:

If the matrix is singular, then at least one of the eigenvalue is zero and vice versa.

5. If $\lambda_1, \lambda_2, \dots \lambda_n$ are the latent roots of A, then

   (i) $k\lambda_1, k\lambda_2, \dots k\lambda_n$ will be the latent roots of kA, k being non-zero scalar.

   (ii) $\dfrac{1}{\lambda_1}, \dfrac{1}{\lambda_2}, \dots \dfrac{1}{\lambda_n}$ are the latent roots of A^{-1} if $\lambda_n \neq 0$.

   (iii) $\lambda_1^p, \lambda_2^p, \dots \lambda_n^p$ are the latent roots of A^p where p is a positive integer.

6. The sum of the eigenvalues of a matrix A is equal to the sum of its main diagonal or principal diagonal or leading diagonal. This is otherwise called as **Trace of the matrix *A*.**
7. The eigenvalues of a triangular matrices are its corresponding diagonal elements.
8. The eigenvalues of a given diagonal matrix are the elements of main diagonal.
9. If $|A| = 0$, then zero is the eigenvalue of the square matrix A.
10. If λ is a eigenvalue of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is a eigenvalue of the matrix Adj A if $\lambda \neq 0$.
11. If A and B are any two non-singular square matrices, then AB and BA have the same eigenvalues.
12. If A and B are any two square matrices and if A is non-singular then $A^{-1}B$ and BA^{-1} have the same eigenvalues.
13. The eigenvalues of a real symmetric matrix are real.
14. The eigenvalues of an identity matrix are all equal and it is equal to 1.
15. The eigenvectors corresponding to distinct eigenvalues of real symmetric matrix and unitary matrix are orthogonal.
16. A square matrix A is said to be orthogonal if $AA' = A'A = I$. Where I is the identity matrix.

WORKED EXAMPLES

Example 1: Show that the eigenvalues of a null matrix are zero. **[AU. Apr/ May 2018]**

Solution:

Let A be a zero matrix of order 3, then $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then the characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0 . \qquad \text{(ie)} \qquad (0-\lambda)\,[\lambda^2] = 0.$$

$$\Rightarrow \lambda^3 = 0$$

Therefore the eigenvalues of A are 0, 0, 0.

Example 2: If λ is an eigenvalue of a matrix A, then prove that $\frac{1}{\lambda}$ is the eigenvalue of A^{-1} **[A.U. Jan. 2009, Jan. 2015]**

Solution:

Let λ be an eigenvalue of A.

Then $AX = \lambda X$ (since X is the eigenvector and $X \neq 0$)

$$\Rightarrow X = \lambda A^{-1} X$$

$$\Rightarrow A^{-1} X = \frac{1}{\lambda} X$$

$\Rightarrow \frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Example 3: If λ is an eigenvalue of A, then prove that λ^2 is an eigenvalue of A^2.

Solution:

Let λ be an eigenvalue of A, then

$AX = \lambda X$ ($\because X$ is an eigenvector and $X \neq 0$)

Premultiplying both sides by A, we get

$$A\,(AX) = A\,(\lambda X)$$

$$A^2 X = \lambda\,(AX)$$

$$A^2X = \lambda\,(\lambda X)\ (\because AX = \lambda X)$$

$$A^2\,X = \lambda^2 X$$

$\Rightarrow \lambda^2$ is an eigenvalue of A^2.

Example 4: If λ is an eigenvalue of an orthogonal matrix then prove that $\frac{1}{\lambda}$ is also its eigenvalue.

Solution:

We know that if λ is an eigenvalue of a matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Since A is an orthogonal matrix, A^{-1} is same as its transpose A.

$\therefore \frac{1}{\lambda}$ is an eigenvalue of A'.

But the matrix A and A' have the same eigenvalues, since $|A - \lambda I|$ and $|A' - \lambda I|$ are the same.

Hence $\frac{1}{\lambda}$ is also an eigenvalue of A.

1.4.1 Eigen values and Eigen vectors of a non-symmetric matrix with non-repeated Eigen values

WORKED EXAMPLES

Example 1: Find the eigen values and eigen vectors of

$$\boldsymbol{A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}}$$

Solution:

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

S_1 = Sum of main diagonal elements

$= 1 + 2 + 3$

$$\boxed{\mathbf{S_1 = 6}}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$= (2-0) + (6-2) + (3+2)$

$= 2 + 4 + 5$

$$\boxed{\mathbf{S_2 = 11}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$= 1\,(6-2) - 0\,(3-2) - 1\,(2-4)$

$= 4 + 2$

$$\boxed{\mathbf{S_3 = 6}}$$

Put S_1, S_2 and S_3 in (1)

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}} \quad \text{... (2)}$$

solving equation (2)

Using synthetic division

Put $\lambda = 1$

$1^3 - 6 + 11 - 6 = 0$

$\Rightarrow \lambda = 1$ is a root.

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)\,(\lambda - 2) = 0$$

We get the eigen values are

$$\boxed{\begin{array}{c} \boldsymbol{\lambda_1 = 3} \\ \boldsymbol{\lambda_2 = 1} \\ \boldsymbol{\lambda_3 = 2} \end{array}}$$

To find eigen vector

The eigen vector is given by $(A - \lambda\, I)\, X = 0$

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots (1)$$

Case (i): Put $\lambda = 3$ in (1)

$$\begin{pmatrix} 1-3 & 0 & -1 \\ 1 & 2-3 & 1 \\ 2 & 2 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 0x_2 - 1x_3 = 0 \quad \text{... (i)}$$

$$1x_1 - 1x_2 + 1x_3 = 0 \quad \text{... (ii)}$$

$$2x_1 + 2x_2 + 0x_3 = 0 \quad \text{... (iii)}$$

From (i) and (ii) using cross multiplication we get

$$\begin{array}{cccc} 0 & -1 & -2 & 0 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{0-1} = \frac{x_2}{-1+2} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Case (ii): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + 0x_2 - 1x_3 = 0 \quad \text{... (i)}$$

$$1x_1 + 1x_2 + 1x_3 = 0 \quad \text{... (ii)}$$

$$2x_1 + 2x_2 + 2x_3 = 0 \quad \text{... (iii)}$$

Using cross multiplication rule

Consider (i) and (ii)

$$\begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x_1}{0+1} = \frac{x_2}{-1-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Case (iii): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 + 0x_2 - 1x_3 = 0 \quad \text{... (i)}$$

$$1x_1 + 0x_2 + 1x_3 = 0 \quad \text{... (ii)}$$

$$2x_1 + 2x_3 + 1x_3 = 0 \quad \text{... (iii)}$$

Using cross multiplication we get,

Consider (ii) and (iii)

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{array}$$

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$X_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$			
Eigen values	3	1	2
Eigen vectors	$X_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$	$X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	$X_3 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

Example 2: Find the characteristic values and characteristic vectors of $A = \begin{bmatrix} \mathbf{11} & \mathbf{-4} & \mathbf{-7} \\ \mathbf{7} & \mathbf{-2} & \mathbf{-5} \\ \mathbf{10} & \mathbf{-4} & \mathbf{-6} \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 11 & 7 & 10 \\ -4 & -2 & -4 \\ -7 & -5 & -6 \end{bmatrix}$$

$A \neq A^T \; \therefore A$ is Non-symmetric matrix

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of main diagonal elements

$= 11 - 2 - 6$

$$\boxed{\mathbf{S_1 = 3}}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix} + \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix}$$

$$= (-22 + 28) + (12 - 20) (-66 + 70)$$

$$\boxed{\mathbf{S_2 = 2}}$$

$$S_3 = |A|$$

$$= \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

$$= 11\,(12 - 20) + 4\,(-42 + 50) - 7\,(-28 + 20)$$

$$= 11\,(-8) + 4\,(8) - 7\,(-8)$$

$$= -88 + 32 + 56$$

$$\boxed{\mathbf{S_3 = 0}}$$

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 3\lambda^2 - 2\lambda - 0 = 0}} \qquad \text{... (2)}$$

solving (2)

$$\lambda^3 - 3\lambda^2 + 2\lambda - 0 = 0$$

Put $\lambda = 1$

$$1 - 3 + 2 - 0 = 0$$

$\Rightarrow \lambda = 1$ is a root.

By using synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -3 & 2 & 0 \\ & 0 & 1 & -2 & 0 \\ \hline & 1 & -2 & 0 & \boxed{0} \end{array}$$

$$\lambda^2 - 2\lambda + 0 = 0$$

$$\lambda = 0, \lambda = 2$$

We get the eigen values are

$$\boxed{\begin{array}{c}\boldsymbol{\lambda_1 = 2} \\ \boldsymbol{\lambda_2 = 0} \\ \boldsymbol{\lambda_3 = 1}\end{array}}$$

To find eigen vector

The eigen vector is given by $(A - \lambda I)\, X = 0$

$$\begin{pmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \ldots (1)$$

Case (i): Put $\lambda = 2$ in (1),

$$\begin{pmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$9x_1 - 4x_2 - 7x_3 = 0 \qquad \ldots (i)$$

$$7x_1 - 4x_2 - 5x_3 = 0 \qquad \ldots (ii)$$

$$10x_1 - 4x_2 - 8x_3 = 0 \qquad \ldots (iii)$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -4 & -7 & 9 & -4 \\ -4 & -5 & 7 & -4 \end{array}$$

$$\frac{x_1}{20-28} = \frac{x_2}{-49+45} = \frac{x_3}{-36+28}$$

$$\frac{x_1}{-8} = \frac{x_2}{-4} = \frac{x_3}{-8}$$

$$X_1 = \begin{bmatrix} -8 \\ -4 \\ -8 \end{bmatrix} \div -4$$

$$X_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Case (ii): Put $\lambda = 0$ in (1)

$$\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$11x_1 - 4x_2 - 7x_3 = 0 \qquad \text{... (i)}$$

$$7x_1 - 2x_2 - 5x_3 = 0 \qquad \text{... (ii)}$$

$$10x_1 - 4x_2 - 6x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -4 & -7 & 11 & -4 \\ -2 & -5 & 7 & -2 \end{array}$$

$$\frac{x_1}{20 - 14} = \frac{x_2}{-49 + 55} = \frac{x_3}{-22 + 28}$$

$$= \frac{x_1}{6} = \frac{x_2}{6} = \frac{x_3}{6}$$

$$X_2 = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} \div 6$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$10x_1 - 4x_2 - 7x_3 = 0 \qquad \text{... (i)}$$

$$7x_1 - 3x_2 - 5x_3 = 0 \qquad \text{... (ii)}$$

$$10x_1 - 4x_2 - 7x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -4 & -7 & 10 & -4 \\ -3 & -5 & 7 & -3 \end{array}$$

$$\frac{x_1}{20-21} = \frac{x_2}{-49+50} = \frac{x_3}{-30+28}$$

$$X_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$			
Eigen values	2	0	1
Eigen vectors	$X_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$	$X_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$	$X_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Example 3: Find the latent roots and latent vectors of the matrix

$$\boldsymbol{A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}}$$

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

$A \neq A^T \therefore A$ is non-symmetric matrix

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \quad \text{... (1)}$$

S_1 = Sum of main diagonal elements

$= 1 + 2 - 1$

$$\boxed{\mathbf{S_1 = 2}}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix}$$

$= (2 + 1) + (-2 - 1) + (-1 + 0)$

$$\boxed{\mathbf{S_2 = -1}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= 1(-2 - 1) - 1(1 + 0) - 2(-1 - 0)$

$= 1(-3) - 1(1) - 2(-1)$

$= -3 - 1 + 2$

$$\boxed{\mathbf{S_3 = -2}}$$

The characteristic equation

$$\lambda^3 - 2\lambda^2 - 1\lambda + 2 = 0 \quad \text{... (2)}$$

Solving (2)

Put $\lambda = 1$

$1 - 2 - 1 + 2 = 0$

$\Rightarrow \lambda = 1$ is a root

By using synthetic division

$$\begin{array}{c|rrr|r} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$$\lambda^2 - 1\lambda - 2 = 0$$

$$(\lambda + 1)\,(\lambda - 2) = 0$$

$$\lambda = -1, 2$$

We get the eigen values are

$$\boxed{\begin{array}{l} \boldsymbol{\lambda_1 = 1} \\ \boldsymbol{\lambda_2 = -1} \\ \boldsymbol{\lambda_3 = 2} \end{array}}$$

To find eigen vectors:

The eigen vectors is given by $(A - \lambda\, I)\, X = 0$

$$\left[\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \ldots (1)$$

Case (i): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 1-1 & 1 & -2 \\ -1 & 2-1 & 1 \\ 0 & 1 & -1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + 1x_2 - 2x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 + 1x_2 - 2x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) & (ii)

$$\begin{array}{cccc} 1 & -2 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{2-0} = \frac{x_3}{0+1}$$

$$X_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = -1$ in (1)

$$\begin{pmatrix} 1+1 & 1 & -2 \\ -1 & 2+1 & 1 \\ 0 & 1 & -1+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 + 1x_2 - 2x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 + 3x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 + 1x_2 + 0x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccccc} 1 & -2 & 2 & 1 \\ 3 & 1 & -1 & 3 \end{array}$$

$$\frac{x_1}{1+6}=\frac{x_2}{2-2}=\frac{x_3}{6+1}$$

$$\frac{x_1}{7}=\frac{x_2}{0}=\frac{x_3}{7}$$

$$X_2=\begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix} \div 7$$

$$X_2=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Case (iii): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} 1-2 & 1 & -2 \\ -1 & 2-2 & 1 \\ 0 & 1 & -1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1+1x_2-2x_3=0 \qquad \text{... (i)}$$

$$-1x_1+0x_2+1x_3=0 \qquad \text{... (ii)}$$

$$0x_1+1x_2-3x_3=0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccccc} 1 & -2 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{2+1} = \frac{x_3}{0+1}$$

$$X_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 2\lambda^2 - 1\lambda + 2 = 0$			
Eigen values	1	-1	2
Eigen vectors	$X_1 = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$	$X_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$	$X_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

Example 4: Find the eigen values and eigen vectors of the matrix $\mathbf{A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}}$

Solution:

Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

$A^T = \begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$

$A \neq A^T$ $\therefore A$ **is non-symmetric matrix**

The characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = Sum of the main diagonal elements

$= 2 + 1 - 3$

$$\boxed{\mathbf{S_1 = 0}}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & -3 \end{vmatrix}$$

$$= (2-4) + (-3-2) + (-6+0)$$

$$= -2-5-6$$

$$\boxed{\mathbf{S_2 = -13}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{vmatrix}$$

$$= 2(-3-2) - 2(-6+7) + 0$$

$$= 2(-5) - 2(1)$$

$$= -10-2$$

$$\boxed{\mathbf{S_3 = -12}}$$

The characteristic equation

$$\boxed{\boldsymbol{\lambda^3 - 0\lambda^2 - 13\lambda + 12 = 0}} \quad \text{... (2)}$$

Put $\lambda = 1$ in (2)

$$1 - 0 - 13 + 12 = 0$$

$$0 = 0$$

$\Rightarrow \lambda = 1$ is a root.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \multicolumn{1}{|r}{0} \end{array}$$

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda + 4)\,(\lambda - 3) = 0$$

$$\lambda = -4, 3$$

On solving equation (2) we get the eigen values are

$$\boxed{\begin{array}{l} \boldsymbol{\lambda_1 = 3} \\ \boldsymbol{\lambda_2 = -4} \\ \boldsymbol{\lambda_3 = 1} \end{array}}$$

To find eigen vectors

The eigen vector is given by

$$(A - \lambda I)\,X = 0$$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{... (1)}$$

Case (i): Put $\lambda = 3$ in (1)

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 + 2x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$2x_1 - 2x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

$$-7x_1 + 2x_2 - 6x_3 = 0 \qquad \text{... (iii)}$$

Consider (i) and (ii)

By using cross multiplication rule,

$$\begin{array}{cccc} 2 & 0 & -1 & 2 \\ -2 & 1 & 2 & -2 \end{array}$$

$$\frac{x_1}{2-0}=\frac{x_2}{0+1}=\frac{x_3}{2-4}$$

$$\Rightarrow \frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$$X_1=\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case (ii): Put $\lambda = -4$ in (1)

$$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$6x_1 + 2x_2 + 0x_3 = 0 \quad \text{... (i)}$$

$$2x_1 + 5x_2 + 1x_3 = 0 \quad \text{... (ii)}$$

$$-7x_1 + 2x_2 + 7x_3 = 0 \quad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{matrix} 2 & & 0 & & 6 & & 2 \\ & \times & & \times & & \times & \\ 5 & & 1 & & 2 & & 5 \end{matrix}$$

$$\frac{x_1}{2-0}=\frac{x_2}{0-6}=\frac{x_3}{30-4}$$

$$\frac{x_1}{2}=\frac{x_2}{-6}=\frac{x_3}{26}$$

$$X_2=\begin{bmatrix} 2 \\ -6 \\ 26 \end{bmatrix} \div 2 \quad X_2=\begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix}$$

Case (iii): Put $\lambda = 1$ in (1)

$$1x_1 + 2x_2 + 0x_3 = 0 \quad \text{... (i)}$$

$$2x_1 + 0x_2 + 1x_3 = 0 \quad \text{... (ii)}$$

$$-7x_1 + 2x_2 - 4x_3 = 0 \quad \text{... (iii)}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By cross multiplication rule, consider (i) and (ii)

$$\begin{matrix} 2 & & 0 & & 1 & & 2 \\ & \times & & \times & & \times & \\ 0 & & 1 & & 2 & & 0 \end{matrix}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0-1} = \frac{x_3}{0-4}$$

$$X_3 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 0\lambda^2 - 13\lambda + 12 = 0$			
Eigen values	3	-4	1
Eigen vectors	$X_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$	$X_2 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$	$X_3 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$

1.4.2 Eigen values and eigen vectors of symmetric matrix with non-repeated eigen values:

WORKED EXAMPLES

Example 1: Find the eigen values and eigen vectors of

$$A = \begin{pmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

$$A = A^T$$

$\therefore$ A is symmetric matrix

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \dots (1)$$

S_1 = Sum of main diagonal elements

$$\boxed{S_1 = 8}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix}$$

$$= (6 - 1) + (6 - 1) + (9 - 0)$$

$$= 5 + 5 + 9$$

$$\boxed{S_2 = 19}$$

$S_3 = |A|$

$$= \begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{vmatrix}$$

$$= 3\,(6 - 1) + 1\,(-3 + 0) - 0$$

$$= 3\,(5) + 1\,(-3)$$

$$= 15 - 3$$

$$\boxed{S_3 = 12}$$

The characteristic equation

$$\boxed{\boldsymbol{\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0}} \quad \text{... (2)}$$

Put $\lambda = 1$

$$1 - 8 + 19 - 12 = 0$$

$$20 - 20 = 0$$

$\Rightarrow \lambda = 1$ is a root.

Now using synthetic division.

$$\begin{array}{c|cccc} 1 & 1 & -8 & 19 & -12 \\ & 0 & 1 & -7 & 12 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 4)(\lambda - 3) = 0$$

$$\lambda = 4, 3$$

Solving equation (2) we get the eigen values are,

$$\boxed{\begin{array}{c} \boldsymbol{\lambda_1 = 4} \\ \boldsymbol{\lambda_2 = 1} \\ \boldsymbol{\lambda_3 = 3} \end{array}}$$

To find eigen vectors:

The eigen vectors is given by $(A - \lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{... (1)}$$

Case (i): Put $\lambda = 4$ in (1)

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 - 1x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 - 2x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii),

$$\begin{array}{cccc} -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -2 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{2-1}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 - 1x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 + 1x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 - 1x_2 + 2x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 1 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0+2} = \frac{x_3}{2-1}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Case (iii): Put $\lambda = 3$ in (1)

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - 1x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 - 1x_2 + 0x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$			
Eigen values	4	1	3
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Example 2: **Find the eigen values and eigen vectors of**

$$A = \begin{pmatrix} \mathbf{10} & \mathbf{-2} & \mathbf{-5} \\ \mathbf{-2} & \mathbf{2} & \mathbf{3} \\ \mathbf{-5} & \mathbf{3} & \mathbf{5} \end{pmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$$

$$A = A^T$$

$\therefore A$ is symmetric matrix

The characteristic equation

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \ldots (1)$$

$S_1 =$ Sum of main diagonal elements

$= 10 + 2 + 5$

$$\boxed{S_1 = 17}$$

$S_2 =$ Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix}$$

$= (20 - 4) + (10 - 9) + (50 - 25)$

$= 16 + 1 + 25$

$$\boxed{S_2 = 42}$$

$S_3 = |A|$

$$= \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{vmatrix}$$

$= 10\,(10 - 9) + 2\,(-10 + 15) - 5\,(-6 + 10)$

$= 10\,(1) + 2\,(5) - 5\,(4)$

$= 10 + 10 - 20$

$$\boxed{\mathbf{S_3 = 0}}$$

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 17\lambda^2 + 42\lambda - 0 = 0}} \qquad \text{... (2)}$$

Put $\lambda = 3$ in (2)

$$27 - 17\,(3) + 42\,(3) - 0 = 0$$

$$0 = 0$$

$\Rightarrow \lambda = 3$ is a root.

By using synthetic division.

$$\begin{array}{c|cccc} 3 & 1 & -17 & 42 & 0 \\ & 0 & 3 & -42 & 0 \\ \hline & 1 & -14 & 0 & \multicolumn{1}{|c}{0} \\ \cline{5-5} \end{array}$$

$$\lambda^2 - 14\lambda + 0 = 0$$

$$\lambda = 0,\ 14$$

Solving equation (2) we get eigen values are

$$\boxed{\begin{aligned}\lambda_1 &= 14\\ \lambda_2 &= 0\\ \lambda_3 &= 3\end{aligned}}$$

To find eigen vectors

The eigen vectors is given by $(A - \lambda I)X = 0$

$$\begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{... (1)}$$

Case (i): Put $\lambda = 14$ in (1)

$$\begin{pmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x_1 - 2x_2 - 5x_3 = 0 \quad \text{... (i)}$$

$$-2x_1 - 12x_2 + 3x_3 = 0 \quad \text{... (ii)}$$

$$-5x_1 + 3x_2 - 9x_3 = 0 \quad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{matrix} -2 & -5 & -4 & -2 \\ -12 & 3 & -2 & -12 \end{matrix}$$

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$X_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Case (ii): Put $\lambda = 0$ in (1)

$$\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$10x_1 - 2x_2 - 5x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 + 2x_2 + 3x_3 = 0 \qquad \text{... (ii)}$$

$$-5x_1 + 3x_2 + 5x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -2 & -5 & 10 & -2 \\ 2 & 3 & -2 & 2 \end{array}$$

$$\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

$$X_2 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Case (iii): Put $\lambda = 3$ in (1)

$$\begin{pmatrix} 7 & -2 & -5 \\ -2 & -1 & 3 \\ -5 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 2x_2 - 5x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 - 1x_2 + 3x_3 = 0 \qquad \text{... (ii)}$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{matrix} -2 & & -5 & & 7 & & -2 \\ & \times & & \times & & \times & \\ -1 & & 3 & & -2 & & -1 \end{matrix}$$

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$X_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 17\lambda^2 + 42\lambda - 0 = 0$			
Eigen values	14	0	3
Eigen vectors	$X_1 = \begin{pmatrix} -3 \\ +1 \\ +2 \end{pmatrix}$	$X_2 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$	$X_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

1.4.3 Eigen values and Eigen vector of non symmetric matrix with repeated eigen values

WORKED EXAMPLES

Example 1: Find the eigen values and eigen vectors of $\boldsymbol{A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}}$

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$A \neq A^T$$

$\therefore A$ is non symmetric matrix.

$$A^T = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

S_1 = Sum of main diagonal elements

$= 2 + 3 + 2$

$$\boxed{\mathbf{S_1 = 7}}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$= (6 - 2) + (6 - 2) + (4 - 1)$

$= 4 + 4 + 3$

$$\boxed{\mathbf{S_2 = 11}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$= 2\,(6 - 2) - 2\,(2 - 1) + 1\,(2 - 3)$

$= 2\,(4) - 2\,(1) + 1\,(-1)$

$$\boxed{\mathbf{S_3 = 5}}$$

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0}} \qquad \ldots (2)$$

Solving equation (2)

Put $\lambda = 1$ in (2),

$$1 - 7 + 11 - 5 = 0$$

$$0 = 0$$

$\Rightarrow \lambda = 1$ is a root,

By using synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \multicolumn{1}{|r}{0} \end{array}$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\lambda = 5, 1$$

We get eigen values are

$$\boxed{\begin{array}{l} \boldsymbol{\lambda_1 = 5} \\ \boldsymbol{\lambda_2 = 1} \\ \boldsymbol{\lambda_3 = 1} \end{array}}$$

To find eigen vectors:

The eigen vectors is given by $(A - \lambda I) X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots (1)$$

Case (i): Put $\boldsymbol{\lambda = 5}$ in (1)

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 2x_2 + 1x_3 = 0 \qquad \text{... (i)}$$

$$1x_1 - 2x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

$$1x_1 + 2x_2 - 3x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} 2 & 1 & -3 & 2 \\ -2 & 1 & 1 & 2 \end{array}$$

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$X_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \div 4$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - 1x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$1x_1 + 2x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

$$1x_1 + 2x_2 + 1x_3 = 0 \qquad \text{... (iii)}$$

Consider (ii)

$$1x_1 + 2x_2 + 1x_3 = 0$$

$$\text{put } x_1 = 0$$

$$2x_2 + 1x_3 = 0$$

$$2x_2 = -1x_3$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

Case (iii):

Consider (ii)

$$1x_1 + 2x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

put $x_2 = 0$ $\Rightarrow 1x_1 + 1x_3 = 0$

$$1x_1 = -1x_3$$

$$X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The characteristic equation $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$			
Eigen values	5	1	1
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$	$X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Example 2: Find the eigen values and eigen vector of the matrix $A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$A^T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$A \neq A^T \; \therefore A$ is non-symmetric matrix

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$= 2 + 2 + 1$

$$\boxed{S_1 = 5}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$= (4 - 1) + (2 - 0) + (2 - 0)$

$= 3 + 2 + 2$

$$\boxed{S_2 = 7}$$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 2\,(2 - 0) - 1\,(1 - 0) + 1\,(0 - 0)$

$= 2\,(2) - 1\,(1) + 0$

$= 4 - 1$

$$\boxed{S_3 = 3}$$

The characteristic equation

$$\boxed{\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0}$$

Put $\lambda = 1$ in (1)

$$1 - 5 + 7 - 3 = 0$$

$$0 - 0 = 0$$

$\lambda = 1$ is a root.

By using synthetic divison

$$\begin{array}{c|cccc} 1 & 1 & -5 & 7 & -3 \\ & 0 & 1 & -4 & 3 \\ \hline & 1 & -4 & 3 & \multicolumn{1}{|c}{0} \end{array}$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 3$$

solving equation (2) we get the eigen values are

$$\boxed{\begin{array}{c} \boldsymbol{\lambda_1 = 3} \\ \boldsymbol{\lambda_2 = 1} \\ \boldsymbol{\lambda_3 = 1} \end{array}}$$

To find eigen vectors

The eigen vectors is given by $(A - \lambda I) X = 0$

$$\left[\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots (1)$$

Case (i): Put $\lambda = 3$ in (1)

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 + 1x_2 + 1x_3 = 0 \quad \ldots \text{(i)}$$

$$1x_1 - 1x_2 + 1x_3 = 0 \quad \ldots \text{(ii)}$$

$$0x_1 + 0x_2 - 2x_3 = 0 \quad \ldots \text{(iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{1+1} = \frac{x_3}{1-1}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$\therefore X_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \div 2$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case (ii): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$1x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (i)}$$

$$1x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 + 0x_2 + 0x_3 = 0 \qquad \text{... (iii)}$$

Consider (i)

$$1x_1 + 1x_2 + 1x_3 = 0$$

put $x_1 = 0$

$$1x_2 + 1x_3 = 0$$

$$1x_2 = -1x_3$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii)

Consider

$1x_1 + 1x_2 + 1x_3 = 0$

put $x_2 = 0$

$1x_1 = -1x_3$

$$X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The characteristic equation $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$			
Eigen values	3	1	1
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

1.4.4 Eigen values and Eigen vectors of a symmetric matrix with repeated eigen values

WORKED EXAMPLES

Example 1: Find the eigen values and eigen vector of a matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$\therefore A = A^T$ $\therefore$ ***A* is symmetric matrix**

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \quad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$= 6 + 3 + 3$

$$\boxed{S_1 = 12}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}$$

$= (18 - 4) + (9 - 1) + (18 - 4)$

$= 14 + 8 + 14$

$$\boxed{S_2 = 36}$$

$S_3 = |A|$

$$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 6\,(9 - 1) + 2\,(-6 + 2) + 2\,(2 - 6)$

$$\boxed{S_3 = 32}$$

The characteristic equation is

$$\boxed{\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0} \quad \text{... (2)}$$

solving equation (2)

Put $\lambda = 2$ in (2)

$$2^3 - 12\,(2)^2 + 36\,(2) - 32 = 0$$

$$8 - 48 + 72 - 32 = 0$$

$$0 - 0 = 0$$

$\Rightarrow \lambda = 2$ is a root

By using synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \multicolumn{1}{|r}{0} \end{array}$$

$$(\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 2)\,(\lambda - 8) = 0$$

$$\lambda = 2, 8$$

We get the eigen values are

$$\boxed{\begin{array}{c} \boldsymbol{\lambda_1 = 8} \\ \boldsymbol{\lambda_2 = 2} \\ \boldsymbol{\lambda_3 = 2} \end{array}}$$

To find eigen vectors:

The eigen vectors is given by $(A - \lambda I)\,X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{... (1)}$$

Case (i): Put $\lambda = 8$ in (1)

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 - 5x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$2x_1 - 1x_2 - 5x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$X_1 = \begin{bmatrix} 12 \\ -6 \\ 6 \end{bmatrix} \div 6$$

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 + 1x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$2x_1 - 1x_2 + 1x_3 = 0 \qquad \text{... (iii)}$$

Consider (i)

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$\textbf{put } \boldsymbol{x_1 = 0}$$

$$-2x_2 + 2x_3 = 0$$

$$-2x_2 = -2x_3$$

$$X_2 = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} \div (-2)$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow 2l - 1m + 1n = 0$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 0l + 1m + 1n = 0$$

$$X_3 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ l & m & n \end{pmatrix}$$

$$2x_1 - 1x_2 + 1x_3 = 0 \qquad \text{... (i)}$$

$$0x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

Consider (i) and (ii)

$$\begin{array}{cccc} -1 & 1 & 2 & -1 \\ 1 & 1 & 0 & 1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0-2} = \frac{x_3}{2-0}$$

$$X_3 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \div 2$$

$$X_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$			
Eigen values	8	2	2
Eigen vectors	$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Example 2: Find the Eigen values and Eigen vectors of $\boldsymbol{A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}}$

Solution:

$$\text{Let } A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

$A = A^T$ $\therefore$ ***A* is symmetric matrix**

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of main diagonal elements

$= 3 + 3 + 3$

$$\boxed{S_1 = 9}$$

$$S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}$$

$= (9 - 1) + (9 - 1) + (9 - 1)$

$= 8 + 8 + 8$

$$\boxed{S_2 = 24}$$

$S_3 = |A|$

$$= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3\,(9 - 1) - 1\,(3 + 1) + 1\,(-1 - 3)$

$= 3\,(8) - 1\,(4) + 1\,(-4)$

$$\boxed{S_3 = 16}$$

The characteristic equation is

$$\boxed{\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0} \qquad \dots (2)$$

Put $\lambda = 1$ in (2)

$$1 - 9 + 24 - 16 = 0$$

$$25 - 25 = 0$$

$\therefore \lambda = 1$ is a root

Now by synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline & 1 & -8 & 16 & \boxed{0} \end{array}$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)\,(\lambda - 4) = 0$$

$$\lambda = 4, 4$$

Solving equation (2) we get the eigen values are

$$\boxed{\begin{array}{c} \boldsymbol{\lambda_1 = 1} \\ \boldsymbol{\lambda_2 = 4} \\ \boldsymbol{\lambda_3 = 4} \end{array}}$$

To find eigen vectors:

The eigen vectors is given by $(A - \lambda I)\,X = 0$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{... (1)}$$

Case (i): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (i)}$$

$$1x_1 + 2x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$1x_1 - 1x_2 + 2x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$X_1 = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} \div 3 \quad X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = 4$ in (1)

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (i)}$$

$$1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (iii)}$$

Consider (i)

$$-1x_1 + 1x_2 + 1x_3 = 0$$

$$\textbf{put } \boldsymbol{x_1 = 0}$$

$$1x_2 + 1x_3 = 0$$

$$1x_2 = -1x_3$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow 1l - 1m - 1n = 0$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow 0l - 1m + 1n = 0$$

$$X_3 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 1 & m & n \end{pmatrix}$$

$$1l - 1m - 1n = 0 \qquad \text{... (i)}$$

$$0l - 1m + 1n = 0 \qquad \text{... (ii)}$$

By using cross multiplication

Consider (i) and (ii)

$$\begin{array}{cccc} -1 & -1 & 1 & -1 \\ -1 & 1 & 0 & -1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X_3 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$			
Eigen values	1	4	4
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$

Example 3: Find the eigen values and eigen vectors of $\boldsymbol{A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}}$

Solution:

$$\text{Let } A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$A = A^T$ $\therefore A$ is symmetric matrix.

The characteristic equation

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$$\boxed{\boldsymbol{S_1 = 0}}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1) + (0-1) + (0-1)$$

$$= -1 - 1 - 1$$

$$\boxed{\boldsymbol{S_2 = -3}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 0\,(0 - 1) - (0 + 1) + 1\,(-1 - 0)$$

$$= -1 + 1\,(-1)$$

$$= -1 - 1$$

$$\boxed{\boldsymbol{S_3 = -2}}$$

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 0\lambda^2 - 3\lambda + 2 = 0}} \qquad \ldots (2)$$

Put $\lambda = 1$ in (2)

$$1 - 0 - 3 + 2 = 0$$

$$-3 + 3 = 0$$

$\Rightarrow \lambda = 1$ is a root.

Now using synthetic division method

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \multicolumn{1}{|c}{0} \end{array}$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda - 1)\,(\lambda + 2) = 0$$

$$\lambda = 1, -2$$

solving equation (2) we get the eigen values

$$\boxed{\begin{array}{l} \boldsymbol{\lambda_1 = -2} \\ \boldsymbol{\lambda_2 = 1} \\ \boldsymbol{\lambda_3 = 1} \end{array}}$$

To find eigen vectors:

The eigen vectors is given by $(A - \lambda I)\, X = 0$

$$\begin{pmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & -1 \\ 1 & -1 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{... (1)}$$

Case (i): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (i)}$$

$$1x_1 + 2x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$1x_1 - 1x_2 + 2x_3 = 0 \qquad \text{... (iii)}$$

By using cross multiplication rule,

Consider (i) and (ii)

$$\begin{array}{ccccccc} 1 & & 1 & & 2 & & 1 \\ & \times & & \times & & \times & \\ 2 & & -1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$X_1 = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} \div 3$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = 1$ in (1)

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 + 1x_2 + 1x_3 = 0 \quad \text{... (i)}$$

$$1x_1 - 1x_2 - 1x_3 = 0 \quad \text{... (ii)}$$

$$1x_1 - 1x_2 - 1x_3 = 0 \quad \text{... (iii)}$$

Consider (i)

$$-1x_1 + 1x_2 + 1x_3 = 0$$

$$\textbf{put } \boldsymbol{x_1 = 0}$$

$$1x_2 + 1x_3 = 0$$

$$1x_2 = -1x_3$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow 1l - 1m - 1n = 0$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow 0l - 1m + 1n = 0$$

$$1l - 1m - 1n = 0 \quad \text{... (i)}$$

$$0l - 1m + 1n = 0 \quad \text{... (ii)}$$

Consider (i) and (ii)

$$X_3 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ l & m & n \end{pmatrix}$$

$$\begin{array}{cccc} -1 & -1 & 1 & -1 \\ -1 & 1 & 0 & -1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0-2} = \frac{x_3}{-1+0}$$

$$X_3 = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$$

The characteristic equation $\lambda^3 - 0\lambda^2 - 3\lambda + 2 = 0$			
Eigen values	-2	1	1
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$

Example 4: Find the eigen values and eigen vectors of

$$\boldsymbol{A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^T = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$A = A^T$ $\therefore$ ***A* symmetric matrix**

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \quad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$$\boxed{S_1 = 3}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= (1 - 1) + (1 - 1) + (1 - 1)$$

$$\boxed{S_2 = 0}$$

$$S_3 = |A|$$

$$= \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 1\,(1 - 1) + 1\,(-1 - 1) - 1\,(1 + 1)$$

$$= 1\,(0) + 1\,(-2) - 1\,(2)$$

$$= -2 - 2$$

$$\boxed{S_3 = -4}$$

The characteristic equation is

$$\boxed{\lambda^3 - 3\lambda^2 + 0\lambda + 4 = 0} \quad \text{... (2)}$$

Put $\lambda = -1$ in (2)

$-1 \quad -3 + 0 + 4 = 0$

$-4 + 4 = 0$

$\Rightarrow \lambda = -1$ is a root.

Now by synthetic division method

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \multicolumn{1}{|r}{0} \end{array}$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 2, 2$$

solving equation (2) we get the eigen values

$$\boxed{\begin{array}{l} \boldsymbol{\lambda_1 = -1} \\ \boldsymbol{\lambda_2 = 2} \\ \boldsymbol{\lambda_3 = 2} \end{array}}$$

To find eigen vectors:

The eigen vector is given by $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{... (1)}$$

Case (i) Put $\lambda = -1$ in (1)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 + 2x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$-1x_1 - 1x_2 - 2x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{ccccccc} -1 & & -1 & & 2 & & -1 \\ & \times & & \times & & \times & \\ 2 & & -1 & & -1 & & 2 \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$X_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \div 3$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$-1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (iii)}$$

Consider (i)

$$-1x_1 - 1x_2 - 1x_3 = 0$$

$$\text{put } x_1 = 0$$

$$-1x_2 - 1x_3 = 0$$

$$-1x_2 = 1x_3$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 1l + 1m + 1n = 0$$

$$X_2 = \begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix} \Rightarrow 0l + 1m - 1n = 0$$

$$X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & m & n \end{pmatrix}$$

$$1l + 1m + 1n = 0 \qquad \text{... (i)}$$

$$0l + 1m - 1n = 0 \qquad \text{... (ii)}$$

Consider (i) and (ii)

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0+1} = \frac{x_3}{1-0}$$

$$X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 3\lambda^2 + 0\lambda + 4 = 0$			
Eigen values	-1	2	2
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	$X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Example 5: Find the eigen values and eigen vectors of $A = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$A = A^T$ $\therefore$ **A is symmetric matrix**

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

S_1 = Sum of the main diagonal elements

$= 0 + 0 + 0$

$$\boxed{S_1 = 0}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= (0 - 1) + (0 - 1) + (0 - 1)$

$= -1 - 1 - 1$

$$\boxed{S_2 = -3}$$

$S_3 = |A|$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$= 0 - 1\,(0 - 1) + 1\,(1 - 0)$

$$= -1\,(-1) + 1\,(1)$$

$$= 1 + 1$$

$$\boxed{S_3 = 2}$$

The characteristic equation

$$\boxed{\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0} \qquad \dots (2)$$

Put $\lambda = -1$ in (2)

$$-1 - 0 + 3 - 2 = 0$$

$$3 - 3 = 0$$

$\Rightarrow \lambda = -1$ is a root

By synthetic division method,

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda + 1)\,(\lambda - 2) = 0$$

$$\lambda = -1, 2$$

Solving equation (2) we get the eigen values are

$$\boxed{\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = -1 \\ \lambda_3 = -1 \end{array}}$$

To find eigen vectors

The eigen vectors is given by $(A - \lambda I)\,X = 0$

$$\begin{pmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots (1)$$

Case (i): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 1x_2 + 1x_3 = 0 \quad \ldots (i)$$

$$1x_1 - 2x_2 + 1x_3 = 0 \quad \ldots (ii)$$

$$1x_1 + 1x_2 - 2x_3 = 0 \quad \ldots (iii)$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$X_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \div 3$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = -1$ in (1)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$1x_1 + 1x_2 + 1x_3 = 0 \quad \ldots (i)$$

$$1x_1 + 1x_2 - 1x_3 = 0 \quad \ldots (ii)$$

$$1x_1 + 1x_2 + 1x_3 = 0 \quad \ldots (iii)$$

Consider (i)

$$1x_1 + 1x_2 + 1x_3 = 0$$

$$\textbf{put } \boldsymbol{x_1 = 0}$$

$$1x_2 + 1x_3 = 0$$

$$1x_2 = -\,1x_3$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 1l + 1m + 1n = 0$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow 0l - 1m + 1n = 0$$

$$X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & m & n \end{pmatrix}$$

$$1l + 1m + 1n = 0 \qquad \text{... (i)}$$

$$0l - 1m + 1n = 0 \qquad \text{... (ii)}$$

Consider (i) and (ii)

$$\begin{array}{cccccccc} 1 & \times & 1 & \times & 1 & \times & 1 \\ -1 & & 1 & & 0 & & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_1}{0-1} = \frac{x_3}{-1-0}$$

$$X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

The characteristic equation $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$			
Eigen values	2	-1	1
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

1.5 DIAGONALISATION OF A MATRIX

1.5.1 Similar Matrices

Two matrices A and B are said to be similar if there exist a matrix P such that $B = P^{-1}AP$.

When $B = P^{-1}AP$ we say that matrix B is said to be obtained from A by a similarity transformation. The situation will be interesting when B becomes a diagonal matrix. The process of reducing A to a diagonal matrix is known as diagonalisation.

Note: Similar matrices will have the same set of eigen values.

1.5.2 Diagonalization of a matrix by Orthogonal Transformation

If A is **real symmetric matrix,** then we can diagonalise the matrix by orthogonal transformation (Orthogonal reduction).

Procedure to diagonalize the matrix by orthogonal transformation

1. Normalise each eigenvector X_r. For this, divide each element of the eigenvector X_r by the square root of the sum of the square of all the elements of X_r.

2. Form the normalised Modal Matrix N by using the normalised eigenvectors of A.

3. Find $D = N^TAN$. Transforming the matrix A into D by means of the transformation $N^TAN = D$ is known as **orthogonal transformation.**

Note:

For the orthogonal matrix A, $\boldsymbol{A^{-1} = A^T}$

Orthogonal Reduction of a Real Symmetric Matrix

If A is a real symmetric matrix, then its eigenvalues are orthogonal in pairs. If we normalize each eigenvector X_r, then the resulting modal matrix N will be orthogonal, $\therefore N^{-1} = N^T$.

Hence the similarity transformation $N^{-1}AN$ becomes $N^T AN = D$, where D is the diagonal matrix whose diagonal elements are the eigenvalues of A. Transforming A into D by means of the transformations $N^T AN = D$ is known as *orthogonal transformation or orthogonal reduction* and is possible only for real symmetric matrices.

Result:

If A is a square matrix of order n, having n linearly independent eigen vectors then, a matrix B can be found such that $B^{-1}AB = D$ is a diagonal matrix.

The matrix B has the eigen vectors as its column and the diagonal matrix consists of eigen values along the diagonal.

WORKED EXAMPLES

Example 1: Diagonalize the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **using orthogonal transformation.**

Solution:

$$\text{Let } A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

$A = A^T \;\therefore A$ symmetric matrix

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}} \quad \text{... (1)}$$

$$S_1 = 6 + 3 + 3$$

$$\boxed{\boldsymbol{S_1 = 12}}$$

$$S_2 = \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= (18 - 4) + (9 - 1) + (18 - 4)$$

$$= 14 + 8 + 14$$

$$\boxed{\boldsymbol{S_2 = 36}}$$

$S_3 = |A|$

$$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6\,(9 - 1) + 2\,(-6 + 2) + 2\,(2 - 6)$$

$$= 6\,(8) + 2\,(-4) + 2\,(-4)$$

$$= 48 - 8 - 8$$

$$\boxed{\boldsymbol{S_3 = 32}}$$

The characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0}} \quad \text{... (2)}$$

Solving equation (2) we get eigen values [Refer Example 1 section 1.4.4]

$$\boxed{\begin{aligned} \boldsymbol{\lambda_1 = 8} \\ \boldsymbol{\lambda_2 = 2} \\ \boldsymbol{\lambda_3 = 2} \end{aligned}}$$

To find eigen vectors

The eigen vectors is given by $(A - \lambda I)\, X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{... (1)}$$

Case (i): Put $\lambda = 8$ in (1)

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 - 5x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$2x_1 - 1x_2 - 5x_3 = 0 \qquad \text{... (iii)}$$

Consider (i) and (ii)

Using cross multiplication rule,

$$\begin{array}{cccc} -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$X_1 = \begin{bmatrix} 12 \\ -6 \\ 6 \end{bmatrix} \div 6$$

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$

$$-2x_1 + 1x_2 - 1x_3 = 0 \quad \text{... (ii)}$$

$$2x_1 - 1x_2 + 1x_3 = 0 \quad \text{... (iii)}$$

Since all the three equations are same.

Consider any equation

$$\text{(i)} \Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$$

$$\textbf{put } \boldsymbol{x_1 = 0}$$

$$-2x_2 + 2x_3 = 0$$

$$-2x_2 = -2x_3$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow 2l - 1m + 1n = 0$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 0l + 1m + 1n = 0$$

$$X_3 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ l & m & n \end{pmatrix}$$

$$2l - 1m + 1n = 0 \qquad \text{... (i)}$$

$$0l + 1m + 1n = 0 \qquad \text{... (ii)}$$

Consider (i) and (ii)

Using cross multiplication, we get

$$\begin{array}{cccc} -1 & 1 & 2 & -1 \\ 1 & 1 & 0 & 1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0-2} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$X_3 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \div 2$$

$$X_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$			
Eigen values	8	2	2
Eigen vectors	$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Diagonalisation by Orthogonal Transformation

$$D = N^T AN$$

$A =$ Given matrix

$N =$ Normalized modal matrix

$N^T =$ Transpose of N

Modal matrix

$$\begin{array}{cc} & \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ P = & \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

Normalization

$$X_1 = \sqrt{2^2 + (-1)^2 + 1^2} \quad X_2 = \sqrt{0^2 + 1^2 + 1^2} \quad X_3 = \sqrt{(-1)^2 + (-1)^2 + 1^2}$$

$$= \sqrt{4 + 1 + 1} \quad = \sqrt{0 + 1 + 1} \quad = \sqrt{1 + 1 + 1}$$

$$= \sqrt{6} \quad = \sqrt{2} \quad = \sqrt{3}$$

N - Normalized modal matrix

$$N = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{array}{ccc} N^T & A & N \end{array}$$

$$D = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 2: Diagonalize $A = \begin{pmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$ using orthogonal transformation

Solution:

Given $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$A = A^T \;\therefore\; A$ is symmetric matrix

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

S_1 = Sum of the main diagonal elements

$= 1 + 1 + 1$

$$\boxed{S_1 = 3}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$= (1 - 1) + (1 - 1) + (1 - 1)$

$$\boxed{S_2 = 0}$$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$= 1\,(1 - 1) + 1\,(-1 - 1) - 1\,(1 + 1)$

$= 1\,(0) + 1\,(-2) - 1\,(-2)$

$= -2 - 2$

$$\boxed{S_3 = -4}$$

The characteristic equation

$$\boxed{\lambda^3 - 3\lambda^2 + 0\lambda + 4 = 0} \qquad \dots (2)$$

Solving equation (2) we get the eigen values are [Refer Example 4 section 1.44]

$$\boxed{\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = 2 \\ \lambda_3 = 2 \end{array}}$$

To find eigen vectors.

The eigen vectors is given by $(A - \lambda I)\, X = 0$

$$\begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \ldots (1)$$

Case (i): Put $\lambda = -1$ in (1)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 - 1x_2 - 1x_3 = 0 \qquad \ldots \text{(i)}$$

$$-1x_1 + 2x_2 - 1x_3 = 0 \qquad \ldots \text{(ii)}$$

$$-1x_1 - 1x_2 + 2x_3 = 0 \qquad \ldots \text{(iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & 2 \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$X_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \div 3$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (i)}$$

$$-1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (ii)}$$

$$-1x_1 - 1x_2 - 1x_3 = 0 \qquad \text{... (iii)}$$

Since all the three equations are same. Consider any (1) equation.

Consider

$$(i) \Rightarrow -1x_1 - 1x_2 - 1x_3 = 0$$

$$\text{put } x_1 = 0$$

$$-1x_2 - 1x_3 = 0$$

$$-1x_2 = 1x_3$$

$$-2x_2 = -2x_3$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 1l + 1m + 1n = 0 \qquad \text{... (i)}$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ +1 \end{pmatrix} \Rightarrow 0l + 1m - 1n = 0 \qquad \text{... (ii)}$$

$$X_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ l & m & n \end{bmatrix}$$

Consider (i) and (ii) By cross multiplication rule,

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$			
Eigen values	-1	2	2
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Diagonalization

$D = N^T AN$

$A-$ Given matrix

N – Normalized modal matrix

N^T – Transpose of N

Modal matrix

$$p = \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

Normalization

$X_1 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ $\quad X_2 = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$ $\quad X_3 = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

***N* – Normalized modal matrix**

$$N=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

$$N^T=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}$$

Diagonalization

$$D=N^T AN$$

$$D=\underbrace{\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}}_{N^T}\underbrace{\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{pmatrix}}_{N}$$

$$D=\begin{pmatrix}1 & 0 & 2\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$$

Example 3: $A=\begin{pmatrix}\mathbf{0} & \mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{1} & \mathbf{0}\end{pmatrix}$ **using orthogonal transformation.**

Solution:

Let $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$

$$A^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$A = A^T$ **$\therefore$ *A* is symmetric matrix.**

The characteristic equation is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \quad \text{... (1)}$$

S_1 = Sum of the main diagonal elements

$= 0 + 0 + 0$

$$\boxed{S_1 = 0}$$

S_2 = Sum of minors of main diagonal elements

$$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= (0 - 1) + (0 - 1) + (0 - 1)$

$= -1 - 1 - 1$

$$\boxed{S_2 = -3}$$

$S_3 = |A|$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\boxed{S_3 = 2}$$

The characteristic equation

$$\boxed{\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0} \quad \text{... (2)}$$

Solving equation (2) we get the eigen values are [Refer Example 5 Section 1.4.4]

$$\boxed{\begin{array}{l} \boldsymbol{\lambda_1 = 2} \\ \boldsymbol{\lambda_2 = -1} \\ \boldsymbol{\lambda_3 = -1} \end{array}}$$

To find eigen vectors

The eigen vectors is given by $(A - \lambda I)\,X = 0$

$$\begin{pmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{... (1)}$$

Case (i): Put $\lambda = 2$ in (1)

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + 1x_2 + 1x_3 = 0 \qquad \text{... (i)}$$

$$1x_1 - 2x_2 + 1x_3 = 0 \qquad \text{... (ii)}$$

$$1x_1 + 1x_2 - 2x_3 = 0 \qquad \text{... (iii)}$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{ccccccc} 1 & & 1 & & -2 & & 1 \\ & \times & & \times & & \times & \\ -2 & & 1 & & 1 & & -2 \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$X_1 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \div 3$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): Put $\lambda = -1$ in (1)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$1x_1 + 1x_2 + 1x_3 = 0 \quad \text{... (i)}$$

$$1x_1 + 1x_2 + 1x_3 = 0 \quad \text{... (ii)}$$

$$1x_1 + 1x_2 + 1x_3 = 0 \quad \text{... (iii)}$$

Since all the three equations are same consider any equation

$$1x_1 + 1x_2 + 1x_3 = 0$$

$$\text{put } x_1 = 0$$

$$1x_2 = -1x_3$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii): Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be the third eigen vector

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 1l + 1m + 1n = 0 \quad \text{... (i)}$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \Rightarrow 0l - 1m + 1n = 0 \quad \text{... (ii)}$$

$$X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ l & m & n \end{pmatrix}$$

By cross multiplication rule

$$1x_1 + 1x_2 + 1x_3 = 0$$

$$0x_1 - 1x_2 + 1x_3 = 0$$

Using cross multiplication rule, consider (i) and (ii)

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$$\frac{x_2}{2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

The characteristic equation $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$			
Eigen values	-1	2	-1
Eigen vectors	$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$	$X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

Diagonalization

$$\boxed{\boldsymbol{D = N^T A N}}$$

$A -$ Given matrix

N – Normalized modal matrix

N^T – Transpose of N

Modal matrix

$$p = \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \end{array}$$

Normalization

$$X_1 = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \qquad X_2 = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2} \qquad X_3 = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$$

$N -$ Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{16}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$\boxed{\boldsymbol{D = N^T A N}}$$

$$\begin{array}{cccc} & N^T & A & N \\ = & \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \end{array}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1.6 CAYLEY - HAMILTON - THEOREM

Statement

Every square matrix satisfies its own characteristic equation.

Uses of Cayley - Hamilton theorem

To find

(i) The positive integral powers of A

(ii) The inverse of a square matrix A

1.6.1 Problems - based on Cayley - Hamilton theorem [2×2 matrix]

WORKED EXAMPLES

Example 1: Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation.

Solution:

Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

Characteristic equation of A is given by

$$\lambda^2 - S_1 \lambda - S_2 = 0 \qquad \text{... (1)}$$

$S_1 =$ sum of main diagonal elements

$= 1 + 1 = 2$

$S_2 = |A|$

$= 1 + 4 = 5$

$\therefore \lambda^2 - 2\lambda + 5 = 0 \qquad \text{... (2)}$

Replace λ by A in (2) we get

$$A^2 - 2A + 5I = 0$$

To prove, by Cayley - Hamilton theorem

$$A^2 = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & -2-2 \\ 2+2 & -4+1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\therefore\ A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A^2 - 2A + 5I = 0$ is proved.

$\therefore$ Given matrix satisfies its own characteristic equation.

Example 2: $A = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} \end{bmatrix}$ **write** A^2 **interms of** A **and** I **using Cayley-Hamilton theorem.**

Solution:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

Characteristic equation of A is

$$\lambda^2 - S_1\lambda + S_2 = 0 \qquad \dots (1)$$

S_1 = sum of main diagonal elements

$= 1 + 5$

$S_1 = 6$

$S_2 = |A|$

$S_2 = 5$

Characteristic Equation $\lambda^2 - 6\lambda + 5\lambda = 0$ ... (2)

by Cayley-Hamilton theorem, put $\lambda = A$ in (2)

$$A^2 - 6A + 5I = 0$$

$$\therefore A^2 = 6A - 5I$$

Example 3: If $A = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$ express A^3 interms of A and I using Cayley-Hamilton theorem.

Solution:

Characteristic equation of A is

$$\lambda^2 - S_1 \lambda + S_2 = 0 \qquad \text{... (1)}$$

S_1 = sum of main diagonal elements

$= 1 + 5$

$\boxed{S_1 = 6}$

$S_2 = |A|$

$\boxed{S_2 = 5}$

The characteristic equation $\lambda^2 - 6\lambda + 5 = 0$ ... (2)

by Cayley-Hamilton theorem, put $\lambda = A$ in (2),

$$A^2 - 6A + 5I = 0$$

$$A^2 = 6A - 5I \qquad \text{... (*)}$$

To find A^3 multiplying both sides by A

$$A \cdot A^2 = A\,(6A - 5I)$$

$$A^3 = 6A^2 - 5A$$

$$= 6\,(6A - 5I) - 5A \qquad \text{(by *)}$$

$$= 36A - 30I - 5A$$

$$A^3 = 31A - 30I$$

Example 4: If $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ find A^{-1} using Cayley-Hamilton theorem.

Solution:

Given: $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Characteristic equation of A is

$$\lambda^2 - S_1\,\lambda + S_2 = 0 \qquad \text{... (1)}$$

S_1 = sum of main diagonal elements

$= 4 + 2$

$\boxed{S_1 = 6}$

$S_2 = |A|$

$= 8 - 3$

$\boxed{S_2 = 5}$

The characteristic equation $\lambda^2 - 6\lambda + 5 = 0$ ... (2)

Using Cayley-Hamilton theorem, put $\lambda = A$ in (2),

$$A^2 - 6A + 5I = 0$$

To find A^{-1}

$\div$ by A

$$\frac{A^2}{A} - \frac{6A}{A} + \frac{5}{A} = 0$$

$$A = 6I + \frac{5}{A} = 0$$

$$\frac{5}{A} = 6I - A$$

$$5A^{-1} = 6I - A$$

$$A^{-1} = \frac{1}{5}[6I - A]$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

1.6.2 Problems based on Cayley-Hamilton theorem [for 3×3 matrix]

WORKED EXAMPLES

Example 1: Using Cayley-Hamilton theorem find A^{-1} when $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = sum of main diagonal elements

$= 1 + 1 + 1$

$$\boxed{S_1 = 3}$$

S_2 = sum of minors of diagonal elements

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$= (1 - 1) + (1 - 3) + (1 - 0)$

$= 0 - 2 + 1$

$$\boxed{S_2 = -1}$$

$S_3 = |A|$

$= 1\,(1 - 1) - 0\,(2 + 1) + 3\,(-2 - 1)$

$$\boxed{S_3 = -9}$$

The characteristic equation $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$ ... (1)

by Cayley-Hamilton theorem [Replace λ by A]

$$A^3 - 3A^2 - A + 9I = 0$$

$\div$ by A

$$A^2 - 3A - I + 9A^{-1} = 0$$

$$-9A^{-1} = A^2 - 3A - I$$

$$A^{-1} = \frac{-1}{9}[A^2 - 3A - I]$$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix}$$

$$\therefore\ A^2 - 3A - I = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

Example 2: Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **and hence find** A^{-1}**.**

Solution:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Characteristic equation of A is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$$

S_1 = sum of main diagonal elements

$$= 2 + 2 + 2$$

$$\boxed{\mathbf{S_1 = 6}}$$

S_2 = sum of minors of diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4 - 1) + (4 - 1) + (4 - 1)$$

$$= 3 + 3 + 3$$

$$\boxed{\mathbf{S_2 = 9}}$$

$S_3 = |A|$

$$= 2\,(4 - 1) + 1\,(-2 + 1) + 1\,(1 - 2)$$

$$= 6 - 1 - 1$$

$$\boxed{\mathbf{S_3 = 4}}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

by Cayley-Hamilton theorem [Replace λ by A]

$$A^3 - 6A^2 + 9A - 4I = 0 \qquad \text{... (1)}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$9A = 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

$$9A = \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix}$$

$$4I = 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Consider $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & +30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $A^3 - 6A^2 + 9A - 4I = 0$ is verified.

To find A^{-1}

$$A^3 - 6A^2 + 9A - 4I = 0$$

$\div$ by A

$A^2 - 6A + 9I - 4A^{-1} = 0$

$4A^{-1} = A^2 - 6A + 9I$

$A^{-1} = \frac{1}{4}[A^2 - 6A + 9I]$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 0 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 3: Verify Cayley-Hamilton theorem and use it $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ to find A^{-1}.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Characteristic equation of 3×3 matrix is given by

$$\lambda^3 - S_1 \lambda^2 + S_3 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = sum of diagonal elements

$= 1 + 4 + 6$

$\boxed{\mathbf{S_1 = 11}}$

S_2 = sum of minors of diagonal elements

$$= \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$= (24 - 25) + (6 - 9) + (4 - 4)$

$= -1 - 3 + 0$

$\boxed{\mathbf{S_2 = -4}}$

$S_3 = |A|$

$= 1\,(24 - 25) - 2\,(12 - 15) + 3\,(10 - 12)$

$= -1 + 6 - 6$

$\boxed{\mathbf{S_3 = -1}}$

The characteristic equation $\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$ ... (2)

by Cayley-Hamilton theorem [Replace λ by A]

$$A^3 - 11A^2 - 4A + I = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+4+9 & 2+8+15 & 3+10+18 \\ 2+8+15 & 4+16+25 & 6+20+30 \\ 3+10+18 & 6+20+30 & 9+25+36 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 14+50+93 & 28+100+155 & 42+125+186 \\ 25+90+168 & 50+180+280 & 75+225+336 \\ 31+112+210 & 62+224+350 & 93+280+420 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

Consider $A^3 - 11A^2 - 4A + I$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - 11 \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - \begin{bmatrix} 154 & 275 & 341 \\ 275 & 495 & 616 \\ 341 & 616 & 779 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 20 \\ 12 & 20 & 24 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 11A^2 - 4A + I = 0$ $\therefore$ Cayley-Hamilton theorem is verified.

To find A^{-1}:

$$A^3 - 11A^2 - 4A + I = 0$$

$\div A$

$$A^2 - 11A - 4I + A^{-1} = 0$$

$$A^{-1} = 11A + 4I - A^2$$

$$= \begin{bmatrix} 154 & 275 & 341 \\ 275 & 495 & 616 \\ 341 & 616 & 770 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 134 & 250 & 310 \\ 250 & 454 & 560 \\ 310 & 560 & 696 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 134 & 250 & 310 \\ 250 & 454 & 560 \\ 310 & 560 & 696 \end{bmatrix}$$

Example 4: Verify Cayley-Hamilton theorem $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ and hence find A^{-1}.

Solution:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Characteristic equation

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \ldots (1)$$

S_1 = sum of main diagonal elements

$= 1 + 2 + 2$

$\boxed{S_1 = 5}$

S_2 = sum of minors of diagonal elements

$$= \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}$$

$= (4 - 0) + (2 - 0) + (2 - 0)$

$= 4 + 2 + 2$

$\boxed{S_2 = 8}$

$S_3 = |A|$

$= 1\,(4-0) - 0\,(4-0) + (-2)\,(0-0)$

$\boxed{\mathbf{S_3 = 4}}$

The characteristic equation $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$ ... (2)

Cayley-Hamilton theorem [Replace λ by A]

$$A^3 - 5A^2 + 8A - 4I = 0$$

$$A^2 = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & -2+0-4 \\ 2+4+0 & 0+4+0 & -4+8+8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & -2+0-12 \\ 6+8+0 & 0+8+0 & -12+16+24 \\ 0+0+0 & 0+0+0 & 0+0+8 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix}$$

Consider $\mathbf{A^3 - 5A^2 + 8A - 4I}$

$$= \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -30 \\ 30 & 20 & 60 \\ 0 & 0 & 20 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -16 \\ 16 & 16 & 32 \\ 0 & 0 & 16 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 5A^2 + 8A - 4I = 0$ Cayley-Hamilton theorem is verified.

To find A^{-1}

$$A^3 - 5A^2 + 8A - 4I = 0$$

$\div A$

$$A^2 - 5A + 8I - 4A^{-1} = 0$$

$$\mathbf{4A^{-1} = A^2 - 5A + 8I}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -10 \\ 10 & 10 & 20 \\ 0 & 0 & 10 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 4 \\ -4 & 2 & -8 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Example 5: If $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ verify Cayley-Hamilton theorem and find A^4.

Solution:

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

Characteristic equation

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = sum of main diagonal elements

$$\boxed{\mathbf{S_1 = 2 + 2 + 2 = 6}}$$

S_2 = sum of minors of diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4 + 1) + (4 - 2) + (4 - 1)$$

$$\boxed{\mathbf{S_2 = 6 + 2 + 3 = 10}}$$

$$S_3 = |A|$$

$$= 2\,(4 + 1) + 1\,(-2 + 1) + 2\,(-1 - 2)$$

$$\boxed{\mathbf{S_3 = 10 - 1 - 6 = 3}}$$

The characteristic equation $\lambda^3 - 6\lambda^2 + 10\lambda - 3 = 0$ ... (2)

Using Cayley-Hamilton theorem [Replace λ by A]

$$A^3 - 6A^2 + 10A - 3I = 0$$

$$A^3 = 6A^2 - 10A + 3I$$

Multiplying by A

$$A^4 = 6A^3 - 10A^2 + 3A$$

$$= 6\,[6A^2 - 10A + 3I] - 10A^2 + 3A$$

$$= 36A^2 - 60A + 18I - 10A^2 + 3A$$

$$A^4 = 26A^2 - 57A + 18I$$

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+2 & -2-2+2 & 4+1+4 \\ -2-2-1 & 1+4-1 & -2-2-2 \\ 2-1+2 & -1+2+2 & 2-1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -2 & 9 \\ -5 & 4 & -6 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\boldsymbol{A^4 = 26A^2 - 57A + 18I}$$

$$= 26\begin{bmatrix} 7 & -2 & 9 \\ -5 & 4 & -6 \\ 3 & 3 & 5 \end{bmatrix} - 57\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 182 & -52 & 234 \\ -130 & 104 & -156 \\ 78 & 78 & 130 \end{bmatrix} + \begin{bmatrix} -114 & 57 & -114 \\ 57 & -114 & 57 \\ -57 & -57 & -114 \end{bmatrix}$$

$$+ \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 86 & 5 & 120 \\ 73 & 8 & -99 \\ 21 & 21 & 34 \end{bmatrix}$$

Example 6: Verify Cayley-Hamilton theorem using it to find the inverse of $A = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

Characteristic equation of A is

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = sum of main diagonal elements

$= 1 + 2 + 3$

$\boxed{S_1 = 6}$

S_2 = sum of minors of diagonal elements

$$= \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$= (6 - 3) + (3 - 2) + (2 - 1)$

$= 3 + 1 + 1$

$\boxed{S_2 = 5}$

$S_3 = |A|$

$= 1\,(6 - 3) - 1\,(3 + 6) + 1\,(-1 - 4)$

$= 3 - 9 - 5$

$\boxed{S_3 = -11}$

The characteristic equation $\lambda^3 - 6\lambda^2 + 5\lambda + 11 = 0$ ... (2)

by Cayley-Hamilton theorem [Replace λ by A]

$$A^3 - 6A^2 + 5A + 11I = 0$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1-2-6 & 1+4+3 & 1-6-9 \\ 2-1-6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$\boldsymbol{A^3 - 6A^2 + 5A + 11I}$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$+ \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$ Cayley-Hamilton theorem is verified.

To find A^{-1}:

$$A^3 - 6A^2 + 5A + 11I = 0$$

$$\div A \qquad A^2 - 6A + 5I + 11A^{-1} = 0$$

$$11A^{-1} = 6A - A^2 - 5I$$

$$11A^{-1} = 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Example 7: Use Cayley-Hamilton theorem to find the value of the matrix given by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Characteristic equation

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \qquad \dots (1)$$

S_1 = sum of main diagonal elements

$= 2 + 1 + 2$

$\boxed{S_1 = 5}$

S_2 = sum of minors of diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$= (2 - 0) + (4 - 1) + (2 - 0)$

$= 2 + 3 + 2$

$\boxed{\mathbf{S_2 = 7}}$

$S_3 = |A|$

$= 2\,(2 - 0) - 1\,(0 - 0) + 1\,(0 - 1)$

$= 4 - 1$

$\boxed{\mathbf{S_3 = 3}}$

The characteristic equation $\lambda^3 - 5\lambda^2 + 7\lambda - 3I = 0$ ... (3)

By Cayley-Hamilton theorem [Replace λ by A]

$$A^3 - 5A^2 + 7A - 3I = 0$$

By division algorithm

Dividend = [Quotient × Divisor] + Remainder

Long division method

Let $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$\begin{array}{r|l}
 & A^5 + A \quad Q \\
\hline
A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\
 & A^8 - 5A^7 + 7A^6 - 3A^5 \\
\hline
 & \qquad\qquad\qquad\qquad A^4 - 5A^3 + 8A^2 - 2A + I \\
 & \qquad\qquad\qquad\qquad (+)\ (+)\ (-)\ (+) \\
 & \qquad\qquad\qquad\qquad A^4 - 5A^3 + 7A^2 - 3A \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad A^2 + A + 1 \qquad R \\
\cline{2-2}
\end{array}$$

$$\therefore f(A) = (A^5 + A)(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$= (A^5 + A)(0) + (A^2 + A + I)$$

$$\text{Since } [A^3 - 5A^2 + 7A - 3I = 0]$$

$$f(A) = A^2 + A + I$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore f(A) = A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

QUADRATIC FORM

A homogeneous polynomial of the second degree in any number of variables is called a Quadratic form.

Example

$3x^2 + 2xy + 7y^2$ and $x_1^2 + 2x_2^2 + 4x_3^2 + 6x_1 x_2 + 7x_1 x_3 + 5x_2 x_3$ are Quadratic forms in 2 and 3 variables respectively.

The general form of a Quadratic form, denoted by Q in 'n' variables is

$$Q = \sum_{j=1}^{n} \int_{i=1}^{n} a_{ij} X_i X_j$$

The matrix corresponding to the Quadratic form is

$$\begin{bmatrix} \text{Co.eff } x_1^2 & \frac{1}{2}\text{Co.eff} \cdot x_1 x_2 & \frac{1}{2}\text{Co.eff } x_1 x_3 \\ \frac{1}{2}\text{Co.eff} \cdot x_2 x_1 & \text{Co.eff } x_2^2 & \frac{1}{2}\text{Co.eff } x_2 x_3 \\ \frac{1}{2}\text{Co.eff} \cdot x_3 x_1 & \frac{1}{2}\text{Co.eff } x_3 x_2 & \text{Co.eff } x_3^2 \end{bmatrix}$$

WORKED EXAMPLES

Case 1: Problems based on matrix of the Quadratic form.

Example 1: Write the matrix of the Quadratic form $Q = 2x^2 + 8z^2 + 4xy + 10xz - 2yz$

Solution:

$$A = \begin{bmatrix} \text{Co.eff } x^2 & \frac{1}{2}\text{Co.eff } xy & \frac{1}{2}\text{Co.eff } xz \\ \frac{1}{2}\text{Co.eff} \cdot yx & \text{Co.eff } y^2 & \frac{1}{2}\text{Co.eff } yz \\ \frac{1}{2}\text{Co.eff } zx & \frac{1}{2}\text{Co.eff } zy & \text{Co.eff } z^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

Note

$$A = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} x^2 & \frac{1}{2}xy & \frac{1}{2}xz \\ \frac{1}{2}xy & y^2 & \frac{1}{2}yz \\ \frac{1}{2}xz & \frac{1}{2}zy & z^2 \end{bmatrix} \end{array}$$

Example 2: Write down the matrix of Quadratic form $Q = x^2 + y^2 + 2z^2 + 3xy + 4yz - zx$

Solution:

$$A = \begin{bmatrix} \text{Co.eff } x^2 & \frac{1}{2}\text{Co.eff } xy & \frac{1}{2}\text{Co.eff } xz \\ \frac{1}{2}\text{Co.eff} \cdot yx & \text{Co.eff } y^2 & \frac{1}{2}\text{Co.eff } yz \\ \frac{1}{2}\text{Co.eff } zx & \frac{1}{2}\text{Co.eff } zy & \text{Co.eff } z^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \frac{3}{2} & \frac{-1}{2} \\ \frac{3}{2} & 1 & 2 \\ -\frac{1}{2} & 2 & 2 \end{bmatrix}$$

Example 3: Write down the matrix corresponding to the Quadratic form $Q = 2x^2 + 2y^2 + 3z^2 + 2xy - 4xz - 4yz$

Solution:

$$A = \begin{bmatrix} \text{Co.eff } x^2 & \frac{1}{2}\text{Co.eff } xy & \frac{1}{2}\text{Co.eff } xz \\ \frac{1}{2}\text{Co.eff} \cdot yx & \text{Co.eff } y^2 & \frac{1}{2}\text{Co.eff } yz \\ \frac{1}{2}\text{Co.eff } zx & \frac{1}{2}\text{Co.eff } zy & \text{Co.eff } z^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

NATURE OF THE QUADRATIC FORM

Rank of *A*

When the Quadratic form is reduced to the canonical form it contains only *r* terms which is the rank of *A*.

Note: [For 3×3 matrix]

If $S_3 = 0$ then Rank $= 2$

If $S_3 \neq 0$ then Rank $= 3$

Index of the Quadratic form

The number of positive terms in the canonical form is called the index (p) of the Quadratic form.

Signature of the Quadratic form

The difference between the number of positive and negative terms is called signature(s) of the Quadratic form [i.e $S = 2p - r$]

The Quadratic form $Q = X^T AX$ in n variables is said to be

(i)	Positive definite	If $r = n$ and $p = n$ or if **all the eigen values of A are positive.**
(ii)	Negative definite	If $r = n$ and $p = 0$ or if **all the eigen values of A are negative.**
(iii)	Positive semi-definite	If $r < n$ and $p = r$ or if all the eigen values of $A \geq 0$ and **atleast one eigen value is zero.**
(iv)	Negative semi-definite	If $r < n$ and $p = 0$ or if all the eigen values of $A \leq 0$ and **atleast one eigen value is zero.**
(v)	Indefinite	In all other cases or if A has **positive** as well as **negative** eigen values.

Rules for finding nature of Quadratic form using principal sub-determinants

Let A be square matrix of order n

$D_1 = |a_{11}|$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= \dots \ \dots \ \dots \ \dots$

$= \dots \ \dots \ \dots \ \dots$

$D_n = |A|$

Here $D_1, D_2 \dots D_n$ are called the principal subdeterminants of A. From $D_1, D_2 \dots D_n$ the nature of the Quadratic form can be determined.

- A Quadratic form is positive definite if $D_1, D_2 \dots D_n$ are all positive i.e $D_n > 0$ for all n.
- A Quadratic form is negative definite if $D_1, D_3, D_5 \dots$ are all negative and $D_2, D_4, D_6 \dots$ are all positive i.e. $(-1)^n D_n > 0$ for all n.
- A Quadratic form is positive semi-definite if $D_n \geq 0$ and alteast one $D_i = 0$.
- A Quadratic form is negative semi-definite if $(-1)^n D_n \geq 0$ and atleast one $D_i = 0$.
- A Quadratic form is indefinite in all other cases.

WORKED EXAMPLES

Problems based on nature of the Quadratic Form

Example 1: State the nature of Quadratic form $Q = 2xy + 2yz + 2zx$.

Solution:

The matrix of Quadratic form is

$$A = \begin{bmatrix} \text{Co.eff } x^2 & \frac{1}{2}\text{Co.eff } xy & \frac{1}{2}\text{Co.eff } xz \\ \frac{1}{2}\text{Co.eff} \cdot yx & \text{Co.eff } y^2 & \frac{1}{2}\text{Co.eff } yz \\ \frac{1}{2}\text{Co.eff } zx & \frac{1}{2}\text{Co.eff } zy & \text{Co.eff } z^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|D_1| = 0$$

$$|D_2| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$|D_3| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0\,(0-1) - 1\,(0-1) + 1\,(1-0) = 2$$

$$\therefore D_2 = -1 < 0$$

$$D_3 = 2 > 0$$

$$\boxed{\therefore \textbf{ It is indefinite in nature}}$$

Example 2: Show that the Quadratic form $Q = 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ is positive definite.

Solution:

The matrix of the Quadratic form is

$$A = \begin{bmatrix} \text{Co.eff } x_1^2 & \frac{1}{2}\text{Co.eff} \cdot x_1 x_2 & \frac{1}{2}\text{Co.eff } x_1 x_3 \\ \frac{1}{2}\text{Co.eff} \cdot x_2 x_1 & \text{Co.eff } x_2^2 & \frac{1}{2}\text{Co.eff } x_2 x_3 \\ \frac{1}{2}\text{Co.eff} \cdot x_3 x_1 & \frac{1}{2}\text{Co.eff } x_3 x_2 & \text{Co.eff } x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$|D_1| = 3 \text{ (+ ve)}$$

$$|D_2| = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = (9 - 1) = 8 \text{ (+ ve)}$$

$$|D_3| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} \begin{aligned} &= 3\,(9-1) - 1\,(3+1) + 1\,(-1-3) \\ &= 24 - 4 - 4 \qquad \text{(+ ve)} \\ &= 16 \end{aligned}$$

Here D_1, D_2, D_3 are the positive and $D_n > 0$ for all n.

$\boxed{\therefore \text{ Quadratic form is positive definite}}$

Example 3: Determine the nature of the following Quadratic form without reducing them to canonical form $Q = 2x_1^2 + x_2^2 - 3x_3^2 + 12x_1 x_2 - 8x_2 x_3 - 4x_3 x_1$.

Solution:

The matrix of the Quadratic form is

$$A = \begin{bmatrix} \text{Co.eff } x_1^2 & \frac{1}{2}\text{Co.eff} \cdot x_1 x_2 & \frac{1}{2}\text{Co.eff } x_1 x_3 \\ \frac{1}{2}\text{Co.eff} \cdot x_2 x_1 & \text{Co.eff } x_2^2 & \frac{1}{2}\text{Co.eff } x_2 x_3 \\ \frac{1}{2}\text{Co.eff} \cdot x_3 x_1 & \frac{1}{2}\text{Co.eff } x_3 x_2 & \text{Co.eff } x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & 3 \end{bmatrix}$$

$|D_1| = 2$ (+ ve)

$$|D_2| = \begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} \begin{matrix} = (-6-4) \\ = -10 \end{matrix} \; (-\text{ ve})$$

$$|D_3| = \begin{vmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{vmatrix}$$

$$= 2(-3-16) - 6(-18-8) - 2(-24+2)$$

$$= -38 + 156 + 44$$

$$= 162 \; (+\text{ ve})$$

$D_1 > 0, D_2 < 0, D_3 > 0$

$\therefore$ The Quadratic form is indefinite

ORTHOGONAL REDUCTION OF QUADRATIC FORM TO ITS CANONICAL FORM

Step 1	:	Write the matrix of the given Quadratic form.
Step 2	:	Find the characteristic equation
Step 3	:	Find the eigen values
Step 4	:	Find the eigen vectors
Step 5	:	Find the eigen vectors orthogonal to each other
Step 6	:	Form normalised modal matrix N and N^T
Step 7	:	Find $D = N^T AN$
Step 8	:	Canonical form $[Y_1 \; Y_2 \; Y_3] \, [D] \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$

WORKED EXAMPLES

Problems based on Orthogonal Reduction of Quadratic Form to its Canonical Form

Example 1: Reduce the given Quadratic form Q to its canonical form using orthogonal transformation $Q = x^2 + 3y^2 + 3z^2 - 2yz$.

Solution:

$$Q = x^2 + 3y^2 + 3z^2 - 2yz$$

Step 1: The matrix of the Quadratic form is

$$A = \begin{bmatrix} \text{Co.eff } x^2 & \frac{1}{2}\text{ Co.eff } xy & \frac{1}{2}\text{ Co.eff } xz \\ \frac{1}{2}\text{ Co.eff} \cdot yx & \text{Co.eff } y^2 & \frac{1}{2}\text{ Co.eff } yz \\ \frac{1}{2}\text{ Co.eff } zx & \frac{1}{2}\text{ Co.eff } zy & \text{Co.eff } z^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Step 2: Find the characteristics equation of A

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \ldots (1)$$

S_1 = sum of main diagonal elements

$= 1 + 3 + 3$

$\boxed{S_1 = 7}$

S_2 = sum of the minors of the main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$$

$= (9 - 1) + (3 - 0) + (3 - 0)$

$\boxed{S_2 = 8 + 3 + 3 = 14}$

$S_3 = |A|$

$= 1\,(9 - 1)$

$\boxed{\mathbf{S_3 = 8}}$

The characteristic equation $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$ ... (2)

Step 3: Find the eigen values

To find the eigen values solve the equation (2)

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

Put $\lambda = 1$

$$1 - 7 + 14 - 8 = 0$$

$$0 = 0$$

$\therefore$ $(\lambda - 1)$ is a factor.

$\lambda = 1$ is a root

using synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -7 & 14 & -8 \\ & 0 & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & \multicolumn{1}{|c}{0} \end{array}$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 4)\,(\lambda - 2) = 0$$

$$\lambda = 4 \quad \lambda = 2$$

$\therefore$ Eigen values 1, 2, 4

Step 4: Find the eigen vectors $(A - \lambda I)\,X = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{... (1)}$$

Case 1: Put $\lambda = 1$ in equation (1)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 + 0x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$0x_1 + 2x_2 - x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 - x_2 + 2x_3 = 0 \qquad \text{... (iii)}$$

From (ii) & (iii) solve by cross multiplication

$$\begin{array}{cccc} 2 & -1 & 0 & 2 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{4-1} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0}$$

i.e $$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

Hence the eigen vector $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case 2: When $\lambda = 2$ put in equation (1)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$-x_1 + 0x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$0x_1 + x_2 - x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 - x_2 + x_3 = 0 \qquad \text{... (iii)}$$

From (i) and (ii) solve by cross multiplication

$$\begin{array}{cccc} 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

i.e $$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, the eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case 3: When $\lambda = 4$ put in equation (1)

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x_1 + 0x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$0x_1 - x_2 - x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 - x_2 - x_3 = 0 \qquad \text{... (iii)}$$

From (i) and (ii) solve by cross multiplication

$$\begin{array}{cccc} 0 & 0 & -3 & 0 \\ -1 & -1 & 0 & -1 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0-3} = \frac{x_3}{3-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-3} = \frac{x_3}{3}$$

i.e $$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence, the eigen vector $X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Step 5: Verify eigen vectors are orthogonal to each other.

$$X_1 X_2^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (0 \;\; 1 \;\; 1) = 0$$

$$X_2 X_3^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} (0 \;\; -1 \;\; 1) = 0$$

$$X_3 X_1^T = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} (1 \;\; 0 \;\; 0) = 0$$

$$\therefore X_1 X_2^T = X_2 X_3^T = X_3 X_1^T = 0$$

Hence eigen vectors are orthogonal to each other.

Step 6: Find the Normalised modal matrix $P\,(X_1\, X_2\, X_3)$

$$\text{Modal matrix, } P = \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Normalization

$$X_1 = \sqrt{1^2 + 0^2 + 0^2} = \sqrt{1} = 1 \qquad X_2 = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \qquad X_3 = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$$

Normalised modal matrix N

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step 7: Find $D = N^T AN$

$$AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}} & 0-\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}} \\ 0+0+0 & 0-\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}} & 0+\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}} \end{bmatrix}$$

$$AN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}$$

$D = N^T AN$

$$\begin{array}{cc} & \quad N^T \qquad\qquad\qquad AN \\ D = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{-4}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+1 & 0-\frac{4}{\sqrt{2}}+\frac{4}{\sqrt{2}} \\ 0+0+0 & 0-\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}} & 0+2+2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Step 8: Canonical form

$$C = Y^T DY$$

$$C = [Y_1\ Y_2\ Y_3]\ [D] \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= [Y_1\ Y_2\ Y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$\boxed{\mathbf{C.F} = \boldsymbol{Y_1^2 + 2Y_2^2 + 4Y_3^2}}$$

Canonical form of the given Quadratic form is

$$C = Y_1^2 + 2Y_2^2 + 4Y_3^2$$

Example 2: Reduce the Quadratic form $Q = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$ into canonical form by an orthogonal transformation.

Solution:

$$Q = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$$

Step 1: The matrix form of given Quadratic form

$$A = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \end{array}$$

Step 2: Find characteristics equation of given matrix A

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \ldots (1)$$

S_1 = sum of main diagonal elements

$= 6 + 3 + 3$

$\boxed{S_1 = 12}$

S_2 = sum of the minors of main diagonal elements

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$= (9 - 1) + (18 - 4) + (18 - 4)$

$= 8 + 14 + 14$

$\boxed{S_2 = 36}$

$S_3 = |A|$

$= 6\,(9 - 1) + 2\,(-6 + 2) + 2\,(2 - 6)$

$= 48 - 8 - 8$

$\boxed{S_3 = 32}$

Hence characteristic equation is

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \qquad \ldots (2)$$

Step 3: Find the eigen values $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

Put $\lambda = 2$

$$(2)^3 - 12\,(2)^2 + 36\,(2) - 32 = 0$$

$$8 - 48 + 72 - 32 = 0$$

$$80 - 80 = 0$$

$\therefore$ $\lambda = 2$ is one root of this equation using synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \multicolumn{1}{|r}{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 8)\,(\lambda - 2) = 0$$

$$\lambda = 8 \;\; \lambda = 2$$

Hence eigen values are 2, 2, 8

Step 4: Find the eigen vectors $(A - \lambda I)\,X = 0$

$$\left[\begin{array}{rrr} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{array}\right] - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{rrr} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (1)}$$

Case 1: When $\lambda = 8$ put in (1)

$$\left[\begin{array}{rrr} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$

$$-2x_1 - 5x_2 - x_3 = 0 \quad \text{... (ii)}$$

$$2x_1 - x_2 - 5x_3 = 0 \quad \text{... (iii)}$$

From (i) and (ii) solve by cross multiplication

$$\begin{array}{cccc} -2 & 2 & -2 & -2 \\ -5 & -1 & -2 & -5 \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

i.e $$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$ Eigen vector $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case 2: Put $\lambda = 2$ in (1)

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \quad \text{... (i)}$$

$$-2x_1 + x_2 - x_3 = 0 \quad \text{... (ii)}$$

$$2x_1 - x_2 + x_3 = 0 \quad \text{... (iii)}$$

These 3 equations are same

Consider $2x_1 - x_2 + x_3 = 0$

Put $x_1 = 0$

$$x_2 = x_3$$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

Hence, Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case 3: Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ be the third eigen vector which orthogonal to X_1 and X_2.

$X_1^T = [2 \quad -1 \quad 1]$; $\quad X_2^T = [0, \quad 1, \quad 1]$

$$2l - m + n = 0$$

$$m + n = 0$$

Solve the equations

$$\begin{array}{cccc} -1 & 1 & 2 & -1 \\ 1 & 1 & 0 & 1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0-2} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{-2} = \frac{x_3}{2}$$

i.e $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Hence, Eigen vector $X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Step 5: Verify eigen vectors are orthogonal to each other

Here, it is clear that eigen vectors are orthogonal to each other.

Step 6: Find the Normalised modal matrix

Modal matrix $P = (X_1\ X_2\ X_3)$

$$P = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Normalised modal matrix

$$N = \begin{bmatrix} \frac{2}{\sqrt{2^2+1^2+1^2}} & 0 & \frac{1}{\sqrt{1^2+1^2+1^2}} \\ \frac{-1}{\sqrt{2^2+1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2+1^2}} \\ \frac{1}{\sqrt{2^2+1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2}} & \frac{-1}{\sqrt{1^2+1^2+1^2}} \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Step 7: Find $D = N^T$, AN

$$AN = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} & 0 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \\ \frac{-4}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{1}{\sqrt{6}} & 0 + \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ \frac{4}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{3}{\sqrt{6}} & 0 - \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} & \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{3}{\sqrt{3}} \end{bmatrix}$$

$$AN = \begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$$

$$\boxed{\boldsymbol{D = N^T AN}}$$

$$D = \underbrace{\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}}^{N^T} \overbrace{\begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}}^{AN}$$

$$= \begin{bmatrix} \frac{32+8+8}{6} & 0-\frac{2}{\sqrt{12}}+\frac{2}{\sqrt{12}} & \frac{4}{\sqrt{18}}-\frac{2}{\sqrt{18}}-\frac{2}{\sqrt{18}} \\ 0-\frac{8}{\sqrt{12}}+\frac{8}{\sqrt{12}} & \frac{0+2+2}{2} & \frac{0}{\sqrt{6}}+\frac{2}{\sqrt{6}}-\frac{2}{\sqrt{6}} \\ \frac{16}{\sqrt{18}}-\frac{8}{\sqrt{18}}-\frac{8}{\sqrt{18}} & 0+\frac{2}{\sqrt{6}}-\frac{2}{\sqrt{6}} & \frac{2+2+2}{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 8: Canonical form: $C = Y^T DY$

$$C = [Y_1 \; Y_2 \; Y_3] \; [D] \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$C = [Y_1 \; Y_2 \; Y_3] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = 8Y_1^2 + 2Y_2^2 + 2Y_3^2$$

Reduction of the Quadratic form into canonical form is

$$\mathbf{8Y_1^2 + 2Y_2^2 + 2Y_3^2}$$

Example 3: Reduce the Quadratic form $x_1^2 + 2x_2^2 - x_3^2 - 2x_1 x_2 + 2x_2 x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semi-definite. Also give an non-zero set of values $(X_1 \, X_2 \, X_3)$ which makes this quadratic form zero.

Solution:

Step 1: The matrix of the Quadratic form is

$$Q = x_1^2 + 2x_2^2 - x_3^2 - 2x_1 \, x_2 + 2x_2 \, x_3$$

$$\begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ A = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Step 2: Find characteristic equation of given matrix A

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \quad \text{... (1)}$$

S_1 = sum of main diagonal elements

$= 1 + 2 + 1$

$\boxed{S_1 = 4}$

S_2 = sum of the minor of main diagonal elements

$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$

$= (2 - 1) + (1 - 0) + (2 - 1)$

$= 1 + 1 + 1$

$\boxed{S_2 = 3}$

$S_3 = |A|$

$= 1\,(2 - 1) + 1\,(-1 - 0) + 0\,()$

$= 1 - 1$

$\boxed{S_3 = 0}$

$\therefore$ Characteristic equation is

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0 \quad \text{... (2)}$$

Step 3: Find the eigen values

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$$\lambda\,(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda\,(\lambda - 1)\,(\lambda - 3) = 0$$

$$\lambda = 0, 1, 3$$

$\therefore$ Eigen values 0, 1, 3

Step 4: Find the eigen vectors

$$(A - \lambda I)\,X = 0$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{... (1)}$$

Case 1: Put $\lambda = 0$ in (1)

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 + 0x_3 = 0 \qquad \text{... (i)}$$

$$-x_1 + 2x_2 + x_3 = 0 \qquad \text{... (ii)}$$

$$0x_1 + x_2 + x_3 = 0 \qquad \text{... (iii)}$$

From (i) and (ii) solve using cross multiplication

$$\begin{array}{cccc} -1 & 0 & 1 & -1 \\ 2 & 1 & -1 & 2 \end{array}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{0-1} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

i.e $\quad \dfrac{x_1}{1} = \dfrac{x_2}{1} = \dfrac{x_3}{-1}$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Case 2: Put $\lambda = 1$ in equation (1)

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x_1 - x_2 + 0x_3 = 0 \quad \text{... (i)}$$

$$-x_1 + x_2 + x_3 = 0 \quad \text{... (ii)}$$

$$0x_1 + x_2 + 0x_3 = 0 \quad \text{... (iii)}$$

From (i) and (ii) we get

$$\begin{array}{cccc} -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 1 \end{array}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{0+0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Eigen vector $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Case 3: Put $\lambda = 3$ in equation (1)

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$-2x_1 - 1x_2 + 0x_3 = 0 \quad \text{... (i)}$$

$$-1x_1 - 1x_2 + 1x_3 = 0 \quad \text{... (ii)}$$

$$0x_1 + 1x_2 - 2x_3 = 0 \quad \text{... (iii)}$$

From (i) and (ii) we get

$$\begin{array}{cccc} -1 & 0 & -2 & -1 \\ -1 & 1 & -1 & -1 \end{array}$$

$$\frac{x_1}{-1} = \frac{x_2}{+2} = \frac{x_3}{1}$$

$$\text{Eigen vector } X_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Step 5: Find the normalised modal matrix

$$\begin{array}{cccc} & X_1 & X_2 & X_3 \\ \text{Modal matrix, } P = & \multicolumn{3}{c}{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}} \end{array}$$

$$N = \begin{bmatrix} \frac{1}{\sqrt{2^2+1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2}} & \frac{-1}{\sqrt{1^2+1^2+1^2}} \\ \frac{1}{\sqrt{2^2+1^2+1^2}} & 0 & \frac{2}{\sqrt{1^2+1^2+1^2}} \\ \frac{-1}{\sqrt{1^2+1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2+1^2}} \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}; \qquad N^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Diagonalisation

Step 6: Find $D = N^T AN$

$$AN = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + 0 & \frac{1}{\sqrt{2}} + 0 + 0 & \frac{-1}{\sqrt{6}} - \frac{2}{\sqrt{6}} + 0 \\ \frac{-1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} + \frac{4}{\sqrt{6}} + \frac{1}{\sqrt{6}} \\ 0 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} & 0 + 0 + \frac{1}{\sqrt{2}} & 0 + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$AN = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{6}} \\ 0 & 0 & \frac{6}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{6}} \end{bmatrix}$$

Now $D = N^T \cdot AN$

$$\begin{array}{ccc} & N^T & AN \\ D = & \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} & \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{6}} \\ 0 & 0 & \frac{6}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{6}} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 0 + 0 + 0 & \frac{1}{\sqrt{6}} + 0 + \frac{-1}{\sqrt{6}} & \frac{-3}{\sqrt{18}} + \frac{6}{\sqrt{18}} - \frac{3}{\sqrt{18}} \\ 0 + 0 + 0 & \frac{1}{2} + 0 + \frac{1}{2} & \frac{-3}{\sqrt{12}} + 0 + \frac{3}{\sqrt{12}} \\ 0 + 0 + 0 & \frac{-1}{\sqrt{12}} + 0 + \frac{1}{\sqrt{12}} & \frac{3}{6} + \frac{12}{6} + \frac{3}{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Step 7: Canonical form

$$C = Y^T DY$$

$$= (Y_1\ Y_2\ Y_3)\ (D) \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= (Y_1\ Y_2\ Y_3) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$\boxed{\mathbf{C = 0Y_1^2 + Y_2^2 + 3Y_3^2}}$$

$\therefore$ canonical form of Quadratic form is $Y_2^2 + 3Y_2^2$

Step 8: Nature of the Quadratic form

As the canonical form contains only two terms both which are positive the Quadratic form is positive semi-definite.

Example 4: Reduce the Quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2\,x_3 - 10x_3\,x_1 - 4x_1\,x_2$ to a canonical form by orthogonal reduction. Find a set of values of x_1, x_2, x_3 which will make the form vanish.

Solution:

Quadratic form $= 10x_1^2 + 2x_2^2 + 2x_2^2 + 5x_3^2 + 6x_2\,x_3 - 10x_3\,x_1 - 4x_1\,x_2$

Step 1: The matrix of the Quadratic form is

$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \end{array}$$

Step 2: Find characteristics equation of given matrix *A*

$$\lambda^3 - S_1\,\lambda^2 + S_2\,\lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = sum of diagonal elements

$= 10 + 2 + 5$

$\boxed{\mathbf{S_1 = 17}}$

S_2 = sum of minor of the main diagonal elements

$$= \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix}$$

$= (10 - 9) + (50 - 25) + (20 - 4)$

$= 1 + 25 + 16$

$\boxed{\mathbf{S_2 = 42}}$

$S_3 = |A|$

$= 10\,(10 - 9) + 2\,(-10 - 15) - 5\,(-6 + 10)$

$= 10 + 10 - 20$

$\boxed{\mathbf{S_3 = 0}}$

Hence the characteristic equation is

$$\boxed{\boldsymbol{\lambda^3 - 17\lambda^2 + 42\lambda - 0 = 0}} \qquad \text{... (2)}$$

Step 3: Find the eigen values

Using synthetic division method

we get

Eigen values 0, 3, 14

Step 4: Find the eigen vectors $(A - \lambda I)\,X = 0$

$$\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{... (1)}$$

Case 1: Put $\lambda = 0$ in (1)

$$\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$10x_1 - 2x_2 - 5x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 + 2x_2 + 3x_3 = 0 \qquad \text{... (ii)}$$

$$-5x_1 + 3x_2 + 5x_3 = 0 \qquad \text{... (iii)}$$

Solve (i) and (ii) using cross multiplication rules

$$\begin{array}{cccc} -2 & -5 & 10 & -2 \\ 2 & 3 & -2 & 2 \end{array}$$

$$\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

i.e $$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

Hence, Eigen vector $X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$

Case 2: When $\lambda = 3$ put in (1)

$$\begin{bmatrix} 7 & -2 & -5 \\ -2 & -1 & 3 \\ -5 & 3 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$7x_1 - 2x_2 - 5x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 - x_2 + 3x_3 = 0 \qquad \text{... (ii)}$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \qquad \text{... (iii)}$$

Solve (i) and (ii) using cross multiplication rules

$$\begin{array}{cccc} -2 & -5 & 7 & -2 \\ -1 & 3 & -2 & -1 \end{array}$$

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_3}{-11} = \frac{x_3}{-11}$$

i.e $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 3: When $\lambda = 14$ put in (1)

$$\begin{bmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_1 - 2x_2 - 5x_3 = 0 \qquad \text{... (i)}$$

$$-2x_1 - 12x_2 + 3x_3 = 0 \qquad \text{... (ii)}$$

$$-5x_1 + 3x_2 - 9x_3 = 0 \qquad \text{... (iii)}$$

From (i) and (ii)

$$\begin{array}{cccc} -2 & -5 & -4 & -2 \\ -12 & 3 & -2 & -12 \end{array}$$

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

i.e $$\frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence Eigen vector $X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

Step 5: To find normalised matrix

Modal matrix $P = (X_1\ X_2\ X_3)$

$$P = \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{bmatrix} \end{array}$$

Normalised Modal Matrix

$$N = \begin{bmatrix} \frac{1}{\sqrt{1^2+5^2+4^2}} & \frac{1}{\sqrt{1^2+1^2+1^2}} & \frac{-3}{\sqrt{3^2+1^2+2^2}} \\ \frac{-5}{\sqrt{1^2+5^2+4^2}} & \frac{1}{\sqrt{1^2+1^2+1^2}} & \frac{1}{\sqrt{3^2+1^2+2^2}} \\ \frac{4}{\sqrt{1^2+5^2+4^2}} & \frac{1}{\sqrt{1^2+1^2+1^2}} & \frac{2}{\sqrt{3^2+1^2+1^2}} \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{1}{\sqrt{42}} & 1 & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{bmatrix}$$

Step 6: Find $D = N^T AN$

$$AN = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{42}} & 1 & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{\sqrt{42}} + \frac{10}{\sqrt{42}} - \frac{20}{\sqrt{42}} - \frac{20}{\sqrt{42}} & \frac{10}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{5}{\sqrt{3}} & \frac{-30}{\sqrt{14}} - \frac{2}{\sqrt{14}} - \frac{10}{\sqrt{14}} \\ \frac{-2}{\sqrt{42}} - \frac{10}{\sqrt{42}} + \frac{12}{\sqrt{42}} & \frac{-2}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{14}} + \frac{2}{\sqrt{14}} + \frac{6}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} - \frac{15}{\sqrt{42}} + \frac{20}{\sqrt{42}} & \frac{-5}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{5}{\sqrt{3}} & \frac{15}{\sqrt{14}} + \frac{3}{\sqrt{14}} + \frac{10}{\sqrt{14}} \end{bmatrix}$$

$$AN = \begin{bmatrix} 0 & \frac{3}{\sqrt{3}} & \frac{-42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{bmatrix}$$

Now, $D = N^T AN$

$$D = \overset{N^T}{\begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{bmatrix}} \overset{AN}{\begin{bmatrix} 0 & \frac{3}{\sqrt{3}} & \frac{-42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{bmatrix}}$$

$$= \begin{bmatrix} 0+0+0 & \frac{3}{\sqrt{26}} - \frac{15}{\sqrt{126}} + \frac{12}{\sqrt{126}} & \frac{-42}{\sqrt{588}} - \frac{70}{\sqrt{588}} + \frac{112}{\sqrt{588}} \\ 0+0+0 & 1+1+1 & \frac{-42}{\sqrt{588}} + \frac{14}{\sqrt{588}} + \frac{28}{\sqrt{588}} \\ 0+0+0 & \frac{-9}{\sqrt{42}} + \frac{3}{\sqrt{42}} + \frac{6}{\sqrt{42}} & \frac{126}{14} + \frac{14}{4} + \frac{56}{14} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

Step 7: Canonical Form

$$C = Y^T DY$$

$$= [Y_1 \ Y_2 \ Y_3] \ (D) \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= [Y_1 \ Y_2 \ Y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$\boxed{\boldsymbol{C = 0Y_1^2 + 3Y_2^2 + 14Y_3^2}}$$

Canonical form of Quadratic form is

$$3y_2^2 + 14y_3^2$$

Example 5: Reduce the Quadratic form $2xy + 2yz + 2zx$ to canonical form by an orthogonal reduction. Find the rank, index, signature and the nature of the Quadratic form.

Solution:

$$Q = 2xy + 2yz + 2zx$$

Step 1: The matrix of the Quadratic form is

$$A = \begin{array}{c} x \\ y \\ z \end{array} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 2: To find characteristic equation ... (1)

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \qquad \text{... (1)}$$

S_1 = sum of diagonal elements

$= 0 + 0 + 0$

$\boxed{\mathbf{S_1 = 0}}$

S_2 = sum of minor of main diagonal elements

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= (0 - 1) + (0 - 1) + (0 - 1)$

$\boxed{\mathbf{S_2 = -3}}$

$S_3 = |A|$

$\boxed{\mathbf{S_3 = 2}}$

Characteristic Equation is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$ ... (2)

$$\lambda^3 - 3\lambda - 2 = 0$$

Step 3: Find the eigen values

$$\lambda^3 - 3\lambda - 2 = 0$$

Put $\lambda = -1$

$$(-1)^3 - 3(-1) - 2 = 0$$

$$-1 + 3 - 2 = 0$$

$$3 - 3 = 0$$

$$0 = 0$$

$\therefore \lambda = -1$ is a root

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2, \ \lambda = -1$$

Eigen values $-1, -1, 2$

Step 4: To find the eigen vectors

$$(A - \lambda I) X = 0$$

$$\left[\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Case 1: Put $\lambda = 2$ in (1)

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0 \qquad \text{... (i)}$$

$$x_1 - 2x_2 + x_3 = 0 \qquad \text{... (ii)}$$

$$x_1 + x_2 - 2x_3 = 0 \qquad \text{... (iii)}$$

From (i) and (ii)

$$\begin{array}{cccc} 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

i.e

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case 2: Put $\lambda = -1$ in (1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + x_3 = 0$$

Put $x_2 = 0$

$$x_1 = -x_3$$

$$\frac{x_1}{1} = \frac{x_3}{-1}$$

Hence, Eigen vector $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Case 3: Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ be the third eigen vector which is orthogonal to X_1 and X_2

$$X_3 \perp \text{ to } X_1 \longrightarrow l + m + m = 0$$

$$X_3 \perp \text{ to } X_2 \longrightarrow l - n = 0$$

Solve these equation

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{array}$$

$$\frac{x_1}{-1-0} = \frac{x_2}{1+1} = \frac{x_3}{0-1}$$

$$\frac{x_2}{-1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

i.e $$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, Eigen vector $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Step 5: Find the Normalised modal matrix

Modal matrix $P = (X_1\ X_2\ X_3)$

Normalised matrix

$$N = \begin{bmatrix} \dfrac{1}{\sqrt{1^2+1^2+1^2}} & \dfrac{1}{\sqrt{1^2+1^2}} & \dfrac{1}{\sqrt{1^2+1^2+1^2}} \\ \dfrac{1}{\sqrt{1^2+1^2+1^2}} & 0 & \dfrac{-2}{\sqrt{1^2+1^2+1^2}} \\ \dfrac{1}{\sqrt{1^2+1^2+1^2}} & \dfrac{-1}{\sqrt{1^2+1^2}} & \dfrac{1}{\sqrt{1^2+1^2+1^2}} \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Step 6: Find $D = N^T AN$

$$AN = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} & 0 + 0 - \frac{1}{\sqrt{2}} & 0 - \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} + 0 + \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 & \frac{1}{\sqrt{2}} + 0 + 0 & \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

Now, $D = N^T \cdot AN$

$$D = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}+\frac{2}{3}+\frac{2}{3} & \frac{-1}{\sqrt{6}}+0+\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{18}}+\frac{2}{\sqrt{18}}-\frac{1}{\sqrt{18}} \\ \frac{2}{\sqrt{6}}+0-\frac{2}{\sqrt{6}} & -\frac{1}{2}+0-\frac{1}{2} & \frac{-1}{\sqrt{12}}+0+\frac{1}{\sqrt{12}} \\ \frac{2}{\sqrt{18}}-\frac{4}{\sqrt{18}}+\frac{2}{\sqrt{18}} & \frac{-1}{\sqrt{2}}+0+\frac{1}{\sqrt{12}} & \frac{-1}{6}-\frac{4}{6}-\frac{1}{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Step 7: Canonical form

$$[Y_1\ Y_2\ Y_3]\ D] \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = [Y_1\ Y_2\ Y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= 2Y_1^2 - Y_2^2 - Y_3^2$$

Step 8: Rank

Here number of non-zero rows 3

$$\therefore \text{ Rank } = 3$$

Step 9: Index

The number of positive term in the canonical form is 1.

$$\therefore \text{ Index } = 1$$

Step 10: Signature

The difference between the number of positive and negative terms is called signature of the Quadratic form.

$$\therefore S = 2 - 1 = 1$$

$\boxed{\textbf{Nature of Quadratic form is indefinite form.}}$

TWO MARKS QUESTIONS AND ANSWERS

1. State Cayley-Hamilton Theorem.

Solution:

"Every square matrix satisfies its own characteristic equation".

2. Write the applications of Cayley-Hamilton theorem:

Solution:

- To find the inverse of a matrix.
- To find the positive integral powers of matrix.

3. Find the sum and product of the eigen value of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{5} \\ \mathbf{3} & \mathbf{5} & \mathbf{6} \end{bmatrix}$

Solution:

Given: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Sum of the Eigen values

$$= \text{Sum of the main diagonal elements}$$

$$= 1 + 4 + 6$$

$$= 11$$

$\therefore$ Sum of Eigen value $= 11$

Product of the Eigen value $= |A|$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$$

$$= 1\,(24 - 25) - 2\,(12 - 15) + 3\,(10 - 12)$$

$$= 1\,(-1) - 2\,(-3) + 3\,(-2)$$

$$= -1 + 6 - 6 = -1$$

$\therefore S_3 =$ Product of Eigen value $= -1$

4. If $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$, find the eigen values of A^2, A^3, A^{-1}.

Solution:

Given: $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$

(Given matrix is a lower triangular matrix)

$\therefore$ The Eigen values of A are $-1, -3, 2$ (diagonal elements)

The Eigen values of A^2 are 1, 9, 4

The Eigen values of A^3 are $-1, -27, 8$

The Eigen values of A^{-1} are $\frac{-1}{1}, \frac{-1}{3}, \frac{1}{2}$

5. Find the eigen values of inverse of the matrix
$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

Solution:

Since A is a "upper triangular matrix"

Eigen values of A are 2, 3, 4.

$\therefore$ Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

6. Find the eigen values of A and A^2 for $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Solution:

Since A is a upper triangular matrix,

Eigen values of A are 3, 2, 5

Eigen values of A^2 are 9, 4, 25

7. Find the sum and product of the eigen values of

$$A = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution:

$$\text{Sums of the Eigen values} = \text{Sum of the main diagonal elements}$$
$$= 2 + 2 + 1$$
$$= 5$$

$\therefore S_1 =$ Sum of the Eigen values $= 5$

$$\text{Product of Eigen value} = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= 2\,(2 - 0) - 1\,(1 - 0) + 1\,(0 - 0)$$
$$= 4 - 1 + 0$$
$$= 3$$

$S_3 =$ Product of the Eigen value $= 3$

8. If 3 and 15 are the two eigen values of $A = \begin{bmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{6} & \mathbf{-1} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{bmatrix}$**. What is the third eigen value?**

Solution:

Given: $\lambda_1 = 3,\ \lambda_2 = 15,\ \lambda_3 = ?$

Sum of the Eigen values $=$ Sum of the diagonal element

$$\lambda_1 + \lambda_2 + \lambda_3 = 8 - 1 + 3$$
$$3 + 15 + \lambda_3 = 10$$
$$18 + \lambda_3 = 10$$
$$\lambda_3 = 10 - 18$$
$$\lambda_3 = -8$$

$\therefore$ The third eigen values $\lambda_3 = -8$

9. If 3 and 6 are two eigen values of A write down the eigen value of A^{-1} and A^2.

Solution:

$$\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = ?$$

Sum of Eigen values = Sum of diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1$$

$$3 + 6 + \lambda_3 = 7$$

$$9 + \lambda_3 = 7$$

$$\lambda_3 = 7 - 9$$

$$\lambda_3 = -2$$

$\therefore$ The Eigen values of A are 3, 6, -2

The Eigen values of A^{-1} are $\frac{1}{3}, \frac{1}{6}, \frac{-1}{2}$

10. The product of two eigen value of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigen value.

Solution:

Given: $\lambda_1 \lambda_2 = 16$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Product of Eigen values $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$16\lambda_3 = 32$$

$$\lambda_3 = \frac{32}{16}$$

$$\lambda_3 = 2$$

$\therefore$ The third eigen value is 2.

11. For a given matrix A, $|A| = 32$. The two of its eigen value are 8 and 12. Find the 3rd eigen value.

Solution:

Given: $\lambda_1 = 8$

$\lambda_2 = 12$

$|A| = 32$

Product of Eigen values $= |A|$

$$\lambda_1 \lambda_2 \lambda_3 = 32$$

$$\lambda_3 = \frac{32}{8 \times 12}$$

$$\lambda_3 = \frac{4}{12}$$

$$\lambda_3 = \frac{1}{3}$$

$\therefore$ The third eigen value is $\frac{1}{3}$

12. If $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$, find the eigen value of Adj A.

Solution:

Eigen value of Adj $= |A|\, A^{-1}$

For a matrix $|A| = 6$

Eigen values of A are $-1, -3, 2$

Eigen values of A^{-1} are $\frac{-1}{1}, \frac{-1}{3}, \frac{1}{2}$

Eigen values of Adj A = (Eigen value of A^{-1}) 6

$$= 6 \times \frac{-1}{1}, 6 \times \frac{-1}{3}, 6 \times \frac{1}{2}$$

$$= -6, -2, 3$$

$\therefore$ Eigen values of Adj A are $-6, -2, 3$

13. Define Orthogonal Matrix.

Solution:

A real square matrix A is said to be orthogonal if $AA^T = A^T A = I$

14. Prove that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal matrix.

Solution:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}; \; A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore AA^T = I$ and $A^T A = I$

$\therefore A$ is Orthogonal matrix

15. Check whether the matrix B is Orthogonal, Justify

$$B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

$$B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \; B_T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BB^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore \; BB^T = I$ and $B^T B = I$

$\therefore B$ is Orthogonal matrix

16. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be diagonalised? Why?

Solution:

Yes, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalised because it has distinct eigen values and also it is a diagonal matrix.

17. Find the characteristic equation of $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

Solution:

Given: $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

$S_1 =$ Sum of diagonal elements

$= 1 + 4$

$S_1 = 5$

$S_2 = |A|$

$= \begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = 4 + 6$

$S_2 = 10$

$\therefore$ The characteristic equation $\lambda^2 - S_1 \lambda + S_2 = 0$

$$\therefore \lambda^2 - 5\lambda + 10 = 0$$

18. Find the eigen values of matrix of $A = \begin{bmatrix} \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} \end{bmatrix}$

Solution:

$S_1 =$ sum of diagonal elements;

$= 4 + 2$

$\boxed{S_1 = 6}$

$S_2 = |A| = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$

$= 8 - 3$

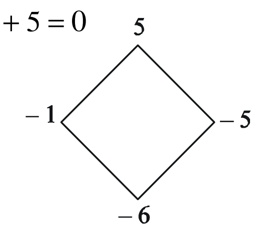
$\boxed{S_2 = 5}$

$\therefore$ The characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$

$$(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 5$$



$\therefore$ The Eigen values are 1 and 5.

19. If 2, – 1, 3 are the eigen values of the matrix, then find the eigen values of matrix $A^2 - 2I$.

Solution:

Eigen values of A are $2, -1, 3$

Eigen values of A^2 are 4, 1, 9

Eigen values of $2I$ are 2, 2, 2

Eigen values of $A^2 - 2I$ are $2, -1, 7$

20. Find the index, signature and nature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$.

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Index = No. of (+ ve Eigen values) $= 2$

Signature $= 2I - R = 2\,(2) = 3 = 4 - 3 = 1$

Nature = Indefinite.

21. Write down the matrix of quadratic form:

Solution:

(i) $2x_1^2 - 8x_1\,x_2 + 4x_2^2 \Rightarrow A = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(ii) $4x^2 + 2y^2 - 3z^2 + 2xy + 4zx \Rightarrow A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix}$

(iii) $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1\,x_2 - 6x_1\,x_3 + 6x_2\,x_3$

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

22. If 3 and 8 are the two eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then find $|A|$.

Solution:

Sum of eigen value = sum of the main diagonal element

$$\lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$$

$$3 + 8 + \lambda_3 = 18$$

$$11 + \lambda_3 = 18$$

$$\lambda_3 = 18 - 11$$

$$\lambda_3 = 7$$

Eigen values of A are 3, 8, 7

$$|A| = \text{Product of eigen values}$$

$$= 3 \times 8 \times 7$$

$$|A| = 168$$

23. Determine the nature of the following quadratic form $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$.

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigen values of A are 1, 2, 0

Nature: Positive semi definite

24. Find the nature of $-1x^2 - 1y^2 - 2z^2$

Solution:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigen values of A are $-1, -1, -2$

Nature: Negative Definite

25. Find the nature of $8x_1^2 + 8x_2^2 + 3x_3^2 + 2x_1 x_2 - 4x_1 x_3 - 4x_2 x_3$

Solution:

$$A = \begin{bmatrix} 8 & 1 & -2 \\ 1 & 8 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

$$D_1 = 8 \Rightarrow +\text{ve}$$

$$D_2 = 64 - 1 = 63 \ +\text{ve}$$

$$D_3 = |A| = \begin{vmatrix} 8 & 1 & -2 \\ 1 & 8 & -2 \\ -2 & -2 & 3 \end{vmatrix} = 153 \Rightarrow +\text{ve}$$

$\therefore$ The Nature is Positive Definite.

26. If λ is the eigen value of A then prove that $\frac{1}{\lambda}$ is the eigen value of A^{-1}.

Solution:

Let λ be the eigen value of A

Then $\quad AX = \lambda X$

$$X = \lambda A^{-1} X$$

$$\frac{1}{\lambda} X = A^{-1} X$$

$\therefore \frac{1}{\lambda}$ is the eigen value of A^{-1}.

27. If λ is the eigen value of A then prove that λ^2 is an eigen value of A^2.

Solution:

Let λ be the Eigen value of A

Then $\quad AX = \lambda X \qquad \ldots (1)$

Multiply by A $\quad A\,(AX) = A\,(\lambda\, X)$

$$A^2\, X = \lambda\,(AX)$$

$$A^2\, X = \lambda\,(\lambda\, X)$$

(by (1) $AX = \lambda X$)

$$A^2\, X = \lambda^2\, X$$

$\therefore\ \lambda^2$ is the Eigen value of A^2

28. Find the eigen values of $3A + 2I, A^2, A^3, A^{-1}$ where $A = \begin{bmatrix} 5 & 4 \\ 0 & 2 \end{bmatrix}$.

Solution:

Eigen values of A are 5, 2

Eigen values of $3A + 2I$ are 17

Eigen values of A^2 are 25, 4

Eigen values of A^3 are 125, 8

Eigen values of A^{-1} are $\frac{1}{5}, \frac{1}{2}$

EXERCISE

Characteristic Equation

1. Find the characteristic equation of

(a) $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (f) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & -1 & 6 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ (j) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. Find the characteristic polynomial of

(a) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

Eigen values and eigen vector of non symmetric matrix its non repeated eigen values

Find the Eigenvalues and Eigenvectors of

Sl.No.	Matrix	Eigenvalues	Eigenvectors
1.	$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$	1, 5	$\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2.	$\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$	− 1, 6	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix}$
3.	$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$	− 1, 1, 2	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$
4.	$\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$	1, 2, 5	$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
5.	$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$	− 2, 1, 3	$\begin{bmatrix} 11 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
6.	$\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$	− 1, 1, 2	$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$
7.	$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$	0, 1, 2	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$
8.	$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$	1, 3, − 4	$\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix}$

9.	$\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$	$1, \sqrt{5}, -\sqrt{5}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \sqrt{5}+1 \\ -1 \\ 1 \end{bmatrix}$
10.	$\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$	$1, 3, -4$	$\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$

Find the eigen values and eigen vector of Non-symmetric matrix with repeated Eigenvalues.

Sl.No.	Matrix	Eigenvalues	Eigenvectors
1.	$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$	$1, 1, 5$ $1, 1, 7$	$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
2.	$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$		$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
3.	$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$	$-1, -1, 3$	$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$
4.	$\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$	$2, 2, -2$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

Find the eigen values and eigen vectors of Symmetric matrices with non-repeated Eigenvalues.

Sl.No.	Matrix	Eigenvalues	Eigenvectors
1.	$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	1, 2, 3 $-2, 4, 6$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
2.	$\begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$	$-2, 4, 6$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
3.	$\begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{bmatrix}$	$-2, 9, -18$	$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$
4.	$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$	$-2, 3, 6$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
5.	$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$	2, 3, 6	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

Find the eigen values and eigen vectors of Symmetric matrices with repeated Eigenvalues

Sl.No.	Matrix	Eigenvalues	Pairwise orthogonal Eigenvectors
1.	$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$	14, 0, 0	$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$

2.	$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	1, 3, 3 [x_2 is arbitrary no unique Eigenvectors]	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
3.	$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$	8, 2, 2	$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Problems on properties of Eigen values

1. Find the sum and product of all Eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 1 & 2 & 7 \end{bmatrix}$. Is the matrix singular?

2. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & 3 \end{bmatrix}$, then find the sum and product of all Eigenvalues of A.

3. Find the sum of the Eigenvalues of $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 4 \\ 1 & 2 & 7 \end{bmatrix}$

4. Find the product of the Eigenvalues of A

   (a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$

5. If the Eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are $-2, 3, 6$, then the Eigenvalues of A^T.

6. Find the Eigenvalues of $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

7. If the Eigenvalues of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ are 2, 2, 3, then find the Eigenvalues of A^{-1}.

8. Find the Eigenvalues of the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

9. Find the Eigenvalues of A^2 if $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

10. Obtain the Eigenvalues of A^3 where $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

11. Show that the Eigenvalues of a real symmetric matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ is real.

Cayley-Hamilton Theorem

1. Verify Cayley - Hamilton Theorem and find its inverse, and a^4

(a) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ - & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

Quadratic form of canonical form

1. Reduce the quadratic form $8x_1^2+7x_2^2+3x_3^2-12x_1\,x_2-8x_2\,x_3+4x_3\,x_1$ to the canonical form through an orthogonal transformation and hence, show that it is positive semi-definite.

2. Reduce the quadratic form $x_1^2+5x_2^2+x_3^2+2x_1\,x_2+2x_2\,x_3+6x_3\,x_1$ to canonical form through an orthogonal transformation.

3. Reduce the quadratic form $10x_1^2+2x_2^2+5x_3^2+6x_2\,x_3-10x_3\,x_1-4x_1\,x_2$ to a canonical form by orthogonal reduction. Find a set of values of x_1, x_2, x_3 which will make the form vanish.

4. Reduce the Q.F $2x_1^2+6x_2^2+2x_3^2+8x_1\,x_2$ to C.F by orthogonal reduction. Find also the nature of the Q.F

5. Reduce the Q.F $2x_1^2+5x_2^2+3x_3^2+4x_1\,x_2$ to C.F by an orthogonal transformation. Also find the rank, index and signature of the Q.F.

6. $3x_1^2+3x_2^2+3x_3^2+2x_1\,x_2+2x_1\,x_3-2x_2\,x_3$

7. $3x_1^2+5x_2^2+3x_3^2-2x_2\,x_3+2x_3\,x_1-2x_1\,x_2$

8. $2x_1^2+3x_2^2+2x_3^2+2x_1\,x_3$

9. Discuss the nature of the Q.F without reducing them to C.F.
   (a) $6x^2+3y^2+3z^2-4xy-2yz+4xz$
   (b) $xy+yz+zx$
   (c) $x^2+4xy+6xz-y^2+2yz+4z^2$
   (d) $12x^2+3y^2+12z^2+2xy$

Formula

- $\frac{d}{dx}(\text{Constant}) = 0$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(x^2) \longrightarrow 2x$
- $\frac{d}{dx}(x^3) \longrightarrow 3x^2$
- $\frac{d}{dx}(x^n) \longrightarrow nx^{n-1}$
- $\frac{d}{dx}\left(\frac{1}{x}\right) \longrightarrow \frac{-1}{x^2}$
- $\frac{d}{dx}\left(\frac{1}{x^2}\right) \longrightarrow \frac{-2}{x^3}$
- $\frac{d}{dx}(\sqrt{x}) \longrightarrow \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(e^x) \longrightarrow e^x$
- $\frac{d}{dx}(e^{-x}) \longrightarrow -e^{-x}$
- $\frac{d}{dx}(e^{ax}) \longrightarrow ae^{ax}$
- $\frac{d}{dx}(e^{-ax}) \longrightarrow -ae^{-ax}$
- $\frac{d}{dx}(\log x) \longrightarrow \frac{1}{x}$
- $\frac{d}{dx}(\sin x) \longrightarrow \cos x$

- $\frac{d}{dx}(\cos x) \longrightarrow - \sin x$
- $\frac{d}{dx}(\tan x) \longrightarrow \sec^2 x$
- $\frac{d}{dx}(\text{cosec})\, x \longrightarrow - \text{cosec}\, x \cot x$
- $\frac{d}{dx}(\sec x) \longrightarrow \sec x \tan x$
- $\frac{d}{dx}(\cot x) \longrightarrow - \text{cosec}^2 x$
- $\frac{d}{dx}(a^x) \longrightarrow a^x \log a$
- $\log a + \log b = \log ab$
- $\log a - \log b = \log \frac{a}{b}$
- $\log a^m = m \log a$
- $\frac{d}{dx}(\sin ax) \longrightarrow a \cos ax$
- $\frac{d}{dx}(\cos ax) \longrightarrow - a \sin ax$
- $\frac{d}{dx}(\tan nx) \longrightarrow a \sec^2 ax$
- $\frac{d}{dx}(\text{cosec}\, ax) \longrightarrow - a\, \text{cosec}\, ax \cot ax$
- $\frac{d}{dx}(\sec ax) \longrightarrow a \sec ax \tan ax$
- $\frac{d}{dx}(\cot ax) \longrightarrow - a\, \text{cosec}^2 ax$

- $\frac{d}{dx}(\sin^{-1} x) \longrightarrow \frac{1}{\sqrt{1-x^2}}$

- $\frac{d}{dx}(\cos^{-1} x) \longrightarrow -\frac{1}{\sqrt{1-x^2}}$

- $\frac{d}{dx}(\tan^{-1} x) \longrightarrow \frac{1}{1+x^2}$

- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

- $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

- $e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

- $\frac{d}{dx}(uv) = uv' + vu'$

- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

- $\frac{d}{dx}(uvw) = u'vw + uv'w + uvw'$

- $\cos^2\theta + \sin^2\theta = 1$

- $\sec^2\theta - \tan^2\theta = 1$

- $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Important values

$e^0 = 1$	$e^{-0} = 1$
$e^{-1} = \frac{1}{e}$	$e^{-2} = \frac{1}{e^2}$
$\cos 0 = 1$	$\cos \frac{\pi}{2} = 0$
$\sin 0 = 0$	$\sin \frac{\pi}{2} = 1$
$\sin \pi = 0$	$\cos \pi = -1$
$\sin 2\pi = 0$	$\cos 2\pi = +1$
$\sin n\pi = 0$	$\cos 3\pi = -1$
	$\cos n\pi = (-1)^n$
$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$\log 1 = 0$	$e^{\infty} = \infty$
$\log 0 = \infty$	$e^{-\infty} = 0$

UNIT - 2

DIFFERENTIAL CALCULUS

2.1 INTRODUCTION

Calculus is the mathematics of motion and change when increasing (or) decreasing quantities are made the subject of mathematical investigation, it frequently becomes necessary to estimate their rates of growth (or) decay. The primary objects of study in differential calculus are the derivative of a function.

The fundamental and important aspects of calculus are depending upon functions. The basic concepts of calculus are concerned with functions, graphs and their transformation.

In this chapter, we will deal with functions, limit of function, continuity and differentiability of functions and simple applications.

2.2 FUNCTION

Functions are very fundamental in mathematics. The term function was coined by Leibnitz in 1673. A function is a tool that scientists and mathematicians use to describe relationship between varying quantities.

Definition

A function 'f': $A \longrightarrow B$ from a set A to a set B is a rule that assigns to each element $x \in A$ a unique element y in B.

The element y in B is called the image of x under f (or) the value of f at x and is written as $f(x)$ (ie) $y = f(x)$.

The set A is called the domain of the function f and B is called the co-domain

The set of all values of the function f is called the range of the function and it is denoted by $f(A)$.

Methods of Representing a function

There are four common methods of representing a function.

1. Verbally (description)
2. Visually (graph)
3. Numerically (table of values)
4. Algebraically (Explicit formula)

1. Algebraically (by an equation)

For example, area A of a circle depends on its radius r and is given by $A = \pi r^2$: A is a function of r, since for each given r there is unique value for A.

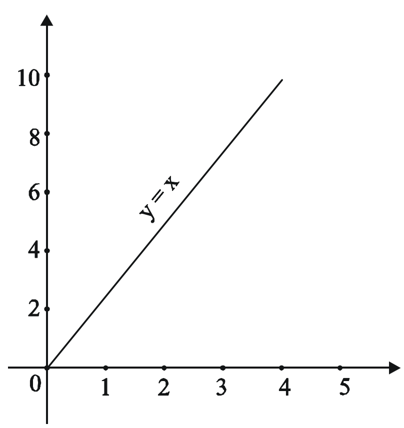
2. Numerically (by table)

x	1	2	3	4	5
y	3	6	9	12	15

represents a function, since for each x there is unique value for y.

3. Geometrically (by graph)

The graph represents a function,



4. Verbally (by words)

For example, Issac Newton's law of universal gravitation is stated as below.

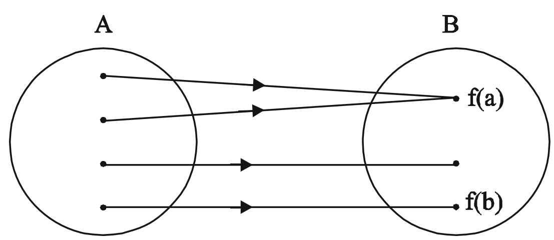
The gravitational force of attraction between two bodies in the universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Verbal description of the formula.

$$F = G\frac{m_1\, m_2}{r^2}$$

Arrow diagram

A function $f : A \longrightarrow B$ can also be represented by an arrow diagram.



Domain and range

Generally, a function is given by an expression

$$f(x) = \frac{x+2}{x-1}$$

When the domain of a function f is not stated explicitly, it is to be understood that the domain is the set of all real number x for which $f(x)$ is real. The set is called the natural domain of the function f.

2.3 REPRESENTATION OF FUNCTION

Types of functions

- Algebraic function (polynomials)
- Rational function
- Real - valued functions
- Trigonometric function
- Inverse trigonometric function

- Exponential function
- Logarithmic function
- Constant function
- Identity function
- One-One function
- Onto function
- Bijective function

Note

The above functions are continuous at every number in their domain.

Special types of function

Transcendental function (combination of function)

Depending upon the nature, functions are classified into 3 types:

- Odd function
- Even function
- Neither even nor odd function

WORKED EXAMPLES

Domain and Range

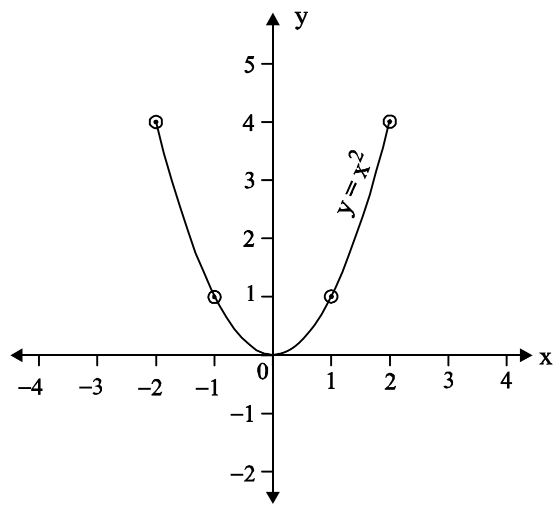
Example 1: Find the domain, range and sketch the graph of $y = x^2$.

Solution:

Verbally	:	It is a parabola.
Algebraically	:	$y = x^2$.
Numerically	:	Table

x	0	1	2	-1	-2
y	0	1	4	1	4

Ordered pairs: (0, 0), (1, 1), (2, 4), (– 1, 1), (– 2, 4)



Example 2: Find the domain and range of $y = \sqrt{25 - x^2}$

Solution:

Given

$$y = \sqrt{25 - x^2}$$

Squaring on both sides

$$y^2 = 25 - x^2$$

$$x^2 + y^2 = 25$$

It is a circle

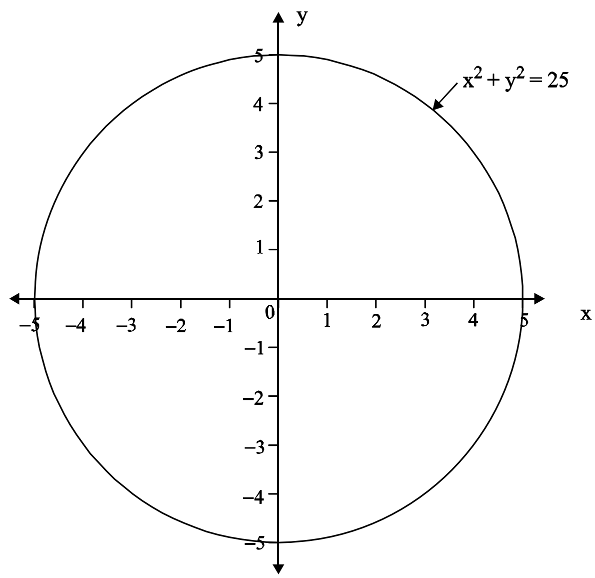
$$\boxed{x^2 + y^2 = 5^2}$$

$$\Rightarrow (x-0)^2 + (y-0)^2 = 5^2$$

Verbally : It is a circle

Algebraically : $x^2 + y^2 = 5^2$

Visually :



Domain $= (-5, 5)$

Range $= [0, 5]$

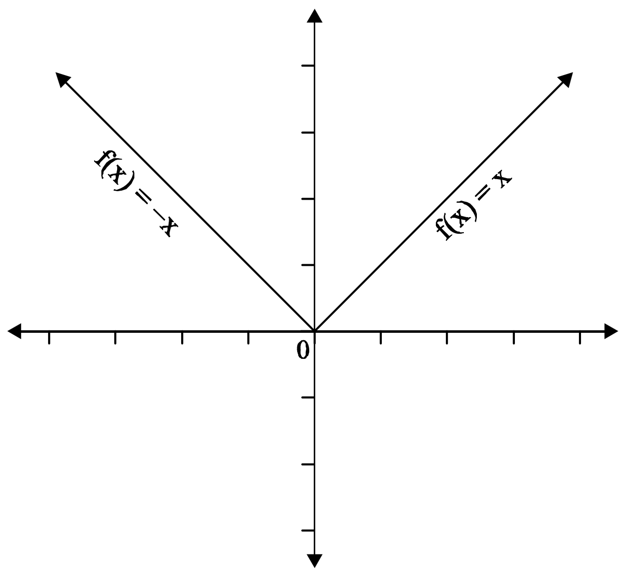
Example 3: Sketch the graph of absolute function $f(x) = |x|$ Also find domain and range.

Solution:

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Domain $= (-\infty, \infty)$

Range $= [0, \infty)$



Example 4: Find the domain and sketch the graph of the function

$$f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 1-x & \text{if } x \geq 0 \end{cases}$$

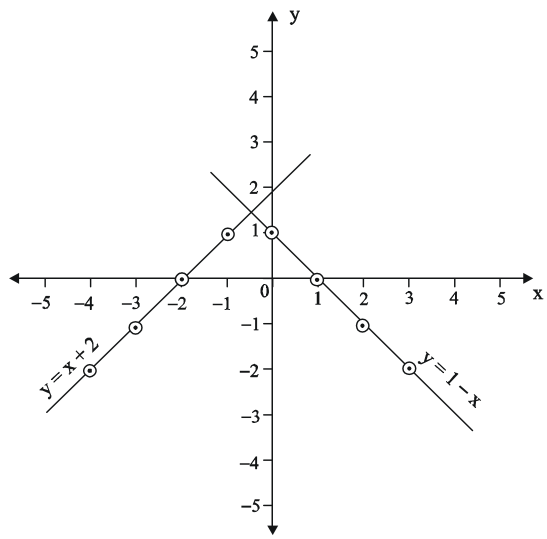
Solution:

$$y = x + 2$$

x	-1	-2	-3	-4
y	1	0	-1	-2

$$y = 1 - x$$

x	0	1	2	3
y	1	0	-1	-2



Domain $= (-\infty, \infty)$

Find the domain of the following functions

Example 1: $f(x) = \dfrac{x+4}{x^2-9}$

Solution:

Consider $x^2 - 9$,

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$\sqrt{x^2} = \pm\sqrt{9}$$

$$x = \pm 3$$

$$\Rightarrow x = -3, 3$$

Domain : $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Example 2: $f(x) = \dfrac{1}{x^2 - x}$

Solution:

Consider, $x^2 - x$

$$x^2 - x = 0$$

$x(x - 1) = 0$

$x = 0,\ x - 1 = 0$

$x = 0, 1$

Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Example 4: $f(x) = \dfrac{4}{3 - x}$

Solution:

$$3 - x = 0$$

$$x = 3$$

Domain: $(-\infty, 3) \cup (3, \infty)$

2.4 LIMITS AND CONTINUITY

Definition

Suppose $f(x)$ is defined when x is near the number a. (This means that f defined on some open interval that contains a, except possibly at a.)

Then the limit of that function is

$$\lim_{x \to a} f(x) = L$$

It is defined as "the limit of $f(x)$, as x approaches a, equals L"

Result

$$\lim_{x \to a} f(x) = L \quad \text{if and only if}$$

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Infinite Limits

Let f be a function defined on both sides of 'a' except possibly at 'a' itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a.

Definition

Let f be defined on both sides of 'a' except possibly at 'a' itself. Then,

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 'a', but not equal to 'a'.

2.4.1 Limit laws

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim\limits_{x \to a} [f(x)\, g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

5. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ provided $\lim\limits_{x \to a} g(x) \neq 0$

Important Results

1. $\lim\limits_{x \to a} f(x) = L$ if and only if

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

2. If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$

   then, $\lim\limits_{x \to a} g(x) = L$

Definition

A function f is continuous at a number 'a' if

$$\lim_{x \to a} f(x) = f(a)$$

In otherwords,

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at if

$$\lim_{x \to a^-} f(x) = f(a)$$

Note

1. A function f is continuous on an interval if it is continuous at every number in the interval.

2. If f and g are continuous at 'a' and c is a constant, then the following functions are also continuous at 'a':

   1. $f+g$ 2. $f-g$

   3. cf 4. fg

   5. f/g if $g \neq 0$

3. Any polynomial is continuous everywhere: that it is continuous on $R = (-\infty, \infty)$

4. The following types of functions are continuous at every number in their domains:

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions and logarithmic functions.

5. **The intermediate value theorem:**
   Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number in (a, b) such that $f(c) = N$

2.4.2 Derivatives

Definition

The derivatives of a function f at a number, a denoted by $f(a)$ is

$$f(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ if this limit exists.}$$

Definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

In otherwords,

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Results

- **If f is differentiable at a, then f continuous at a**
- **Derivatives of constant function**

$$\frac{d}{dx}(c) = 0$$

- **Power functions**

$$\frac{d}{dx}(x) = 1$$

- **Power rule**

  If n is a positive integer then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

- **Power rule (general version)**

  If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

- **The constant multiple rule**

  If c is a constant and f is a differential function, then

$$\frac{d}{dx}[cf(x)] = c\,\frac{d}{dx}f(x)$$

- **The sum rule**

  If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

- **The difference rule**

If f and g are both diffferentiable then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

- **The product rule**

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)\, g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)]$$

- **The Quotient rule**

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,[f'(x)] - f(x)\, g'(x)}{[g(x)]^2}$$

- **The chain rule**

If g is differentiable at x and f is differentiable at $g(x)$, then the compositive function $F = fog$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In leibnitz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

2.4.3 Methods of differentiation

WORKED EXAMPLES

Type 1: Sum Rule

Example 1: Find $\frac{dy}{dx}$ if $y = 3x^2 + 12x - 7x^7 + e^x - \sin x + 2\cos x$

$+ e^{-2x} + e^{7x} + \cos 3x + \log x + e^{-9x} + 100$

Solution:

Given: $y = 3x^2 + 12x - 7x^7 + e^x - \sin x + 2\cos x + e^{-2x} + e^{7x}$

$$+ \cos 3x + \log x + e^{-9x} + 100$$

$$\frac{dy}{dx} = 6x + 12 - 49x^6 + e^x - \cos x - 2\sin x - 2e^{-2x}$$

$$+ 7e^{7x} - 3\sin 3x + \frac{1}{x} - 9e^{-9x} + 0$$

Example 2: Find the derivative of y if $y =$

$y = \sin 2x + \tan x + \sec x + \cot x + 5x^2 + \sin 7x$

Given:

$$y = \sin 2x + \tan x + \sec x + \cot x + 5x^2 + \sin 7x$$

$$y' = 2\cos 2x + \sec^2 x + \sec x \tan x - \operatorname{cosec}^2 x + 10x + 7\cos 7x$$

Example 3: Find y' if $y = x^8 + 10x^3 + \frac{1}{x^2} - \frac{1}{x} + x^{90} + 83 + \sqrt{x}$

Given: $y = x^8 + 10x^3 + \frac{1}{x^2} + \frac{1}{x} + x^{90} + 83 + \sqrt{x}$

$$y' = 8x^7 + 30x^2 - \frac{2}{x^3} - \frac{1}{x^2} + 90x^{89} + 0 + \frac{1}{2\sqrt{x}}$$

Type 2: Product Rule

Example 1: Find $f'(x)$ if $f(x) = xe^x$

Solution:

Given: $f(x) = xe^x = y$

$$y = xe^x$$

$$\boxed{\frac{d}{dx}(uv) = vu' + uv'}$$

$$\boxed{\begin{array}{ll} u = x & v = e^x \\ u' = 1 & v = e^x \end{array}}$$

$$\frac{d}{dx}(xe^x) = e^x(1) + x(e^x)$$

$$\frac{dy}{dx} = e^x + xe^x = e^x(1 + x)$$

Example 2: $y = (x^3 + 2x)(e^x)$

Solution:

$$\boxed{\frac{d}{dx}(uv) = vu' + uv'}$$

$$\begin{array}{ll} u = x^3 + 2x & v = e^x \\ u' = 3x^2 + 2 & v' = e^x \end{array}$$

$$\frac{d}{dx}[(x^3 + 2x)(e^x)] = e^x(3x^2 + 2) + (x^3 + 2x)e^x$$

$$\frac{dy}{dx} = e^x[3x^2 + 2 + x^3 + 2x]$$

Example 3: $\mathbf{y = (x^2 + 1)\,(x^3 + 3)}$

Solution:

Given: $y = (x^2 + 1)\,(x^3 + 3)$

$$\frac{d}{dx}(uv) = vu' + uv'$$

$$u = x^2 + 1 \qquad v = x^3 + 3$$

$$u' = 2x \qquad v' = 3x^2$$

$$\frac{d}{dx}[(x^2 + 1)\,(x^3 + 3)] = (x^3 + 3)\,(2x) + (x^2 + 1)\,(3x^2)$$

$$\frac{dy}{dx} = 2x^4 + 6x + 3x^4 + 3x^2$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x$$

Example 4: $\mathbf{y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)}$

Solution:

Given: $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$

$$u = x^2 + 1 \qquad v = x + 5 + \frac{1}{x}$$

$$u' = 2x \qquad v' = 1 - \frac{1}{x^2}$$

$$\frac{d}{dx}\left[(x^2 + 1)\left(x + 5 + \frac{1}{x}\right)\right] = \left(x + 5 + \frac{1}{x}\right)(2x) + (x^2 + 1)\left(1 - \frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = \left[x + 5 + \frac{1}{x}\right](2x) + (x^2 + 1)\left(1 - \frac{1}{x^2}\right)$$

Example 5: Find $y = 3x^5 \log x$

$$\frac{d}{dx}(uv) = vu' + uv'$$

$$u = 3x^5 \qquad v = \log x$$

$$u' = 15x^4 \qquad v = \frac{1}{x}$$

$$= \log x \; 15x^4 + 3x^5 \; 1/x$$

$$= (\log x) \; 15x^4 + 3x^4$$

Type 3: UVW rule

Example 1: Find $\frac{dy}{dx}$ if $y = x^2 \sin x \tan x$

Given: $y = \frac{x^2}{u} \frac{\sin x}{v} \frac{\tan x}{w}$

$$\boxed{\frac{d}{dx}(uvw) = uvw' + uv'w + u'vw}$$

$$\frac{dy}{dx} = x^2 \sin x \sec^2 x + x^2 \cos x \tan x + 2x \sin x \tan x$$

Example 2: Find y' if $y = xe^x \operatorname{cosec} x$

Solution:

Given: $y = xe^x \operatorname{cosec} x$

$$\frac{dy}{dx} = xe^x (-\operatorname{cosec} x \cot x) + xe^x \operatorname{cosec} x + (1)\, e^x \operatorname{cosec} x$$

$$= -xe^x \operatorname{cosec} x \cot x + xe^x \operatorname{cosec} x + e^x \operatorname{cosec} x$$

Example 3: Find y' if $y = x^5 \log x \operatorname{cosec} x$

Given: $y = x^5 \log x \operatorname{cosec} x$

$$\frac{dy}{dx} = x^5 \log x\,(-\operatorname{cosec} x \cot x) + x^5 \left(\frac{1}{x}\right) \operatorname{cosec} x$$

$$+\, 5x^4 \log x \operatorname{cosec} x$$

$$y' = -\,x^5 \log x \operatorname{cosec} x \cot x + x^4\, cosec\, x + 5x^4 \log x\, cosec\, x$$

Type 4: Quotient rule

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}}$$

Example 1: Find $f'(x)$ if $f(x) = \dfrac{e^x}{x^2}$

Solution:

Given: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{vu' - uv'}{v^2}$

$u = e^x$	$v = x^2$	$v^2 = (x^2)^2 = x^4$
$u = e^x$	$v' = 2x$	

$$\frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{x^2\,(e^x) - e^x\,(2x)}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{x^2\, e^x - e^x\, 2x}{x^4}$$

$$= \frac{xe^x\,(x-2)}{x^4}$$

$$\frac{dy}{dx} = \frac{e^x\,(x-2)}{x^3}$$

Example 2: Find $\frac{dy}{dx}$ if $f(x) = \frac{\sin x}{x^3}$

Solution:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$u = \sin x$ $u' = \cos x$	$v = x^3$ $v' = 3x^2$	$v^2 = (x^3)^2 = x^6$

$$\frac{dy}{dx} = \frac{x^3 (\cos x) - \sin x\,(3x^2)}{x^6} = \frac{x^2\,(x \cos x - 3 \sin x)}{x^6}$$

$$= \frac{x \cos x - 3 \sin x}{x^4}$$

Example 3: Find $f'(x)$ if $f(x) = \frac{x^2}{1 + 2x}$

Solution:

(i) $f(x) = \frac{x^2}{1 + 2x}$

Given:

$$y = f(x) = \frac{x^2}{1 + 2x}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$u = x^2$ $u' = 2x$	$v = 1 + 2x$ $v = 0 + 2$	$v^2 = (1 + 2x)^2$

$$\frac{dy}{dx} = \frac{(1+2x)\,(2x) - (x^2)\,(2)}{(1+2x)^2}$$

$$\frac{dy}{dx} = \frac{2x + 4x^2 - 2x^2}{(1+2x)^2}$$

$$\frac{dy}{dx} = \frac{2x + 2x^2}{(1+2x)^2}$$

(ii) $\boldsymbol{y = \dfrac{x^2 - 1}{x^2 + x + 1}}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$u = x^2 - 1$	$v = x^2 + x + 1$
$u' = 2x$	$v' = 2x + 1$
	$v^2 = (x^2 + x + 1)^2$

$$\frac{dy}{dx} = \frac{(x^2 + x + 1)\,(2x) - (x^2 - 1)\,(2x + 1)}{(x^2 + x + 1)^2}$$

$$\frac{dy}{dx} = \frac{2x^3 + 2x^2 + 2x - [2x^3 - 2x + x^2 - 1]}{(x^2 + x + 1)^2}$$

$$\frac{dy}{dx} = \frac{2x^3 + 2x^2 + 2x - 2x^3 + 2x - x^2 + 1}{(x^2 + x + 1)^2} = \frac{x^2 + 4x + 1}{(x^2 + x + 1)^2}$$

Example 4: Find $\boldsymbol{\dfrac{dy}{dx}}$ if $\boldsymbol{y = \dfrac{x^3}{3x - 2}}$

Solution:

Given: $y = \dfrac{x^3}{3x - 2}$

$u = x^3$	$v = 3x - 2$
$u' = 3x^2$	$v' = 3$

$$\frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \frac{(3x-2)\,(3x^2) - (x^3)\,(3)}{(3x-2)^2}$$

$$\frac{dy}{dx} = \frac{9x^3 - 6x^2 - 3x^3}{(3x-2)^2}$$

$$\boxed{y' = \frac{6x^3 - 6x^2}{(3x-2)^2}}$$

Example 5: Find $y = \dfrac{x^3 - 4x}{x^2 - 3x - 4}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^2 - 3x - 4)\,(3x^2 - 4) - (x^3 - 4x)\,(2x - 3)}{(x^2 - 3x - 4)^2}$$

$u = x^3 - 4x$	$u' = 3x^2 - 4$
$v = x^2 - 3x - 4$	$v' = 2x - 3$
	$v^2 = (x^2 - 3x - 4)^2$

$$\frac{dy}{dx} = \frac{(x^2 - 3x - 4)\,(3x^2 - 4) - (x^3 - 4x)\,(2x - 3)}{(x^2 - 3x - 4)^2}$$

$$= \frac{3x^4 - 9x^3 - 12x^2 - 4x^2 + 12x + 16 - 2x^4 + 3x^3 + 8x^2 - 12x}{(x^2 - 3x - 4)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{x^4 - 6x^3 - 8x^2 + 16}{(x^2 - 3x - 4)^2}}$$

Example 6: **Find** $y = \frac{1 - 2x}{3 + x}$

$u = 1 - 2x$	$u' = -2$	
$v = 3 + x$	$v' = 1$	$v^2 = (3 + x)^2$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \frac{(3 + x)(-2) - (1 - 2x)(1)}{(3 + x)^2}$$

$$= \frac{-6 - 2x - 1 + 2x}{(3 + x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-7}{(3 + x)^2}}$$

Example 7: **Find** $y = x^2 e^{2x}$

$u = x^2$	$v = e^{2x}$
$u' = 2x$	$v' = 2e^{2x}$

$$\frac{d}{dx}(uv) = vu' + uv'$$

$$= e^{2x}(2x) + (x^2)\, 2e^{2x}$$

$$= e^{2x}(2x + 2x^2)$$

$$= 2xe^{2x}(1 + 2x)$$

Type 5: Chain rule

Chain rule is otherwise called as function of the function rule

Example 1: **Find** $\frac{dy}{dx}$ **if** $y = (2x+3)^2$

Solution:

$$y = (2x+3)^2$$

$$\frac{dy}{dx} = 2\,(2x+3)\,(2)$$

$$\frac{dy}{dx} = 4\,(2x+3)$$

$$\boxed{\frac{dy}{dx} = 8x + 12}$$

Example 2: **Find** $\frac{dy}{dx}$ **if** $y = \sqrt{2x-5}$

Solution:

$$y = \sqrt{2x-5}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{2x-5}}\,(2)$$

$$= \frac{2}{2\sqrt{2x-5}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{2x-5}}}$$

Example 3: **Find** $\frac{dy}{dx}$ **if** $y = \sin\,(4x^2 + 3x + 2)$

Solution:

$$\frac{dy}{dx} = \cos\,(4x^2 + 3x + 2)\,(8x+3)$$

Example 4: Find $\frac{dy}{dx}$ if $y = \sin\sqrt{x^2 + x + 1}$

Solution:

$$\frac{dy}{dx} = \cos\sqrt{x^2 + x + 1} \times (2x + 1)$$

$$\frac{d}{dx}(\sin x) = \cos x$$

Example 5: Find $\frac{dy}{dx}$ if $y = \log(\sin x)$

Solution:

$$y' = \frac{1}{\sin x} \times \cos x$$

$$= \frac{\cos x}{\sin x}$$

$$\boxed{y' = \cot x}$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

Example 6: Find the derivative of $f(x) = \log(x^3 + 1)$

Solution:

$$f(x) = \log(x^3 + 1)$$

$$f'(x) = \frac{1}{x^3 + 1} \times 3x^2$$

$$\boxed{f'(x) = \frac{3x^2}{x^3 + 1}}$$

Hint : $X = x^3 + 1$

$$f(x) = \log x$$

$$f'(x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

Example 7: Find $\frac{dy}{dx}$ if $f(x) = \log(\cos x)$

Solution:

$$f(x) = \frac{1}{\cos x}(-\sin x)$$

$$= \frac{-\sin x}{\cos x}$$

$$\boxed{f'(x) = -\tan x}$$

Type 6: Implicit differentiation

Find $\frac{dy}{dx}$ of the following functions

Example 1: $x^2 + y^2 = c^2$

Solution: .... (1)

$$x^2 + y^2 = c^2$$

Differentiate (1) on both sides

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Example 2: $x^3 + y^3 = a^3$

Solution:

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} = -3x^2$$

$$\boxed{\frac{dy}{dx} = -\frac{x^2}{y^2}}$$

Example 3: $x^2 + y^2 = 1$

Solution:

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = 0 - 2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 4: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Solution:

$$2ax + 2h\left[y\,(1) + x\frac{dy}{dx}\right] + b2y\frac{dy}{dx} + 2g\,(1) + 2f\frac{dy}{dx} + 0 = 0$$

$$2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\,[2hx + 2by + 2f] = -\,2ax - 2hy - 2g$$

$$\frac{dy}{dx} = \frac{-\,2ax - 2hy - 2g}{2hx + 2by + 2f}$$

$$\frac{dy}{dx} = \frac{2\,(-\,ax - hy - g)}{2\,(hx + by + f)}$$

$$\frac{dy}{dx} = \frac{-\,(ax + hy + g)}{(hx + by + f)}$$

MISCELLANEOUS EXAMPLES

Find $\frac{dy}{dx}$ for the following functions

Example 1: $y = (x + 1)\,(x - 1)\,(x + 3)$

Solution:

$$\boxed{\frac{d}{dx}\,(uvw) = u'\,vw + uv'\,w + uvw'}$$

$$u = x + 1 \qquad v = x - 1 \qquad w = x + 3$$

$$u' = 1 \qquad v' = 1 \qquad w' = 1$$

$$\frac{dy}{dx} = 1\,(x - 1)\,(x + 3) + (x + 1)\,(1)\,(x + 3) + (x + 1)\,(x - 1)\,(1)$$

$$\frac{dy}{dx} = (x + 3)\,(x - 1) + (x + 1)\,(x + 3) + (x + 1)\,(x - 1)$$

$$\frac{dy}{dx} = x^2 + 3x - x - 3 + x^2 + 3x + x + 3 + x^2 - x + x - 1$$

$$\boxed{\frac{dy}{dx} = 3x^2 + 6x - 1}$$

Example 2: $y = \sqrt{x} \sin x$

Solution:

$$\frac{d}{dx}(uv) = vu' + uv'$$

$$\frac{dy}{dx} = \sin x \frac{1}{2\sqrt{x}} + \sqrt{x} \cos x$$

$$\boxed{y' = \frac{\sin x}{2\sqrt{x}} + \sqrt{x} \cos x}$$

Example 3: $y = (2x - 3)^2$

Solution:

$$y' = 2(2x - 3)(2)$$

$$= 4(2x - 3)$$

$$\boxed{y' = 8x - 12}$$

Example 4: $y = \sin 5x + \cos 3x + \cot x$

Solution:

$$y' = 5 \cos 5x - 3 \sin 3x - \operatorname{cosec}^2 x$$

Example 5: $f(x) = \dfrac{\log x}{x^2}$

Solution:

$u = \log x$	$v = x^2$	$v^2 = x^4$
$u' = \dfrac{1}{x}$	$v' = 2x$	

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} = \frac{(x^2)\left(\frac{1}{x}\right) - (\log x)\,(2x)}{x^4}$$

$$\boxed{\frac{dy}{dx} = \frac{x - 2x \log x}{x^4}}$$

Example 6: $y \sin 2x - x \cos 2y$

Solution:

$$\left[\sin 2x \frac{dy}{dx} + y\, 2 \cos 2x\right] = \left[\cos 2y\,(1) + x\,(-2 \sin 2y)\frac{dy}{dx}\right]$$

$$\sin 2x \frac{dy}{dx} + 2y \cos 2x - \cos 2y + 2x \sin 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\,[\sin 2x + 2x \sin 2y] = \cos 2y - 2y \cos 2x$$

$$\boxed{\frac{dy}{dx} = \frac{\cos 2y - 2y \cos 2x}{\sin 2x + 2x \sin 2y}}$$

Example 7: $y = \sqrt{\log x}$

Solution:

$$y' = \frac{1}{2\sqrt{\log x}} \times \frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2x\sqrt{\log x}}}$$

Example 8: $y = \dfrac{x \sin x}{1 + x}$

Solution:

$$u = \frac{x}{u}\,\frac{\sin x}{v}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$u' = \sin x\,(1) + x \cos x$$

$$\frac{dy}{dx} = \frac{(1+x)\,(\sin x + x\cos x) - (x\sin x)\,(1)}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{(1+x)\,(\sin x) + (1+x)\,(x\cos x) - (x\sin x)}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{\sin x + x\sin x + x\cos x + x^2\cos x - x\sin x}{(1+x)^2}$$

$$\boxed{\mathbf{\frac{dy}{dx} = \frac{\sin x + x\cos x + x^2\cos x}{(1+x)^2}}}$$

Example 9: $\mathbf{y = \dfrac{x^2 \sin x}{1+x}}$

Solution:

$$u = \underset{u}{x^2}\ \underset{v}{\sin x}$$

$$u' = \sin x\,(2x) + x^2\cos x$$

$$\frac{d}{dy}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{(1+x)\,[(\sin x\,(2x) + x^2\cos x)] - x^2\sin x\,(1)}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{(1+x)\,(2x\sin x) + (1+x)\,(x^2\cos x) - x^2\sin x}{(1+x)^2}$$

Example 10: $\mathbf{f(x) = \dfrac{x^2 + x + 1}{x^2 - x - 1}}$

Solution:

$$u = x^2 + x + 1 \qquad v = x^2 - x - 1$$

$$u' = 2x + 1 \qquad v' = 2x - 1$$

$$f'(x) = \frac{(x^2 - x - 1)(2x + 1) - (x^2 + x + 1)(2x - 1)}{(x^2 - x - 1)^2}$$

$$= \frac{(2x^3 - 2x^2 - 2x + x^2 - x - 1) - (2x^3 + 2x^2 + 2x - x^2 - x - 1)}{(x^2 - x - 1)^2}$$

$$= \frac{2x^3 - 2x^2 - 2x + x^2 - x - 1 - 2x^3 - 2x^2 - 2x + x^2 + x + 1}{(x^2 - x - 1)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-2x^2 - 4x}{(x^2 - x - 1)^2}}$$

2.4.4 Limits

Example 1: Evaluate $\lim\limits_{x \to 1} (x^2 + x + 1)$

Solution:

Given: $\lim\limits_{x \to 1} (x^2 + x + 1)$

$$= 1^2 + 1 + 1$$

$$\lim_{x \to 1} x^2 + x + 1 = 3$$

Example 2: Evaluate $\lim\limits_{x \to 2} 2x^2 + 4$

Solution:

Given: $\lim\limits_{x \to 2} 2x^2 + 4$

$$= 2(2)^2 + 4$$

$$\lim_{x \to 2} 2x^2 + 4 = 12$$

Example 3: Evaluate $\lim\limits_{x \to 0} \left(\dfrac{x^2 + 3x + 4}{x^2 + 2x + 1} \right)$

Solution:

Given: $\lim\limits_{x \to 0} \left(\dfrac{x^2 + 3x + 4}{x^2 + 2x + 1} \right)$

$$= \frac{0^2 + 3\,(0) + 4}{0^2 + 2\,(0) + 1}$$

$$\lim_{x \to 0} \left(\frac{x^2 + 3x + 4}{x^2 + 2x + 1} \right) = 4$$

Example 4: Evaluate $\lim\limits_{x \to -2} x^3 - 3x^2 + 4$

Solution:

Given: $\lim\limits_{x \to -2} x^3 - 3x^2 + 4$

$$= (-2)^3 - 3\,(-2)^2 + 4$$

$$= -8 - 12 + 4 = -20 + 4$$

$$\lim_{x \to -2} x^3 - 3x^2 + 4 = -16$$

Indeterminate forms (Limits)

$\dfrac{0}{0}, \dfrac{\infty}{\infty}, 0°, \infty^{\infty}$

Whenever indeterminate form occur we need to use **L'Hospital rule.**

WORKED EXAMPLES

L'Hospital rule

Example 1: Evaluate $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$

Solution:

Given: $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$

$$= \frac{1^3 - 1}{1 - 1} = \frac{0}{0}$$

Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 1} \frac{3x^2 - 0}{1 - 0}$$

$$= \frac{3\,(1)^2}{1} = 3$$

$$\boxed{\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3}$$

Example 2: Evaluate $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

Solution:

Given: $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

$$= \frac{(2)^3 - 8}{2 - 2} = \frac{0}{0}$$

$\Rightarrow$ Indeterminate forms

Apply L'Hospital rule

$$\lim_{x \to 2} \frac{3x^2}{1} = \frac{3\,(2)^2}{1} = 12$$

$$\boxed{\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = 12}$$

Example 3: Evaluate $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$

Solution:

Given: $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$

$$= \frac{\sin 0}{0} = \frac{0}{0}$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{\theta \to 0} \frac{\cos\theta}{1} = \frac{\cos 0}{1} = \frac{1}{1}$$

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

Example 4: Evaluate $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$

Solution:

Given: $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$

$$= \frac{1 - \cos 2\,(0)}{0^2}$$

$$= \frac{1-1}{0} = \frac{0}{0}$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 0} \frac{0 - (-2 \sin 2x)}{2x}$$

$$\lim_{x \to 0} \frac{2 \sin 2x}{2x}$$

$$= \frac{2 \sin 2(0)}{2(0)} = \frac{2(0)}{0} = \frac{0}{0}$$

Apply L'Hospital rule

$$\lim_{x \to 0} \frac{2(2 \cos 2x)}{2}$$

$$\lim_{x \to 0} \frac{4 \cos 2x}{2}$$

$$= \frac{4 \cos 2(0)}{2}$$

$$= \frac{4}{2} = 2$$

$$\boxed{\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = 2}$$

Example 5: Evaluate $\lim\limits_{\theta \to 0} \left[\frac{\sin 5\theta - \sin 3\theta}{\sin 3\theta} \right]$

Solution:

Given: $\lim\limits_{x \to 0} \left[\frac{\sin 5\theta - \sin 3\theta}{\sin 3\theta} \right]$

$$= \frac{\sin 5(0) - \sin 3(0)}{\sin 3(0)} = \frac{0-0}{0}$$

$= \frac{0}{0}$ Indeterminate form

Apply L'Hospital rule

$$\lim_{\theta \to 0} \left[\frac{5 \cos 5\theta - 3 \cos 3\theta}{3 \cos 3\theta} \right]$$

$$= \frac{5 \cos 5 (0) - 3 \cos 3 (0)}{3 \cos 3 (0)}$$

$$= \frac{5 - 3}{3} = \frac{2}{3}$$

$$\boxed{\lim_{\theta \to 0} \left[\frac{\sin 5\theta - \sin 3\theta}{\sin 3\theta} \right] = \frac{2}{3}}$$

Example 6: Evaluate $\displaystyle \lim_{x \to \pi/2} \frac{\cos x}{x - \frac{\pi}{2}}$

Solution:

Given: $\displaystyle \lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2}$

$$= \frac{\cos \pi/2}{\frac{\pi}{2} - \frac{\pi}{2}} = \frac{0}{0}$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to \pi/2} \frac{- \sin x}{1}$$

$$= - \frac{\sin \pi/2}{1}$$

$$= \frac{-1}{1} = -1$$

$$\boxed{\lim_{x \to \pi/2} \frac{\cos x}{x - \pi/2} = -1}$$

Example 7: Evaluate $\lim\limits_{x \to \infty} \left(\dfrac{2x+5}{3x-2} \right)$

Solution:

Given: $\lim\limits_{x \to \infty} \dfrac{2x+5}{3x-2}$

$$= \frac{2\,(\infty)+5}{3\,(\infty)-2} = \frac{\infty}{\infty}$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to \infty} \frac{2}{3} = \frac{2}{3}$$

$$\lim_{x \to \infty} \left(\frac{2x+5}{3x-2} \right) = \frac{2}{3}$$

Example 8: Evaluate $\lim\limits_{x \to \infty} \dfrac{6x^2 - x + 7}{x+3}$

Solution:

Given $\lim\limits_{x \to \infty} \dfrac{6x^2 - x + 7}{x+3}$

$$= \frac{6\,(\infty)^2\, x + 7}{\infty + 3}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Apply L'Hospital rule

$$\lim_{x \to \infty} \frac{12x-1}{1+0}$$

$$= \frac{12\,(\infty) - 1}{1}$$

$$= \frac{\infty}{1} = \infty$$

$\therefore$ **Limit does not exist.**

Example 9: Evaluate $\lim\limits_{x \to \infty} \left[\dfrac{5x^2 + 7x + 10}{3x^2 + 10x + 25} \right]$

Solution:

Given: $\lim\limits_{x \to \infty} \left[\dfrac{5x^2 + 7x + 10}{3x^2 + 10x + 25} \right]$

$$= \frac{5\,(\infty)^2 + 7\,(\infty) + 10}{3\,(\infty)^2 + 10\,(\infty) + 25}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Apply L'Hospital rule

$$\lim_{x \to \infty} \frac{10x + 7}{6x + 10} = \frac{10\,(\infty) + 7}{6\,(\infty) + 10}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Apply L'Hospital rule

$$\lim_{x \to \infty} \frac{10}{6} = \frac{10}{6} = \frac{5}{3}$$

$$\boxed{\lim_{x \to \infty} \left[\frac{5x^2 + 7x + 10}{3x^2 + 10x + 25} \right] = \frac{5}{3}}$$

2.4.5 Continuity

WORKED EXAMPLES

Example 1: Examine the continuity of a function

$$f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2}; & x \neq 2 \\ 12; & x = 2 \end{cases}$$

Solution:

Given: $f(x) = \dfrac{x^3 - 8}{x - 2}$

To check the continuity at $x = 2$

Left limit	Right limit
$\lim\limits_{x \to 2^-} f(x)$	$\lim\limits_{x \to 2^+} f(x)$
$\lim\limits_{x \to 2^-} \dfrac{x^3 - 8}{x - 2}$	$\lim\limits_{x \to 2^+} \dfrac{x^3 - 8}{x - 2}$
$= \dfrac{2^3 - 8}{2 - 2} = \dfrac{0}{0}$ Indeterminate form	$= \dfrac{2^3 - 8}{2 - 2} = \dfrac{0}{0}$ Indeterminate form
Apply L'Hospital rule	**Apply L'Hospital rule**
$\lim\limits_{x \to 2^-} \dfrac{3x^2}{1} = \dfrac{3(2)^2}{1}$	$\lim\limits_{x \to 2^+} \dfrac{3x^2}{1} = \dfrac{3(2)^2}{1}$
$\boxed{\textbf{Left limit} = 12}$ ... (1)	$\boxed{\textbf{Right limit} = 12}$ ... (2)

From (1) & (2), Left limit $=$ Right limit

$\therefore$ The function is continuous at $x = 2$

Example 2: Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^4 - 1}{x - 1} & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$$

Solution:

Given: $f(x) = \dfrac{x^4 - 1}{x - 1}$

To check the continuity at $x = 1$

Left limit

$$\lim_{x \to 1^-} f(x)$$

$$\lim_{x \to 1^-} \frac{x^4 - 1}{x - 1} = \frac{1^4 - 1}{1 - 1} = \frac{0}{0}$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 1^-} \frac{4x^3}{1}$$

$$\text{Left limit } = \frac{4\,(1)^3}{1} = 4 \qquad \ldots (1)$$

Right limit

$$\lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^+} \frac{x^4 - 1}{x - 1} = \frac{1^4 - 1}{1 - 1} = \frac{0}{0}$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 1^+} \frac{4x^3}{1} = \frac{4(1)^3}{1}$$

$$\text{Right limit} = 4 \qquad \text{... (2)}$$

From (1) & (2), Left limit = right limit.

$\therefore f(x)$ is continuous in $x = 1$

Example 3: Examine the continuous of the function at $x = 0$ for $f(x) = \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Solution:

Given: $f(x) = \frac{\sin 2x}{x}$

$$\lim_{x \to 0} \frac{\sin 2x}{x}$$

$$= \frac{\sin 2(0)}{0} = \frac{0}{0}$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 0} \frac{2 \cos 2x}{1}$$

$$= \frac{2 \cos 2(0)}{1} = \frac{2 \cos 0}{1} = \frac{2}{1} = (2)$$

$$\lim_{x \to 0} \frac{\sin 2x}{x} = 2$$

but it is given in the problem $\lim_{x \to 0} \frac{\sin 2x}{x} = 1$

$$\boxed{\because \text{ The given function } f(x) \text{ is discontinuous at } x = 0}$$

Example 4: Determine the value of λ for which the following function is continuous at $x = -1$

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1} & \text{if } x \neq -1 \\ \lambda & \text{if } x = -1 \end{cases}$$

Solution:

Given

$$f(x) = \frac{x^2 - 2x - 3}{x+1}$$

Since $f(x)$ is continuous at $x = -1$

$\Rightarrow$ Left limit = Right limit = λ

$$\lim_{x \to -1} \frac{x^2 - 2x - 3}{x+1} = \lambda$$

$$\Rightarrow \frac{(-1)^2 - 2(-1) - 3}{-1+1} = \lambda$$

$$\Rightarrow \frac{1+2-3}{0} = \frac{0}{0} = \lambda$$

Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to -1} \frac{2x-2}{1} = \lambda$$

$$\frac{2(-1)-2}{1} = \lambda$$

$$\frac{-2-2}{1} = \lambda$$

$$\boxed{-4 = \lambda}$$

Example 5: Determine the value of k for which the following function is continuous at $x = 3$

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ k & \text{if } x = 3 \end{cases}$$

$$f(x) = \frac{x^2 - 9}{x - 3}$$

Solution:

Since $f(x)$ is continuous at $x = 3$

Left = Right limit $= k$

$$\lim_{x \to 3} f(x) = k$$

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = k$$

$$\frac{3^2 - 9}{3 - 3} = k$$

$$\frac{0}{0} = k$$

Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 3} \frac{2x}{1} = k$$

$$2\,(3) = k$$

$$\boxed{6 = k}$$

Example 6: Determine the value k for which the following function is continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$

Solution:

$$f(x) = \frac{1-\cos 2x}{x^2}$$

Since $f(x)$ is continuous on $(-\infty, \infty)$

Left limit = Right limit $= k$

$$\lim_{x \to 0} f(x) = k$$

$$\lim_{x \to 0} \frac{1-\cos 2x}{x^2} = k$$

$$\frac{1-\cos 2(0)}{0^2} = k$$

$$\frac{0}{0} = k$$

Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 0} \frac{0-(-2\sin 2x)}{2x} = k$$

$$\Rightarrow \lim_{x \to 0} \frac{2\sin 2x}{2x} = k$$

$$\Rightarrow \frac{2\sin 2(0)}{2(0)} = k$$

$$\frac{0}{0} = k$$

Indeterminate form

Apply L'Hospital rule

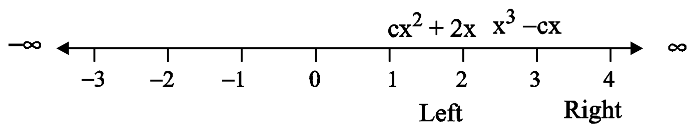
$$\lim_{x \to 0} \frac{2\,(2)\cos 2x}{2} = k$$

$$\Rightarrow \frac{4\cos(0)}{2} = k \quad [\cos 0 = 1]$$

$$\Rightarrow \frac{4}{2} = k$$

$$\boxed{\mathbf{2 = k}}$$

Example 7: For what value of 'c' is the function continuous on $(-\infty, \infty)$ $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$



Solution:

Since $f(x)$ is continuous on $(-\infty, \infty)$

$$\text{Left limit} = \text{Right limit}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} cx^2 + 2x = \lim_{x \to 2^+} x^3 - cx$$

$$c\,(2)^2 + 2\,(2) = (2)^3 - 2c$$

$$4c + 4 = 8 - 2c$$

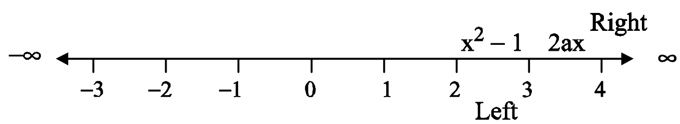
$$4c + 2c = 8 - 4$$

$$6c = 4$$

$$c = \frac{4}{6} \Rightarrow \boldsymbol{c = \frac{2}{3}}$$

Example 8: Determine the value of 'a' for which $f(x)$ continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 3 \\ 2ax & \text{if } x \geq 3 \end{cases}$

Solution:



Since $f(x)$ is continuous on $(-\infty, \infty)$

$$\Rightarrow \text{Left limit} = \text{Right limit}$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

$$\lim_{x \to 3^-} x^2 - 1 = \lim_{x \to 3^+} 2ax$$

$$3^2 - 1 = 2a(3)$$

$$9 - 1 = 6a$$

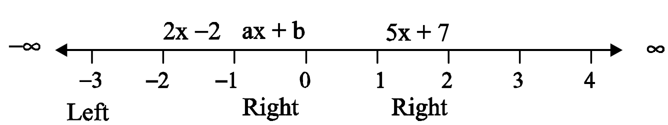
$$\frac{8}{6} = a$$

$$\boxed{\frac{4}{3} = a}$$

Example 9: Find the values of 'a' and 'b' for which the following function is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 2x - 2 & \text{if } x < -1 \\ ax + b & \text{if } -1 \leq x \leq 1 \\ 5x + 7 & \text{if } x > 1 \end{cases}$$

Solution:



Given

$$f(x) = \begin{cases} 2x - 2 & \text{if } x < -1 \\ ax + b & \text{if } -1 \le x \le 1 \\ 5x + 7 & \text{if } x > 1 \end{cases}$$

Since $f(x)$ is continuous on $(-\infty, \infty)$

$\Rightarrow$ Left limit = Right limit

Case (i): To check the continuity at $x = -1$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$$

$$\lim_{x \to -1^-} 2x - 2 = \lim_{x \to -1^+} ax + b$$

$$2(-1) - 2 = a(-1) + b$$

$$2(-1) - 2 = a(-1) + b$$

$$-2 - 2 = -a + b$$

$$-4 = -a + b$$

$$-a + b = -4 \qquad \ldots (1)$$

Case (ii): To check the continuity at $x = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^-} ax + b = \lim_{x \to 1^+} 5x + 7$$

$$a(1) + b = 5(1) + 7$$

$$a + b = 12 \qquad \ldots (2)$$

Solving equations (1) & (2) we get

$$\boxed{a = 8}$$

$$\boxed{b = 4}$$

Example 10: Find the values of a and b that makes $f(x)$ is continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$

Solution:

Given: $$f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

Since $f(x)$ is continuous on $(-\infty, \infty)$

$\Rightarrow$ Left limit = Right limit

Case (i): To check the continuity at $x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2^+} ax^2 - bx + 3$$

$$\frac{8 - 8}{2 - 2} = 4a - 2b + 3$$

$$\frac{0}{0} = 4a - 2b + 3$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 2^-} \frac{3x^2}{1} = 4a - 2b + 3$$

$$\frac{3\,(2)^2}{1} = 4a - 2b + 3$$

$$12 = 4a - 2b + 3$$

$$4a - 2b + 3 = 12$$

$$4a - 2b = 12 - 3$$

$$4a - 2b = 9 \qquad \text{... (1)}$$

Case (ii): To check the continuity at $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

$$\lim_{x \to 3^-} ax^2 - bx + 3 = \lim_{x \to 3^+} 2x - a + b$$

$$9a - 3b + 3 = 6 - a + b$$

$$9a - 3b + a - b = 6 - 3$$

$$10a - 4b = 3 \qquad \text{... (2)}$$

Solving (1) and (2) we get

$$\boxed{a = \frac{-15}{2}}$$

$$\boxed{b = \frac{-39}{2}}$$

Example 11: Find the values of a and b that makes $f(x)$ is continuous on $(-\infty, \infty)$, $\boldsymbol{f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}}$

Solution:

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 22 - a + b & \text{if } x \ge 3 \end{cases}$$

Since $f(x)$ is continuous on $(-\infty, \infty)$

$\Rightarrow$ Left limit = Right limit

Case (i): To check the continuity at $x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^+} ax^2 - bx + 3$$

$$\frac{4-4}{2-2} = 4a - 2b + 3$$

$$\frac{0}{0} = 4a - 2b + 3$$

$\Rightarrow$ Indeterminate form

Apply L'Hospital rule

$$\lim_{x \to 2^-} \frac{2x}{1} = 4a - 2b + 3$$

$$4 = 4a - 2b + 3$$

$$4a - 2b = 1 \qquad \text{... (i)}$$

Case (ii): To check the continuity at $x = 3$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

$$\lim_{x \to 3^-} ax^2 - bx + 3 = \lim_{x \to 3} 2x - a + b$$

$$9a - 3b + 3 = 6 - a + b$$

$$9a + a - b - 3b = 3$$

$$10a - 4b = 3 \qquad \text{... (2)}$$

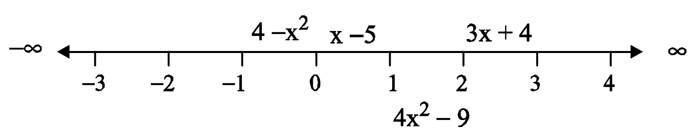
Solving (1) and (2) we get

$$\boxed{a = \frac{1}{2}}$$

$$\boxed{b = \frac{1}{2}}$$

Example 12: Test the continuity of the function

$$f(x) = \begin{cases} 4 - x^2 & ; \ x \le 0 \\ x - 5 & ; \ 0 \le x \le 1 \\ 4x^2 - 9 & ; \ 1 < x \le 2 \\ 3x + 4 & ; \ x \ge 2 \end{cases}$$



Solution:

$$f(x) = \begin{cases} 4 - x^2 & ; \ x \le 0 \\ x - 5 & ; \ 0 \le x \le 1 \\ 4x^2 - 9 & ; \ 1 < x \le 2 \\ 3x + 4 & ; \ x \ge 2 \end{cases}$$

Case (i): To check the continuity at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\lim_{x \to 0^-} 4 - x^2 = \lim_{x \to 0^+} x - 5$$

$$4 \neq -5$$

$f(x)$ is discontinuous at $x = 0$

Case (ii): To check the continuity at $x = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^-} x - 5 = \lim_{x \to 1^+} 4x^2 - 9$$

$$1 - 5 = 4\,(1)^2 - 7$$

$$-4 \neq -3$$

$\therefore f(x)$ is discontinuous at $x = 1$

Case (iii): To check the continuity at $x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} 4x^2 - 9 = \lim_{x \to 2^+} 3x + 4$$

$$4\,(2)^2 - 9 = 3\,(2) + 4$$

$$16 - 9 = 6 + 4$$

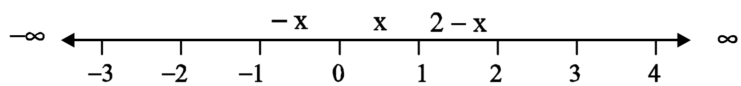
$$7 \neq 10$$

$\therefore f(x)$ is discontinuous at $x = 2$

Example 13: Discuss the continuity of the function

$$f(x)=\begin{cases} -x\,; & x<0 \\ x\,; & 0\le x\le 1 \\ 2-x\,; & x>1 \end{cases}$$

Solution:



Given: $\begin{cases} -x\,; & x<0 \\ x\,; & 0\le x\le 1 \\ 2-x\,; & x>1 \end{cases}$

Case (i): To check the continuity at $x=0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-} -x = \lim_{x\to 0^+} x$$

$$0 = 0$$

$f(x)$ is continuous at $x=0$

Case (ii): To check the continuity at $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\lim_{x\to 1^-} x = \lim_{x\to 1^+} 2-x$$

$$1 = 2-1$$

$$1 = 1$$

$f(x)$ is continuous at $x=1$

2.5 EQUATION OF TANGENT AND NORMAL

Tangent line

Tangent is a line which is touching exactly one point on the curve.

Normal line

Normal is a line which is perpendicular to tangent line.

- **Equation of tangent line**

$$(y - y_0) = m\,(x - x_0)$$

where m is the slope $= \left(\dfrac{dy}{dx}\right)$

- **Equation of normal**

$$(y - y_0) = -\frac{1}{m}\,(x - x_0)$$

WORKED EXAMPLES

Example 1: Find the equation of tangent and normal to the curve $y = x^3 - 3x + 1$ at the point (2, 3).

Solution:

Given: Curve $y = x^3 - 3x + 1$ ... (1)

Points $(x_0, y_0) = (2, 3)$

Equation of tangent

$$y - y_0 = m\,(x - x_0) \quad \text{... (2)}$$

Slope $= m = \dfrac{dy}{dx}$

$$\frac{dy}{dx} = 3x^2 - 3$$

$$\left(\frac{dy}{dx}\right)_{(2,\,3)} = 3\,(2)^2 - 3$$

$$m = 12 - 3$$

$$\boxed{\boldsymbol{m = 9}}$$

$$(y - 3) = 9\,(x - 2)$$

$$y - 3 = 9x - 18$$

$$y - 9x = -\,18 + 3$$

$$\boxed{\boldsymbol{y - 9x = -\,15}}$$

Equation of normal

$$y - y_0 = -\frac{1}{m}\,(x - x_0)$$

$$(y - 3) = -\frac{1}{9}\,(x - 2)$$

$$9\,(y - 3) = -\,1\,(x - 2)$$

$$9y - 27 = -\,x + 2$$

$$x + 9y = 2 + 27$$

$$\boxed{\boldsymbol{x + 9y = 29}}$$

Example 2: Find the equation of tangent and normal to the curve $x^2 + y^2 = 25$ at the point (3, 4)

Solution:

Given: Curve $x^2 + y^2 = 25$ .... (1)

$(x_0, y_0) = (3, 4)$

using Implicit differentiation

differentiate equation (1)

$$2x + 2y\,\frac{dy}{dx} = 0$$

$$2y\,\frac{dy}{dx} = -\,2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\left(\frac{dy}{dx}\right)_{(3,\,4)} = \mathbf{\frac{-3}{4}} = \boldsymbol{m}$$

Equation of tangent	**Equation of normal**
$y - y_0 = m\,(x - x_0)$	$y - y_0 = -\frac{1}{m}\,(x - x_0)$
$y - 4 = \frac{-3}{4}\,(x - 3)$	$(y - 4) = -\frac{1}{\frac{-3}{4}}\,(x - 3)$
$4\,(y - 4) = -3\,(x - 3)$	$(y - 4) = \frac{4}{3}\,(x - 3)$
$4y - 16 = -3x + 9$	$3\,(y - 4) = \frac{4}{3}\,(x - 3)$
$3x + 4y = 9 + 16$	$3y - 12 = 4x - 12$ $4x - 3y - 12 + 12 = 0$
$\mathbf{3x + 4y = 25}$	$\mathbf{4x - 3y = 0}$

Example 3: Find the equation of tangent and normal to the curve $\boldsymbol{y = x + \cos x}$ at (0, 1)

Solution:

Given: Curve $y = x + \cos x$

Point $\underset{x_0\;y_0}{(0,\;1)}$

$$\frac{dy}{dx} = 1 - \sin x$$

$$m = \left(\frac{dy}{dx}\right)_{(0,\ 1)} = 1 - \sin 0$$

$$= 1 - 0$$

$$\boxed{\therefore \sin 0 = 0}$$

$$\boxed{m = 1}$$

Equation of tangent	**Equation of normal**
$y - y_0 = m\,(x - x_0)$	$y - y_0 = \frac{-1}{m}\,(x - x_0)$
$(y - 1) = 1\,(x - 0)$	$y - 1 = -\frac{1}{1}\,(x - 0)$
$y - 1 = x - 0$	$1\,(y - 1) = -1\,(x - 0)$
$-x + y = 0 + 1$	$y - 1 = -x + 0$
$\boldsymbol{-x + y = 1}$	$\boldsymbol{x + y = 1}$

Example 4: Find the equation of tangent and normal to the curve $\boldsymbol{y = \frac{2x+1}{x+1}}$ at (1, 1)

Solution:

Given: $y = \frac{2x+1}{x+1}$

use Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$u = 2x + 1 \qquad v = x + 1$$

$$u' = 2 \qquad v' = 1$$

$$\frac{dy}{dx} = \frac{(x+1)\,(2) - (2x+1)\,(1)}{(x+1)^2}$$

$$= \frac{2x + 2 - 2x - 1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}$$

$$m = \left(\frac{dy}{dx}\right)_{(1,\,1)} = \frac{1}{(1+1)^2}$$

$$\boxed{\boldsymbol{m = \frac{1}{4}}}$$

at point $\begin{matrix}(1, 1)\\ x_0\ y_0\end{matrix}$

Equation of tangent	**Equation of normal**
$y - y_0 = m\,(x - x_0)$	$y - y_0 = \frac{-1}{m}\,(x - x_0)$
$y - 1 = \frac{1}{4}\,(x - 1)$	$y - 1 = -\frac{1}{1/4}\,(x - 1)$
$4\,(y - 1) = 1\,(x - 1)$	$y - 1 = -\frac{4}{1}\,(x - 1)$
$-x + 4y = -1 + 4$	$y - 1 = -4x + 4$
$\boldsymbol{-x + 4y = 3}$	$\boldsymbol{4x + y = 5}$

Example 5: Find the equation of tangent and normal to the curve $y = e^x \cos x$ at (0, 1).

Solution:

Given: $y = e^x \cos x$

Use Product rule:

$$\boxed{\frac{d}{dx}(uv) = uv' + vu'}$$

$u = e^x \qquad v = \cos x$

$u' = e^x \qquad v' = -\sin x$

$$\frac{dy}{dx} = uv' + vu'$$

$$= -e^x \sin x + \cos x e^x$$

$$\frac{dy}{dx} = e^x (\cos x - \sin x)$$

$$m = \left(\frac{dy}{dx}\right)_{(0,\ 1)} = e^0 (\cos 0 - \sin 0)$$

$$m = 1\ (1 - 0)$$

$$\boxed{m = 1}$$

Equation of tangent	**Equation of normal**
$y - y_0 = m\ (x - x_0)$	$y - y_0 = -\frac{1}{m}(x - x_0)$
$y - 1 = 1\ (x - 0)$	$y - 1 = \frac{-1}{1}(x - 0)$
$y - 1 = x$	$y - 1 = -1\ (x)$
$\boldsymbol{-x + y = 1}$	$\boldsymbol{x + y = 1}$

Example 6: Find the equation of tangent and normal for $y = \frac{3}{x}$ at (3, 1).

Solution:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Given: $y = \frac{3}{x}$ (i.e) $y = 3\left(\frac{1}{x}\right)$

$$\frac{dy}{dx} = -\frac{3}{x^2}$$

$$m = \left(\frac{dy}{dx}\right)_{3,\,1} = -\frac{3}{9}$$

$$\boxed{\boldsymbol{m = -\frac{1}{3}}}$$

Equation of tangent	**Equation of normal**
$y - y_0 = m\,(x - x_0)$	$(y - y_0) = -\frac{1}{m}\,(x - x_0)$
$y - 1 = -\frac{1}{3}\,(x - 3)$	$(y - 1) = \frac{-1}{-1/3}\,(x - 3)$
$3\,(y - 1) = -1\,(x - 3)$	$(y - 1) = \frac{3}{1}\,(x - 3)$
$3y - 3 = -x + 3$	$y - 1 = 3x - 9$
$\boldsymbol{x + 3y = 6}$	$-3x + y = -9 + 1$
	$\boldsymbol{-3x + y = -8}$

Example 7: (i) Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$

(ii) Find the equation of tangent and normal to the curve $x^3 + y^3 = 6xy$ at (3, 3)

(iii) At what point in the first quadrant is the tangent line horizontal.

Solution:

Given: $x^3 + y^3 = 6xy$ ... (1)

using Implicit differentiation

$$3x^2 + 3y^2 \frac{dy}{dx} = 6\left[x\frac{dy}{dx} + y\,(1) \right]$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

$$3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{3\,(2y - x^2)}{3\,(y^2 - 2x)}$$

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

At (3, 3)

$$m = \frac{2\,(3) - 3^2}{3^2 - 2\,(3)}$$

$$= \frac{6 - 9}{9 - 6} = \frac{-3}{3}$$

$$\boxed{m = -1}$$

Equation of tangent

$$y - y_0 = m\,(x - x_0)$$

$$y - 3 = -1\,(x - 3)$$

$$y - 3 = -x + 3$$

$$\boxed{x + y = 6}$$

Equation of normal

$$(y - y_0) = \frac{-1}{m}(x - x_0)$$

$$(y - 3) = \frac{-1}{-1}(x - 3)$$

$$y - 3 = 1\,(x - 3)$$

$$y - 3 = x - 3$$

$$-x + y = -3 + 3$$

$$\boxed{-x + y = 0}$$

For a horizontal tangent line

Slope $= 0$, (ie) Put, $\dfrac{dy}{dx} = 0$

$$\frac{2y - x^2}{y^2 - 2x} = 0$$

$$2y - x^2 = 0\ (y^2 - 2x)$$

$$2y - x^2 = 0$$

$$2y = x^2$$

$$\boxed{\boldsymbol{y = \frac{x^2}{2}}}$$

Sub. y in given equ (1) $x^3 + y^3 = 6xy$

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right)$$

$$x^3 + \frac{x^6}{8} = \frac{6x^3}{2}$$

$$\frac{x^6}{8} = 3x^3 - x^3$$

$$\frac{x^6}{8} = 2x^3$$

$$x^6 = 16x^3$$

$$(x^3)\ (x^3) = 16x^3$$

$$x^3 = 16$$

$$x = \sqrt[3]{16}$$

$$x = (2^4)^{1/3}$$

$$\boxed{\boldsymbol{x = 2^{4/3}}}$$

$$\Rightarrow y = \frac{x^2}{2} = \frac{(2^{4/3})^2}{2}$$

$$\boxed{y = \frac{2^{8/3}}{2}}$$

$\therefore$ Required point is

$$\left[2^{4/3}, \frac{2^{8/3}}{2}\right]$$

Example 8: Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

Solution:

Given: $f(x) = y = 2x^3 + 3x^2 - 12x + 1$

$$y' = \frac{dy}{dx} = 2\,(3x^2) + 3\,(2x) - 12\,(1) + 0$$

$$y' = 6x^2 + 6x - 12$$

To find horizontal tangents

Put $\frac{dy}{dx} = y' = 0$

$$6x^2 + 6x - 12 = 0$$

$$x_1 = 1$$

$$x^2 = -2$$

$$y = f(x) = 2x^3 + 3x^2 - 12x + 1$$

$$f(1) = 2\,(1)^3 + 3\,(1)^2 - 12\,(1) + 1$$

$$= -6$$

$$f(-2) = 2\,(-2)^3 + 3\,(-2)^2 - 12\,(-2) + 1$$

$$= 21$$

The points on which the tangent line is horizontal

$$\boxed{\begin{array}{c}(1, -6)\\(-2, 21)\end{array}}$$

Example 9: Does the curve $y = x^4 - 2x^2 + 3$ have any horizontal tangents? If so where?

Solution:

Tangents are horizontal $\Rightarrow \frac{dy}{dx} = 0$

Given:

$$y = x^4 - 2x^2 + 3$$

$$\frac{dy}{dx} = 4x^3 - 4x + 0$$

Put $\frac{dy}{dx} = 0$

$$4x^3 - 4x = 0$$

$$4x\,(x^2 - 1) = 0$$

$4x = 0,\ x + 1 = 0,\ x - 1 = 0$

$\Rightarrow$ $\boldsymbol{x = 0,\ x = -1,\ x = 1}$

at $x = 0,\ y = 0^4 - 2\,(0)^2 + 3$

$y = 3$

Point $(0,\ 3)$

At $x = 1, y = 1^4 - 2\,(1)^2 + 2$

$= 1 - 2 + 2$

$y = 1$

Point $(1,\ 1)$

at $x = -1,\ y = (-1)^4 - 2\,(-1)^2 + 2$

$= 1 - 2 + 2$

$y = 1$

Point $(-1, 1)$

$\therefore$ The curve will have horizontal tangents at

$$(0, 3), (1, 1), (-1, 1)$$

Example 10: Does the curve have any horizontal tangents? If so where?

$$f(x) = \frac{\sec x}{1 + \tan x}$$

Solution:

$$f'(x) = \frac{\sec x \tan x}{\sec^2 x} = \frac{\tan x}{\sec x}$$

To find horizontal tangents

Put $\quad f'(x) = 0$

$$\frac{\tan x}{\sec x} = 0 \Rightarrow \tan x = \sec x$$

By using quotient rule, we have

$$f'(x) = \frac{(1 + \tan x)\frac{d}{dx}(\sec x) - \sec x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= \frac{(1 + \tan x)(\sec x \tan x) - \sec x(\sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x (\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$$

$\because \sec^2 x - \tan^2 x = 1$

Since $\sec x$ is never 0,

$\Rightarrow \tan x - 1 = 0 \Rightarrow \tan x = 1$, and this occurs when $x = n\pi + \frac{\pi}{4}$, where n is an integer.

2.6 CRITICAL NUMBERS OR CRITICAL POINTS

A critical number of a function f is a number 'c' in the domain of 'f' such that $f'(c) = 0$ or does not exist.

WORKED EXAMPLES

Find the critical numbers or critical points for the following functions:

Example 1: $\boldsymbol{f(x) = 5x^2 + 4x}$

Solution:

Given: $f(x) = 5x^2 + 4x$

$$f'(x) = 10x + 4$$

For critical number put $f'(x) = 0$

$$10x + 4 = 0$$

$$10x = -4$$

$$x = \frac{-4}{10}$$

$$x = -\frac{2}{5}$$

Here the critical number is $\frac{-2}{5}$

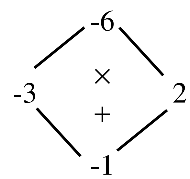
Example 2: $\boldsymbol{f(x) = 2x^3 - 3x^2 - 36x}$

Solution:

$$f'(x) = 6x^2 - 6x - 36$$

For critical number put $f'(x) = 0$

$6x^2 - 6x - 36 = 0$



Divide by 6

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$\Rightarrow x - 3 = 0 \qquad x + 2 = 0$

$x = 3 \qquad x = -2$

$$\left.\begin{array}{l} x_1 = 3 \\ x_2 = -2 \end{array}\right\} \text{critical numbers}$$

Example 3: $\boldsymbol{f(x) = 3x^4 - 4x^3 - 12x^2 + 1}$

Solution:

$f'(x) = 12x^3 - 12x^2 - 24x$

Put $\quad f'(x) = 0$

$12x^3 - 12x^2 - 24x + 0 = 0 \qquad \text{.... (1)}$

Solve using synthetic division method

$$\boxed{\begin{array}{l} \boldsymbol{x_1 = 2} \\ \boldsymbol{x_2 = 0} \\ \boldsymbol{x_3 = -1} \end{array}}$$

Example 4: $\boldsymbol{f(x) = x^2 e^{-3x}}$

Solution:

$$f'(x) = x^2(-3e^{-3x}) + e^{-3x}(2x)$$

$$= -3x^2 e^{-3x} + 2xe^{-3x}$$

$$f'(x) = e^{-3x} x(-3x+2)$$

Put $\quad f'(x) = 0$

$$xe^{-3x}(2-3x) = 0$$

$$e^{-3x} \neq 0$$

$$x(2-3x) = 0 \qquad \Big| \qquad \begin{aligned} 2-3x &= 0 \\ -3x &= -2 \end{aligned}$$

$$\boldsymbol{x = 0} \qquad \Big| \qquad \boldsymbol{x = \frac{2}{3}}$$

The critical numbers are $\left\{0, \dfrac{2}{3}\right\}$

Example 5: $\boldsymbol{f(x) = \dfrac{x-1}{x^2-x+1}}$

Solution:

$$f'(x) = \frac{(x^2-x+1)(1)-(x-1)(2x-1)}{(x^2-x+1)^2}$$

$$= \frac{x^2-x+1-(2x^2-x-2x+1)}{(x^2-x+1)^2}$$

$$f'(x) = \frac{x^2-x+1-2x^2+x+2x-1}{(x^2-x+1)^2}$$

$$f'(x) = \frac{-1x^2 + 2x}{(x^2 - x + 1)^2}$$

Put $\qquad f'(x) = 0$

$$\frac{-1x^2 + 2x}{(x^2 - x + 1)^2} = 0$$

$$-1x^2 + 2x = 0$$

$$x(-1x + 2) = 0$$

$$\boxed{\boldsymbol{x = 0}}$$

$$-1x + 2 = 0$$

$$-1x = -2$$

$$\boxed{\boldsymbol{x = 2}}$$

Critical numbers are

$$\boxed{\begin{array}{c}\boldsymbol{x = 0} \\ \boldsymbol{x = 2}\end{array}}$$

2.7 ABSOLUTE MAXIMUM & ABSOLUTE MINIMUM VALUE

Let c be a number in the domain D of a function f. Then $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

The closed interval method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).

2. Find the values of f at the end-points of the interval.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Local Maximum & Local Minimum Value:

The number $f(c)$ is a

- Local maximum value of f if $f(c) \geq f(x)$ when x is near c.
- Local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Result 1: The extreme value theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Result 2: Fermat's theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Result 3: Rolle's theorem

Let f be a function that satisfies the following three Assumptions:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

then there is a number c in (a, b) such that $f'(c) = 0$

Result 4: The mean value theorem

Let f be a function that satisfies the following hypothesis:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)

then there is a number c in (a, b) such that

1. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

2. $f(b) - f(a) = f'(c)\,(b - a)$

Increasing / Decreasing rest

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The first derivative test

Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example,) if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Concavity test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Inflection point:

A point p on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at p.

The second derivative test

Suppose f'' is continuous near c

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local minimum at c.

2.8 MAXIMA AND MINIMA OF ONE VARIABLE

2.8.1 Absolute maximum and Minimum value

Example 1: Find the absolute maximum and absolute minimum of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ in $[-2, 3]$

Solution:

Given: $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ ... (1)

Interval: $[-2, 3]$

$$f'(x) = 3(4x^3) - 4(3x^2) - 12(2x) + 0$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

To find critical points

Put $f'(x) = 0$

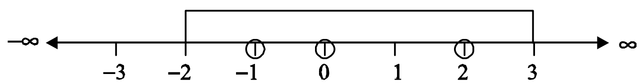
$$12x^3 - 12x^2 - 24x = 0$$

$$12x^3 - 12x^2 - 24x + 0 = 0 \quad \text{... (2)}$$

Solving equation (2) we get

Critical points

$$\boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 0 \\ x_3 = -1 \end{array}}$$



$f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

Put $x = -2, 0, 2, 3, -1$ in equation (1),

$$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1 = 0 + 33$$

$$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1 = -4$$

$$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 1 = 1$$

$$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1 = -31$$

$$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1 = 28$$

Absolute maximum value $= 33$

Absolute minimum value $= -31$

Example 2: Find the absolute maxima and absolute minima of $f(x) = x^3 - 3x + 1$ in [0, 3]

Solution:

Given: $f(x) = x^3 - 3x + 1$ ... (1)

Interval: [0, 3]

$$f'(x) = 3x^2 - 3$$

To find critical points

Put $f'(x) = 0$

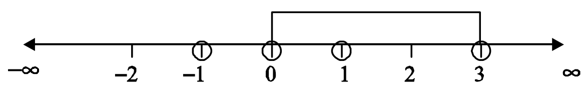
$$3x^2 - 3 = 0$$

$$3x^2 + 0x - 3 = 0 \quad \text{... (2)}$$

Critical points

$$\boxed{\begin{array}{l} \mathbf{x_1 = 1} \\ \mathbf{x_2 = -1} \end{array}}$$

$$f(x) = x^3 - 3x + 1 \qquad \text{.... (1)}$$



Put $x = 1$ & -1 in (1) we get

$$f(-1) = (-1)^3 - 3(-1) + 1 = 3$$

$$f(0) = (0)^3 - 3(0) + 1 = 1$$

$$f(1) = (1)^3 - 3(1) + 1 = -1$$

$$f(3) = (3)^3 - 3(3) + 1 = 19$$

$$\boxed{\begin{array}{l}\text{Absolute maximum value} = 17 \\ \text{Absolute minimum value} = -1\end{array}}$$

Example 3: Find the absolute maximum and absolute minimum of $f(x) = 3x^2 - 12x + 5$ in the interval of $0 \le x \le 3$ or [0, 3]

Solution:

Given: $f\,x) = 3x^2 - 12x + 5$ ... (1)

Interval: [0, 3]

$f'(x) = 3(2x) - 12(1) + 0$

$f'(x) = 6x - 12$

To find critical point

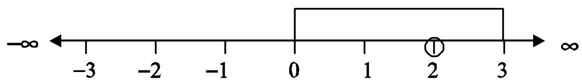
Put $f'(x) = 0$

$6x - 12 = 0$

$6x = 12$

$$x = \frac{12}{6}$$

$$\boxed{x = 2}$$



$$f(x) = 3x^2 - 12x + 5 \qquad \text{.... (1)}$$

Put $x = 0, x = 2, x = 3$ in (1), we get

$$f(0) = 3(0)^2 - 12(0) + 5 = 5$$

$$f(2) = 3(2)^2 - 12(2) + 5 = -7$$

$$f(3) = 3(3)^2 - 12(3) + 5 = -4$$

$$\boxed{\begin{array}{l}\text{Absolute maximum value } = 5 \\ \text{Absolute minimum value } = -7\end{array}}$$

Example 4: Find the exact maximum and exact minimum of values of $f(x) = x^3 - 3x^2 + 1$ in the interval $-\frac{1}{2} \le x \le 4$ by closed interval method.

Solution:

Given: $f(x) = x^3 - 3x^2 + 1$ ... (1)

Interval: $\left[-\frac{1}{2}, 4\right]$

$$\boxed{f'(x) = 3x^2 - 6x}$$

To find critical point

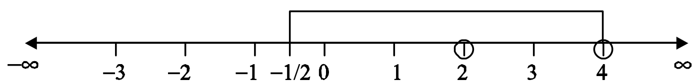
Put $f'(x) = 0$

$$3x^2 - 6x = 0$$

$$3x^2 - 6x + 0 = 0$$

$$\boxed{\begin{array}{l}x_1 = 0 \\ x_2 = 2\end{array}}$$

$$f(x) = x^3 - 3x^2 + 1 \quad \text{.... (1)}$$



Put $x = -\frac{1}{2}$, 0, 2 and 4 in (1) we get

$$f\left(\frac{-1}{2}\right) = \left(-\frac{1}{3}\right)^2 + 1 = \frac{1}{8} = 0.125$$

$$f(0) = (0)^3 - 3\,(0)^2 + 1 = 1$$

$$f(2) = 2^3 - 3\,(2)^2 + 1 = -3$$

$$f(4) = (4)^3 - 3\,(4)^2 + 1 = 17$$

Absolute maximum value $= 17$
Absolute minimum value $= -3$

Example 5: Find the exact maximum and exact minimum value of $f(x) = x - 2\sin x,\ 0 \le x \le 2\pi$.

Solution:

Given: $f(x) = x - 2\sin x$ ... (1)

Interval: $[0, 2\pi]$

$$f'(x) = 1 - 2\cos x$$

To find critical point

Put $f'(x) = 0$

$$1 - 2\cos x = 0$$

$$-2\cos x = -1$$

$$\boxed{\begin{array}{c} \cos x = \dfrac{1}{2} \\[2ex] \cos 60^\circ = \dfrac{1}{2} \\[2ex] \cos \dfrac{\pi}{3} = \dfrac{1}{2} \\[2ex] \sin 0 = 0 \end{array}}$$

degree $x = 60^\circ \quad x = 300^\circ$

$x = \dfrac{\pi}{3}$ & $x = \dfrac{5\pi}{3} \Rightarrow$ critical points

$$f(x) = x - 2 \sin x$$

$$f(0) = 0 - 2 \sin 0 = 0$$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2 \sin\left(\frac{\pi}{3}\right) = -0.684$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2 \sin\left(\frac{5\pi}{3}\right) = 6.968$$

$$f(2\pi) = 2\pi - 2 \sin(2\pi) = 6.283$$

$$\boxed{\begin{array}{l} \text{Absolute maximum value } = 6.968 \\ \text{Absolute minimum value } = -0.684 \end{array}}$$

Example 6: Find the absolute maximum and absolute minimum of $f(x) = 2 \cos x + \sin 2x$ in $\left[0, \dfrac{\pi}{2}\right]$ by closed interval method.

Solution:

Given: $f(x) = 2 \cos x + \sin 2x$

$$f'(x) = 2(-\sin x) + 2 \cos 2x$$

$$f'(x) = -2 \sin x + 2 \cos 2x$$

For critical points $f'(x) = 0$

$$-2 \sin x + 2 \cos 2x = 0$$

$$2 \cos 2x = 2 \sin x \qquad \cos\theta = \sin(90 - \theta)$$

$$\sin(90 - 2x) = \sin x$$

$$90 - 2x = x$$

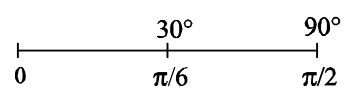
$$90 = x + 2x$$

$$90 = 3x$$

$$\frac{90}{3} = x$$

$$x = 30° = \frac{\pi}{6}$$

Critical point is $x = \dfrac{\pi}{6}$



$f(x) = 2 \cos x + \sin 2x$

$f(0) = 2 \cos 0 + \sin 2(0) = 2$

$$f\left(\frac{\pi}{6}\right) = 2 \cos \frac{\pi}{6} + \sin 2 \frac{\pi}{6} = 2.018$$

$$f\left(\frac{\pi}{2}\right) = 2 \cos \frac{\pi}{2} + \sin 2 \times \frac{\pi}{2} = 2.064$$

Absolute maximum value $= 2.054$
Absolute minimum value $= 2$

Example 7: Find the exact maxima and exact minima of $f(x) = \log(x^2 + x + 1)$ in $-1 \le x \le 1$ or $[-1, 1]$.

Solution:

Given: $f(x) = \log(x^2 + x + 1)$

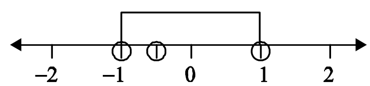
$$f'(x) = \frac{1}{x^2 + x + 1}(2x + 1 + 0)$$

$$f'(x) = \frac{2x + 1}{x^2 + x + 1}$$

To find critical points

Put $\quad f'(x) = 0$

$$\frac{2x + 1}{x^2 + x + 1} = 0$$



$$2x + 1 = (x^2 + x + 1) \times 0$$

$$2x + 1 = 0$$

$$2x = -1$$

$$\boxed{x = \frac{-1}{2}}$$

$$f(x) = \log(x^2 + x + 1)$$

$$f(-1) = \log[(-1)^2 + (-1) + 1] = \log 1 = 0$$

$$f\left(\frac{-1}{2}\right) = \log\left[\left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) + 1\right] = \log\left(\frac{3}{4}\right)$$

$$f(x) = \log(1^2 + 1 + 1) = \log 3$$

Absolute maximum value $= \log 3$
Absolute minimum value $= \log(3/4)$

Example 8: Using closed interval method find the maximum and minimum value of $f(x) = xe^{-x^2/8}$ in $[-1, 4]$.

Solution:

Given:

$$f(x) = xe^{-x^2/8}$$

$$u = x \qquad v = e^{-x^2/8}$$

$$u' = 1 \qquad v' = e^{-x^2/8}\left(\frac{-2x}{8}\right)$$

$$v' = -\frac{x}{4}e^{-x^2/8}$$

$$\text{Hint: } \because \frac{dy}{dx} = uv' + vu'$$

$$f'(x) = x\left(\frac{-x}{4}e^{-x^2/8}\right) + e^{-x^2/8} \text{ (1)}$$

$$f'(x) = e^{-x^2/8}\left(-\frac{x^2}{4} + 1\right)$$

To find critical points

$$\text{Put} \qquad f'(x) = 0$$

$$e^{-x^2/8}, \left(-\frac{x^2}{4} + 1\right) = 0$$

Since $e^{-x^2/8} \neq 0$

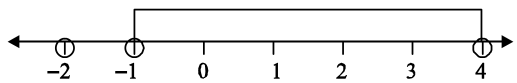
$$-\frac{x^2}{4} + 1 = 0$$

$$-\frac{x^2}{4} = -1$$

$$x^2 = 4$$

$$x = \pm 2$$

Critical points are $x = 2, x = -2$



2.8.2 Local Maximum and Minimum values of a funcion.

- The another name for local maximum and minimum is relative maximum and minimum
- Here the ends points are not given [interval not given]
- The interval in which the sign of $f'(x)$ is +ve then the nature of $f(x)$ is increasing.
- The interval in which the sign of $f'(x)$ is −ve then the nature of $f(x)$ is decreasing.
- The interval in which the sign of $f''(x)$ is +ve then the nature of $f(x$ is concave upward.
- The interval in which the sign of $f''(x)$ is −ve then the nature of $f(x$ is concave downward.

Note

1. The point at which nature of $f(x)$ is decreasing to increasing then $f(x)$ attains local minimum at the point.
2. The point at which nature of $f(x)$ is increasing to decreasing then $f(x)$ attains local maximum at the point.

WORKED EXAMPLES

Example 1: Find the local maxima and minima of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. Find the inflection points, and also find the intervals of concavity.

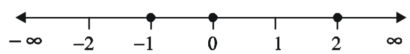
Solution:

Given: $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ ... (1)

Step 1: To find critical numbers

$$f'(x) = 12x^3 - 12x^2 - 24x + 0$$

To find the critical points



Put $\qquad f'(x) = 0$

$$12x^3 - 12x^2 - 24x + 0 = 0 \qquad \text{... (2)}$$

Solving equation (2) by synthetic division method, we get

$$\left.\begin{aligned} x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}\right\} \text{critical points}$$

Step 2: Increasing / Decreasing test:

Interval	**Sign of $f'(x)$ $f'(x) = 12x^3 - 12x^2 - 24x$**	**Nature**	**Conclusion**
$x < -1$ (or) $(-\infty, -1)$	$-$ ve	decreasing	Minimum point at $x = -1$ Local minimum value $= 0$
$-1 < x < 0$ (or) $(-1, 0)$	+ ve	Increasing	Maximum point at $x = 0$ Local maximum value $= 5$
$0 < x < 2$ (or) $(0, 2)$	$-$ ve	decreasing	Minimum point at $x = 2$ Local minimum value $= -27$

Step 3: Inflection points

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$f''(x) = 36x^2 - 24x - 24$$

To find hyper critical points

Put $\quad f''(x) = 0$

$$36x^2 - 24x - 24 = 0 \tag{3}$$

Solving (3) we get

$$\boxed{\begin{aligned} \boldsymbol{x_1} &= \mathbf{1.215} \\ \boldsymbol{x_2} &= \mathbf{-\,0.548} \end{aligned}}$$

Substitute x_1 and x_2 in (1)

To find Inflection points

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5 \qquad \ldots (1)$$

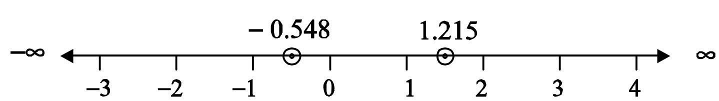
$$f(1.215) = -\,13.351$$

$$f(-\,0.548) = 2.325$$

$\therefore$ Inflection points

(1.215, – 13.351)

(– 0.548, 2.325)



Step 4: Concavity test

Interval	Sign of $f''(x)$ $= 36x^2 - 24x - 24$	Nature
$x < -0.548$	+ ve	concave upward
$-0.548 < x < 1.215$	− ve	concave downward
$x > 1.215$	+ ve	concave upward

Example 2: Find the intervals on which the function $f(x) = x^4 - 2x^2 + 3$ is increasing or decreasing. Also find the local maximum and minimum values of $f(x)$. Find the intervals of concavity and the point of inflection.

Solution:

Given: $f(x) = x^4 - 2x^2 + 3$

Step 1: To find critical points

$$f'(x) = 4x^3 - 4x + 0$$

To find critical points

Put $f'(x) = 0$

$$4x^3 + 0x^2 - 4x + 0 = 0 \qquad \text{... (1)}$$

solving (1) using synthetic division method

we get

$$\left.\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}\right\} \text{critical points}$$

Step 2: Increasing / decreasing test

Interval	Sign of $f'(x) = 4x^3 - 4x$	Nature	Conclusion
$x < -1$ (or) $(-\infty, -1)$	$-$ ve	decreasing	Minimum point at $x = -1$ Local minimum value $= 2$
$-1 < x < 0$ (or) $(-1, 0)$	$+$ ve	Increasing	Maximum point at at $x = 0$ Local maximum value $= 3$
$0 < x < 1$ (or) $(0, 1)$	$-$ ve	decreasing	Minimum point at $x = 1$ Local minimum value $= 2$
$x > 1$ (or) $(1, \infty)$	$+$ ve	Increasing	

Step 3: To find inflection points

$$f'(x) = 4x^3 - 4x$$

$$f''(x) = 12x^2 - 4$$

To find hyper critical points

Put $f''(x) = 0$

$$12x^2 + 0x - 4 = 0 \Rightarrow 12x^2 = 4$$

$$x^2 = \frac{4}{12} \Rightarrow x^2 = \frac{1}{3}$$

$$\Rightarrow x = \pm\sqrt{\frac{1}{3}}$$

$$x = +\sqrt{\frac{1}{3}},\ n = -\sqrt{\frac{1}{3}}$$

$$x = -0.577,\ x = 0.577$$

Put $x = -0.577$ in $f(x)$ we get 2.444

Put $x = 0.577$ in $f(x)$ we get 2.444

Hence inflection points are

$$\boxed{\begin{array}{c}\mathbf{(0.577, 2.444)}\\ \mathbf{(-0.577, 2.444)}\end{array}}$$

Step 4: Concavity test

Interval	**Sign of $f''(x)$ $= 12x^2 - 4$**	**Nature**
$x < -0.577$	+ ve	concave upward
$-0.577 < x < 0.577$	− ve	concave downward
$x > 0.577$	+ ve	concave upward

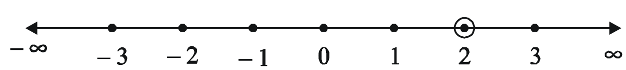
Example 3: Find the local maximum, local minimum value, intervals of concavity and the inflection points of a function $f(x) = x^4 - 4x^3$.

Solution:

Given: $f(x) = x^4 - 4x^3$

Step 1: To find critical points

$$f'(x) = 4x^3 - 12x^2$$



To find critical points

Put $\qquad f'(x) = 0$

$$4x^3 - 12x^2 + 0x + 0 = 0 \qquad .... (1)$$

Solving equation (1) by synthetic division method,

$$\left.\begin{array}{l} x_1 = 0 \\ x_2 = 3 \\ x_3 = 3 \end{array}\right\} \text{critical points}$$

Step 2: Increasing / decreasing test

Interval	Sign of $f'(x)$	Nature	Conclusion
$x < 0$	– ve	Decreasing	No conclusion
$0 < x < 3$	– ve	Decreasing	Local minimum at $x = 3$ Local minimum value $= -27$
$x > 3$	+ ve	Increasing	

Step 3: Inflection points

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x$$

To find hyper critical point

Put $f''(x) = 0$

$$12x^2 - 24x = 0$$

$$12x^2 - 24x + 0 = 0$$

$$x(12x - 24) = 0$$

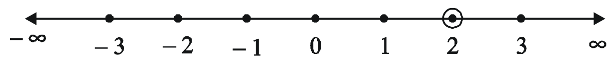
$$\Rightarrow x = 0, \; 12x - 24 = 0$$

$$12x = 24$$

$$x = \frac{24}{12}$$

$$x = 2$$

Put $x = 0$ & $x = 2$ in $f(x)$. we get inflection point (0, 0), (2, – 16)



Step 4: Concavity test

Interval	**Sign of $f''(x)$ $= 12x^2 - 24x$**	**Nature**
$x < 0$	+ ve	concave upward
$0 < x < 2$	– ve	concave downward
$x > 2$	+ ve	concave upward

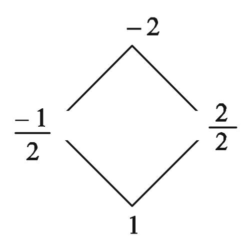
Example 4: Find the local maximum, local minimum value, intervals of concavity and the inflection points of a function $4x^3 + 3x^2 - 6x + 1$

Solution:

Given: $f(x) = 4x^3 + 3x^2 - 6x + 1$

$$f'(x) = 12x^2 + 6x - 6$$

To find critical point



Put $f'(x) = 0$

$$12x^2 + 6x - 6 = 0$$

Divide by 6

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0$$

$2x - 1 = 0$ | $x + 1 = 0$

$2x = 1$ | $x = -1$

Critical points

$$\boxed{\begin{array}{c} x = \frac{1}{2} \\ x = -1 \end{array}}$$

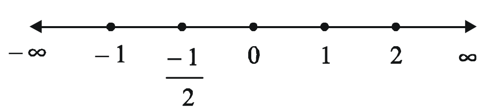
Increasing / decreasing test

Interval	**Sign of $f'(x) = 12x^2 + 6x - 6$**	**Nature**	**Conclusion**
$x < -1$ (or) $(-\infty, -1)$	+ ve	Increasing	Maximum point at $x = -1$ Local maximum value $= -6$
$-1 < x < 0.5$ (or) $(-1, 0.5)$	− ve	decreasing	Minimum point at $x = 0.5$ Local maximum value $= -0.75$
$x > 0.5$ (or) $(0.5, \infty)$	+ ve	Increasing	

Step 3: To find inflection points

$f'(x) = 12x^2 + 6x - 6$

$f''(x) = 24x + 6$



Put $f''(x) = 0 \Rightarrow 24x + 6 = 0$

$24x = -6$

$x = \frac{-6}{24}$

$$x = \frac{-1}{4}$$

Put $x = -\frac{1}{4}$ in $f(x)$

Inflection points

$$\left(\frac{-1}{4}, 2.625\right)$$

Step 4: Concavity test

Interval	**Sign of $f''(x)$ $= 24x + 6$**	**Nature**
$x < -1/4$	$-$ ve	concave downward
$x > -1/4$	$+$ ve	concave upward

Example 5: Find the local maximum, local minimum value, intervals of concavity and the inflection points of a function $f(x) = 2x^3 + 3x^2 - 36x$

Solution:

Given: $f(x) = 2x^3 + 3x^2 - 36x$

Step 1: To find critical points

$$f'(x) = 6x^2 + 6x - 36$$

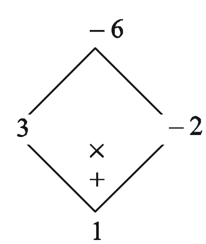
Put $f'(x) = 0$

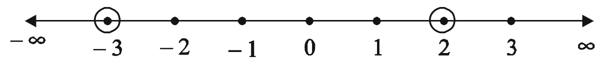
$$6x^2 + 6x - 36 = 0$$

Divide by 6

$$x^2 + x - 6 = 0$$

$$(x + 3)(x - 2) = 0$$





$x + 3 = 0 \quad x - 2 = 0$

Critical points

$$\boldsymbol{x = -3, \quad x = 2}$$

Step 2: Increasing / decreasing test

Interval	**Sign of $f'(x) = 6x^2 + 6x - 36$**	**Nature**	**Conclusion**
$x < -3$ (or) $(-\infty, -3)$	+ ve	Increasing	Maximum point at $x = -3$ Local maximum value $= 81$
$-3 < x < 2$ (or) $(-3, 2)$	− ve	Decreasing	Minimum point at $x = 2$ Local minimum value $= -44$
$x > +2$ (or) $(2, -\infty)$	+ ve	Increasing	

Step 3: To find inflection points

$f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6$

To find hyper critical point

Put $\quad f''(x) = 0$

$$12x + 6 = 0$$

$$x = \frac{-6}{12}$$

$$x = \frac{-1}{2}$$

Put $x = -\frac{1}{2}$ in $f(x)$

Inflection points

$$\left(-\frac{1}{2}, -17\right)$$

Step 4: Concavity test

Interval	Sign of $f''(x)$	Nature
$x < -1/2$	$-$ ve	concave downward
$x > -1/2$	$+$ ve	concave upward

Example 6: Find the local maximum, local minimum value, intervals of concavity and the inflection points of a function $f(x) = x^3 - 3x^2 - 12x$

Solution:

Given: $f(x) = x^3 - 3x^2 - 12x$

$$f'(x) = 3x^2 - 6x - 12$$

Divide by 3

$$x^2 - 2x - 4 = 0$$

To find critical point

Put $\quad f'(x) = 0$

$$3x^2 - 6x - 12 = 0 \quad \text{... (1)}$$

Solving equation (1) we get

$$\boxed{\begin{aligned} x_1 &= 3.236 \\ x_2 &= -1.236 \end{aligned}}$$

Step 2: Increasing / decreasing test

Interval	Sign of $f(x) = 3x^2 - 6x - 12$	Nature	Conclusion
$x < -1.236$ (or) $(-\infty, -1.236)$	+ ve	Increasing	Maximum point at $x = -1.236$ Local maximum value $= 8.3606$
$-1.236 < x < 3.236$ (or) $(-1.236, 3.236)$	– ve	decreasing	Minimum point at $= 3.236$ Local minimum value $= -36.360$
$x > 3.236$ (or) $(3.236, \infty)$	+ ve	Increasing	

Step 3: To find inflection point

$$f'(x) = 3x^2 - 6x - 12$$

$$f''(x) = 6x - 6$$

$$6x - 6 = 0 \Rightarrow 6x = 6$$

$$x = 1$$

Inflection points $(1, -14)$

Step 4: Concavity test

Interval	Sign of $f''(x)$	Nature
$x < 1$	– ve	concave downward
$x > 1$	+ ve	concave upward

2.8.3 Second derivative

If $y = f(x)$ then, the first order derivative is $\dfrac{dy}{dx} = y'$ the second order derivative is $\dfrac{d^2y}{dx^2} = y''$

Example 1: Find y'' if $y = 2x^3 + 3x^2 - 6x + 5$

Solution:

$$y' = 6x^2 + 6x - 6 \Rightarrow \frac{dy}{dx}$$

$$y'' = 12x + 6 \Rightarrow \frac{d^2y}{dx^2}$$

Example 2: Find $\dfrac{d^2y}{dx^2}$ if $y = e^x + \cos x + \sin x + x^5$

Solution:

$$y' = e^x - \sin x + \cos x + 5x^4$$

$$y'' = e^x - \cos x - \sin x + 20x^3$$

Example 3: Find y'' if $x^2 + y^2 = 1$

Solution:

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y} \qquad \text{.... (1)}$$

Differentiate (1) again w.r.t ‘x’

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$\frac{d^2 y}{dx^2} = -\left[\frac{y\,(1) - x\,\frac{dy}{dx}}{y^2}\right]$$

$$= -\left[\frac{y - x\left(\frac{-x}{y}\right)}{y^2}\right]$$

$$= -\left[\frac{y + \frac{x^2}{y}}{y^2}\right] = -\left[\frac{\left(\frac{y^2 + x^2}{y}\right)}{y^2}\right] = -\left[\frac{x^2 + y^2}{y^3}\right]$$

$$\frac{d^2 y}{dx^2} = -\left[\frac{x^2 + y^2}{y^3}\right]$$

Example 4: If $x^3 + y^3 = 1$ then find $\frac{d^2}{dx^2}$

Solution:

Given $x^3 + y^3 = 1$, By implicit differentiation, we have

$$3x^2 + 3y^2\,\frac{dy}{dx} = 0$$

$$3y^2\,\frac{dy}{dx} = -\,3x^2$$

$$\frac{dy}{dx} = \frac{-\,3x^2}{3y^2}$$

$$\frac{dy}{dx} = -\,\frac{x^2}{y^2}$$

$$\frac{d^2 y}{dx^2} = -\left[\frac{y^2\,(2x) - x^2\,(2y)\,\frac{dy}{dx}}{y^4}\right]$$

$$= -\left[\frac{2xy^2 - 2x^2\,y\left(-\frac{x^2}{y^2}\right)}{y^4}\right]$$

$$= -\left[\frac{2xy^2 - \dfrac{2x^4\,y}{y^2}}{y^4}\right]$$

$$= -\left[\frac{\dfrac{2xy^4 - 2x^4\,y}{y^2}}{y^4}\right]$$

$$= -\left[\frac{2xy\,(y^3 - x^3)}{y^6}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-\,2x\,(y^3 - x^3)}{y^5} \qquad \because\ x^3 + y^3 = 1 \quad \Rightarrow\ y^3 - x^3 = -\,1$$

$$= \frac{-\,2x\,(y^3 - x^3)}{y^5}$$

$$\boxed{\frac{d^2\,y}{dx^2} = \frac{2x}{y^5}}$$

Example 5: **Find** y'' **if** $y = x^2 \sin x$

Solution:

$$\frac{dy}{dx} = x^2\,(\cos x) + \sin x\,(2x)$$

$$y' = x^2 \cos x + 2x \sin x$$

$$\frac{d^2\,y}{dx^2} = x^2\,(-\sin x) + \cos x\,(2x) + 2\,[x \cos x + \sin x\,(1)]$$

$$= -\,x^2 \sin x + 2x \cos x + 2x \cos x + 2 \sin x$$

$$y'' = -\,x^2 \sin x + 4x \cos x + 2 \sin x$$

TWO MARKS QUESTIONS AND ANSWERS

1. Find $\lim\limits_{x \to 5} (2x^2 - 3x + 4)$

Solution:

$$\lim_{x \to 5} (2x^2 - 3x + 4) = 2\,(5)^2 - 3\,(5) + 4$$

$$= 2\,(25) - 3\,(5) + 4$$

$$= 50 - 15 + 4$$

$$= 39$$

$$\therefore \lim_{x \to 5} (2x^2 - 3x + 4) = 39$$

2. Find $\lim\limits_{u \to -2} \sqrt{u^4 + 3u + 6}$

Solution:

$$\lim_{u \to -2} \sqrt{u^4 + 3u + 6} = \sqrt{(-2)^4 + 3\,(-2) + 6}$$

$$= \sqrt{16 - 6 + 6}$$

$$= \sqrt{16} = 4$$

$$\therefore \lim_{u \to -2} \sqrt{u^4 + 3u + 6} = 4$$

3. Determine the infinite limit $\lim\limits_{x \to -3} \dfrac{x+2}{x+3}$

Solution:

$$\lim_{x \to -3^-} \frac{x+2}{x+3} = \frac{-3+2}{-3+3} = \frac{-1}{0}$$

$$= \infty$$

$$\therefore \lim_{x \to -3^-} \frac{x+2}{x+3} = \infty$$

4. Evaluate $\lim\limits_{x \to 0} \left(\dfrac{\sin 5x}{\sin 3x}\right) = ?$

Solution:

$$\lim_{x \to 0} \left(\frac{\sin 5x}{\sin 3x}\right) = \frac{\sin 5\,(0)}{\sin 3\,(0)}$$

$$= \frac{\sin 0}{\sin 0}$$

$$= \frac{0}{0} \qquad \text{(Indeterminate Form)}$$

Apply L'Hospital Rule

$$\lim_{x \to 0} \frac{5 \cos 5x}{3 \cos 3x} = \frac{5 \cos 5\,(0)}{3 \cos 3\,(0)}$$

$$= \frac{5}{3}$$

$$\therefore \lim_{x \to 0} \frac{\sin 5x}{\sin 3x} = \frac{5}{3}$$

5. Evaluate $\lim\limits_{t \to 1} \left(\dfrac{t^4 - 1}{t^3 - 1}\right)$

Solution:

$$\lim_{t \to 1} \left(\frac{t^4 - 1}{t^3 - 1}\right) = \frac{(1)^4 - 1}{(1)^3 - 1} = \frac{0}{0} \qquad \text{(Indeterminate Form)}$$

Apply L'Hospital Rule

$$\lim_{t \to 1} \frac{4t^3}{3t^2} = \frac{4\,(1)^3}{3\,(1)^2} = \frac{4}{3}$$

$$\therefore \lim_{t \to 1} \frac{t^4 - 1}{t^3 - 1} = \frac{4}{3}$$

6. Find $\frac{dy}{dx}$ if $xy = c^2$.

Solution:

$$xy = c^2$$

$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

$$\frac{dy}{dx} = \frac{-xy}{x^2} \qquad (\because c^2 = xy)$$

$$\boxed{\therefore \frac{dy}{dx} = -\frac{y}{x}}$$

7. Find $\frac{dy}{dx}$ if $x^2 + y^2 = a^2$

Solution:

$$x^2 + y^2 = a^2$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{-x}{y}}$$

8. State sandwich theorem (squeeze theorem).

Solution:

If $f(x) \le g(x) \le h(x)$ and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \text{ then}$$

$$\lim_{x \to a} g(x) = L$$

9. State Rolle's theorem.

Solution:

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

(iii) If $f(a) = f(b)$ then there exists a point c such that $f'(c) = 0$

10. State (Lagranges) Mean Value Theorem.

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

(iii) Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

11. State Intermediate Value Theorem.

Solution:

(i) $f(x)$ is continuous in $[a, b]$

(ii) N be a number between $f(a)$ and $f(b)$

(iii) There exists a number c in (a, b) such that $f(c) = N$.

12. Check whether $\lim\limits_{x \to -3} \dfrac{3x + 9}{|x + 3|}$ exists.

Solution:

$$\lim_{x \to -3} \frac{3x + 9}{|x + 3|} = \frac{3(-3) + 9}{|-3 + 3|}$$

$$= \frac{-9+9}{0}$$

$$= \frac{0}{0} \qquad \text{(Indeterminate form)}$$

$$\lim_{x \to -3} \left(\frac{3}{1}\right) = 3 \text{ (exists)}$$

13. Given that $\lim\limits_{x \to 2} f(x) = 4$ and $\lim\limits_{x \to 2} g(x) = -2$. Find the limits that exists for $\lim\limits_{x \to 2} \dfrac{3f(x)}{g(x)}$.

Solution:

$$\lim_{x \to 2} \frac{3f(x)}{g(x)} = \frac{3 \lim\limits_{x \to 2} f(x)}{\lim\limits_{x \to 2} g(x)}$$

$$= \frac{3(4)}{-2}$$

$$= \frac{12}{-2}$$

$$= -6$$

$$\therefore \lim_{x \to 2} \frac{3f(x)}{g(x)} = -6$$

14. Find $\lim\limits_{\theta \to 0} \left(\dfrac{\tan\theta}{\theta}\right)$

Solution:

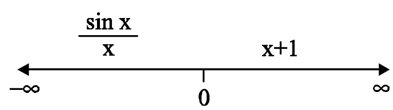
$$\left[\frac{\tan 0}{0}\right] = \frac{0}{0}$$

$$= \lim_{\theta \to 0} \frac{\sec^2}{1}$$

$$= \frac{\sec^2 0}{1} = \frac{1}{1} = 1$$

15. $f(x) = \begin{cases} \dfrac{\sin x}{x}\,; & x < 0 \\ x + 1\,; & x > 0 \end{cases}$ **Test the continuity.**

Solution:



Left limit

$$\lim_{x \to 0^-} \frac{\sin x}{x} = \frac{\sin 0}{0} = 1 \left[\frac{\sin \theta}{\theta} = 1 \right]$$

Right limit:

$$\lim_{x \to 0^+} x + 1 = 0 + 1 = 1$$

Here Left Limit = Right Limit

$\therefore$ The function is continuous at $x = 0$

16. Does the curve $y = x^4 - 2x^2 + 4$ have any horizontal tangents? If so where?

Solution:

For horizontal tangents $\dfrac{dy}{dx} = 0$

Given: $y = x^4 - 2x^2 + 2$

$$\frac{dy}{dx} = 4x^3 - 4x$$

$$4x^3 - 4x = 0$$

$$4x^3 = 4x$$

$$(x^3 - x) = 0$$

$$x\,(x^2 - 1) = 0$$

$$x = 0 \quad \left| \quad \begin{aligned} x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned} \right.$$

$\therefore$ The curve has horizontal tangents at $x = 0, 1, -1$

17. Find $\frac{dy}{dx}$ if $y = \sqrt{\sqrt{\sin x} + \sqrt{\sin x} + \sqrt{\sin x}}$

Solution:

$$y = \sqrt{\sin x + y}$$

$$y^2 = \sin x + y$$

$$2y \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx}(2y - 1) = \cos x$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{\cos x}{2y - 1}}$$

18. Find $\frac{dy}{dx}$ from $y = x^{x^{x \ldots \infty}}$

Solution:

$$y = x^y$$

$$\log y = \log x^y$$

$$\log y = y \log x$$

$$\frac{1}{y} \frac{dy}{dx} = y\left(\frac{1}{x}\right) + \log x \frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - \log x\,\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\boxed{\therefore\ \frac{dy}{dx} = \left(\frac{\frac{y}{x}}{\frac{1}{y} - \log x}\right)}$$

19. Find the critical points of the function $f(x) = 5x^3 - 6x$.

Solution:

$$f(x) = 5x^3 - 6x \qquad \text{... (1)}$$

$$f'(x) = 15x^2 - 6$$

To find critical points,

Put $\quad f'(x) = 0$

$$15x^2 - 6 = 0$$

$$15x^2 + 0x - 6 = 0 \qquad \text{... (2)}$$

Solving (2) we get

$$\boxed{\begin{aligned} \therefore\ x_1 &= 0.632 \\ x_2 &= -0.632 \end{aligned}}$$

20. If $f(x) = \dfrac{x^3}{3x-2}$, find $f'(x)$.

Solution:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$u = x^3 \qquad v = 3x - 2$

$du = 3x^2 \qquad dv = 3$

$\qquad\qquad v^2 = (3x-2)^2$

$$f'(x) = \frac{(3x-2)(3x^2) - (x^3)(3)}{(3x-2)^2}$$

$$= \frac{9x^3 - 6x^2 - 3x^3}{(3x-2)^2}$$

$$\boxed{f'(x) = \frac{6x^3 - 6x^2}{(3x-2)^2}}$$

21. If $f(x) = xe^x$, then find $f'(x)$ and $f''(x)$.

Solution:

$$u = x \qquad v = e^x$$

$$u' = 1 \qquad v' = e^x$$

$$f(x) = xe^x$$

$$\frac{d}{dx}(uv) = vu' + uv'$$

$$f'(x) = e^x(1) + x(e^x)$$

$$\boxed{f'(x) = e^x + xe^x}$$

$$f''(x) = e^x + [e^x(1) + xe^x]$$

$$\boxed{f''(x) = 2e^x + xe^x}$$

22. If $y = \dfrac{x \sin x}{1+x}$ then find y.

Solution:

$$\frac{dy}{dx} = y' = \frac{[(1+x)[x\cos x + \sin x] - x\sin x(1)]}{(1+x)^2}$$

$$y' = \frac{(1+x)x\cos x + (1+x)\sin x - x\sin x}{(1+x)^2}$$

$$= \frac{x\cos x + x^2 \cos x + \sin x + x \sin x - x \sin x}{(1+x)^2}$$

$$\boxed{y' = \frac{x\cos x + x^2 \cos x + \sin x}{(1+x)^2}}$$

23. If $y = (x^4 + 3x^2 - 2)^5$ then find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = 5\,(x^4 + 3x^2 - 2)^4\,(4x^3 + 6x)$$

24. If $y = \sin(\tan 2x)$ then find y'.

Solution:

$$\frac{dy}{dx} = 2\cos(\tan 2x)\sec^2(2x)$$

25. Find the critical point of $f(x) = 2x^3 - 3x^2 - 36x$

Solution:

$$f(x) = 2x^3 - 3x^2 - 36x \qquad \text{... (1)}$$

$$f'(x) = 6x^2 - 6x - 36$$

Put $f'(x) = 0$

$$6x^2 - 6x - 36 = 0 \qquad \text{... (2)}$$

Solving (2) we get

$$\boxed{\therefore x_1 = -2}$$

$$\boxed{x_2 = 3}$$

26. Find y' for $\cos(xy) = 1 + \sin y$.

Solution:

$$\cos(xy) = 1 + \sin y$$

$$-\sin(xy)\left[x\frac{dy}{dx} + y\,(1)\right] = 0 + \cos\frac{dy}{dx}$$

$$-x \sin(xy)\frac{dy}{dx} - y \sin(xy) = \cos y \frac{dy}{dx}$$

$$-x \sin(xy)\frac{dy}{dx} - \cos y \frac{dy}{dx} = y \sin(xy)$$

$$\frac{dy}{dx}(-x \sin(xy) - \cos y) = y \sin(xy)$$

$$\boxed{\frac{dy}{dx} = \frac{y \sin(xy)}{[-x \sin(xy) - \cos y]}}$$

27. Find y' if $y = x^x$

Solution:

Given $y = x^x$

Taking log on both sides

$$\log y = \log x^x$$

$$\log y = x \log x$$

$$\frac{1}{y}\frac{dy}{dx} = \left[x \frac{1}{x} + \log x \right]$$

$$\frac{dy}{dx} = y\,[1 + \log x]$$

$$\boxed{\frac{dy}{dx} = x^x\,[1 + \log x]}$$

28. Find the critical points of $f(x) = \log(x^2 + x + 1)$.

Solution:

$$f(x) = \log(x^2 + x + 1)$$

$$f'(x) = \frac{1}{x^2 + x + 1}(2x + 1)$$

$$f'(x) = \frac{2x + 1}{x^2 + x + 1}$$

Put $\qquad f'(x) = 0$

$$\frac{2x+1}{x^2+x+1} = 0$$

$$2x + 1 = 0$$

$$2x = -1$$

$$\boxed{\boldsymbol{x = \frac{-1}{2}}}$$

29. Prove that $\lim\limits_{x \to 0} |x| = 0$

Solution:

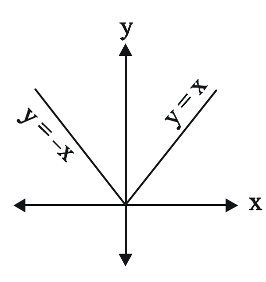
$$f(x) = |x| = \begin{cases} x & \text{if} \quad +n \geq 0 \\ -x & \text{if} \quad +n < 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 \qquad \text{... (1)}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} x = 0 \qquad \text{... (2)}$$

from (1) and (2)

$$\lim_{x \to 0^-} f(x) = 0 \lim_{x \to 0^+} f(x)$$



30. Prove that $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

Solution:

Let $$f(x) = \frac{|x|}{x}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{x}{x}\right) = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{-x}{x}\right) = -1$$

$$\therefore \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$$

$\therefore \lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

EXERCISE

Domain

1. Find the domain of the following functions.

   (a) $f(x) = \sqrt{2x-1}$ (b) $f(x) = \sqrt{3-x} - \sqrt{2+x}$

   (c) $f(x) = \dfrac{1}{\sqrt{x^2 - 5x}}$ (d) $f(x) = \sqrt{2 - \sqrt{x}}$

Sketch the Graph

2. Find the domain and sketch the graph of the function.

   (a) $f(x) = 2 - 0.4x$ (b) $f(t) = 2t + r^2$

   (c) $g(x) = \sqrt{x-5}$ (d) $g(x) = \dfrac{3x + |x|}{x}$

   (e) $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

   (f) $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > 2 \end{cases}$

Odd and Even Functions

3. Determine whether each of the following function is even, odd or neither even nor odd.

   (a) $f(x) = x^5 + x$ (b) $f(x) = 1 - x^4$

   (c) $f(x) = x^2 + 1$ (d) $g(x) = x^3 + x$

Infinite Limit

4. Determine the infinite limit for the following:

   (a) $\lim\limits_{x \to 1} \dfrac{2-x}{(x-1)^2}$ (b) $\lim\limits_{x \to 3^+} \log(x^2 - 9)$

(c) $\lim\limits_{x \to 2^+} \dfrac{x^2 - 2x - 8}{x^2 - 5x + 6}$ (d) $\lim\limits_{x \to 1^-} \dfrac{1}{x^3 - 1}$

(e) $\lim\limits_{x \to 1^+} \dfrac{1}{x^3 - 1}$

Continuity and Discontinuity

1. Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

(a) $f(x) = \dfrac{2x - 3x^2}{1 + x^3}, a = 1$ (b) $f(x) = \dfrac{2x + 3}{x - 2}, (2, \infty)$

(c) $f(x) = 2\sqrt{3 - x}, (-\infty, 3]$ (d) $f(x) = \dfrac{x^2 + 5x}{2x + 1}, a = 2$

2. Explain why the function is discontinuous at the given number a.

(a) $f(x) = \begin{cases} \dfrac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}, a = -2$

(b) $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}, a = 1$

(c) $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}, a = -2$

(d) $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}, a = 3$

3. Use continuity to evaluate the limit.

(a) $\lim\limits_{x \to 4} \dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ (b) $\lim\limits_{x \to 1} e^{x^2 - x}$

Differentiation

1. Differentiate the following function

(a) $f(x) = \dfrac{3x-1}{2x+1}$ (b) $f(x) = (x-\sqrt{x})(x+\sqrt{x})$

(c) $f(x) = \dfrac{x^2}{1-x^2}$ (d) $y = \dfrac{t^2+2}{t^4-3t^2+1}$

(e) $f(x) = \dfrac{2y}{2+\sqrt{x}}$

2. Find $f'(x)$ and $f''(x)$ for the following functions.

(a) $F(x) = x^4 e^x$ (b) $f(x) = \dfrac{x^2}{1+2x}$

(c) $f(x) = \dfrac{(x^2-1)}{(x^2+1)}$

3. Differentiate the following functions.

(a) $f(x) = 3x^2 - 2\cos x$ (b) $f(x) = \sin x + \dfrac{1}{2}\cos x$

(c) $f(x) = \operatorname{cosec} x + e^x \cos x$ (d) $y(x) = \dfrac{x}{2-\tan x}$

(e) $y(t) = \dfrac{t \sin t}{1+t}$ (f) $f(x) = xe^x \operatorname{cosec} x$

4. If $f(x) = \sec x - x$, then find $f'(x)$.

5. If $f(x) = x\sin x$, then find $f'(x)$ and $f''(x)$.

6. Use the quotient rule to differentiate the function

$f(x) = \dfrac{\tan x - 1}{\sec x}$

7. Find the limit for the following

(a) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{3}$ (b) $\lim\limits_{x \to 0} \dfrac{\tan 6t}{\sin 2t}$

Derivatives

1. Differentiate the given functions.

(a) $g(x) = x^2(1 - 2x)$ (b) $y = x^{-2/5}$

(c) $f(s) = -\dfrac{12}{s^5}$ (d) $f(x) = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

(e) $f(x) = \sqrt[3]{x} + 4\sqrt{x^5}$

2. Find the first and second derivatives of the functions.

(a) $f(x) = 10x^{10} + 5x^5 - x$ (b) $s(t) = t^3 - 3t$

(c) $f(t) = t^4 - 2t^3 + t^2 - t$

3. Find the derivative of the given functions.

(a) $y = \tan^{-1}\sqrt{x}$ (b) $G(x) = \sqrt{1 - x^2}\cos x$

(c) $y = \cos^{-1}\left[\dfrac{b + a\cos x}{a + b\cos x}\right]$ (d) $y = \cosh^{-1}\sqrt{x}$

Critical numbers

1. Find the critical numbers of the function.

(a) $f(x) = 5x^2 + 4x$ (b) $f(x) = 2x^3 - 3x^2 - 36x$

(c) $g(t) = t^4 + t^3 + t^2 + 1$ (d) $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

(e) $h(t) = t^{3/4} - 2t^{1/4}$ (f) $f(x) = x^{4/5}(x - 4)^2$

Absolute maximum and minimum values

1. Find the absolute maximum and absolute minimum values of f on the given interval.

   (a) $f(x) = 3x^2 - 12x + 5$, $[0, 3]$

   (b) $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

   (c) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$, $[-2, 3]$

   (d) $f(x) = x + \dfrac{1}{x}$, $[0, 2, 4]$

   (e) $f(t) = t\sqrt{4 - t^2}$, $[-1, 2]$

   (f) $f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$

   (g) $f(x) = xe^{-x^2/8}$, $[-1, 4]$

   (h) $f(x) = \log[x^2 + x + 1)$, $[-1, 1]$

Local maximum and minimum values

1. For the following functions.
   (i) Find the interval on which it is increasing and decreasing
   (ii) Find the local maximum and minimum values.
   (iii) Determine the intervals of concavity
   (iv) Find the inflection points.

   (a) $f(x) = x^2 - 4x + 4$ (b) $f(x) = \dfrac{x}{1 + x^2}$

   (c) $f(x) = 2x^3 + 3x^2 - 12x$ (d) $f(x) = \dfrac{x}{x - 5}$

   (e) $f(x) = 3x^2 - 4x^3 - 12x^2 + 1$

   (f) $f(x) = 4x^3 - 6x^2 - 72x + 30$

   (g) $f(x) = -2x^3 - 9x^2 - 12x + 1$

   (h) $f(x) = 10 - 6x - 2x^2$

UNIT - 3

MULTIVARIABLE CALCULUS

3.1 INTRODUCTION

There are quantities which depend upon two or more variables. If x, y, z are three variables and if there exists a relation between them such that the value of z depends upon the values of x and y, then z is called a function of two variables x and y and this is denoted by $z = f(x, y)$. Here z is a dependent variable and x and y are the independent variables.

For example the area of a rectangle is determined when its length x and breadth y are given. Thus the area of a rectangle is a function of two variables, length and breadth.

3.2 Continuity

A function $f(x, y)$ defined in a region is said to be continuous at the point (a, b) if $\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y)$ exists and is equal to $f(a, b)$ along whatever path $x \to a$, $y \to b$.

If a function is continuous at all points of a region, then it is said to be continuous in that region. If a function is not continuous at a point, then it is said to be discontinuous at that point.

Let us assume that the functions considered are continuous and their partial differential coefficients exist.

3.2.1 Partial Derivatives of First Order

Let $Z = f(x, y)$ be a function of two independent variables x and y.

If we keep y as constant and vary x alone, then z is a function of x only. The derivative of z, with respect to x, treating y as constant, is called the partial derivative of z w.r.t. x and it is denoted by $\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$ or Z_x or f_x.

$$\text{Thus } \frac{\partial z}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

In a similar way, the derivative of z w.r.to y, keeping x as constant, is called the partial derivative of z w.r.to y and is denoted by $\dfrac{\partial z}{\partial y}$ or $\dfrac{\partial f}{\partial y}$ or z_y or f_y.

$$\text{Thus } \frac{\partial z}{\partial y} = \underset{\Delta y \to 0}{Lt} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are called first order partial derivatives of z.

3.2.2 Second Order Partial Derivatives

In general $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are also functions of x and y and hence these can be differentiated further partially w.r.to x and y.

$$\text{Thus } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

In general $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ and it can easily be verified in all ordinary cases.

3.2.3 Homogeneous Functions

If every term of the expression $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$ is of the n^{th} degree, then it is called a homogeneous function of degree n as this expression can also be rewritten as $x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \dots + a_n \left(\frac{y}{x}\right)^n \right] = x^n \phi\left(\frac{y}{x}\right)$.

Hence any function $f(x, y)$ which can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ is called a homogeneous function of degree n in x and y.

In a similar way if $f(x, y)$ can be expressed in the form $y^n \phi\left(\frac{x}{y}\right)$ then it is called a homogeneous function of degree n in x and y.

3.3 Euler's Theorem on Homogeneous Functions

Statement

If u is a homogeneous function of degree n in x and y, then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u$$

Proof: Since u is a homogeneous function of degree n in x and y, it can be expressed as $u = x^n f\left(\frac{y}{x}\right)$.

$$\frac{\partial u}{\partial x} = n x^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$x \frac{\partial u}{\partial x} = n x^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) \qquad \dots\dots (1)$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$y\,\frac{\partial u}{\partial y} = x^{n-1}\, y f'\left(\frac{y}{x}\right) \qquad \ldots\ldots (2)$$

Adding (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,x^{n} f\left(\frac{y}{x}\right) - x^{n-1}\, y f'\left(\frac{y}{x}\right) + x^{n-1}\, y f'\left(\frac{y}{x}\right)$$

$$= n\,x^{n}\, f\left(\frac{y}{x}\right)$$

$$= n\,u$$

$$\boxed{\therefore\ \ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u}$$

3.3.1 Euler's Extension Theorem

If u is a homogeneous function of degree n, then

$$x^2\,\frac{\partial^2 u}{\partial x^2} + 2xy\,\frac{\partial^2 u}{\partial x\,\partial y} + y^2\,\frac{\partial^2 u}{\partial y^2} = n\,(n-1)\,u$$

3.3.2 Problems on Partial differentiation

Example 1: $\boldsymbol{f(x, y) = x^3 + 3x^2 + 4y^2 + 4xy + 8yx + 12y + 13x + 100 = 0}$ **find** $\boldsymbol{f_x, f_y, f_{xx}, f_{yy}, f_{xy}}$.

Solution

$$f = x^3 + 3x^2 + 4y^2 + 4xy + 8yx + 12y + 13x + 100$$

$$f_x = 3x^2 + 6x + 0 + 4y + 8y + 0 + 13 + 0$$

$$\boxed{f_x = 3x^2 + 6x + 4y + 8y + 13}$$

$$f_y = 0 + 0 + 8y + 4x + 8x + 12 + 0 + 0$$

$$\boxed{f_y = 8y + 4x + 8x + 12}$$

$$f_{xx} = 6x + 6$$

$$f_{yy} = 8$$

$$f_{xy} = 4 + 8 = 12$$

$$f_{yx} = 4 + 8 = 12$$

Example 2: Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x\, \partial y}, \frac{\partial^2 f}{\partial y\, \partial x}$ the following problems (i) $f(x, y) = x^3 + y^3 + 6xy + 12$

Solution:

$$f = x^3 + y^3 + 6xy + 12$$

$$\frac{\partial f}{\partial x} = 3x^2 + 0 + 6y + 0$$

$$f_x = 3x^2 + 6y$$

$$f_y = 0 + 3y^2 + 6x + 0$$

$$f_y = 3y^2 + 6x$$

$$f_{xx} = 6x$$

$$f_{yy} = 6y$$

$$f_{xy} = 0 + 6$$

$$f_{yx} = 0 + 6$$

(ii) $f(x, y) = x^3 + 72xy + y^2 + 18x$

Solution:

$$f = x^3 + 72xy + y^2 + 18x$$

$$f_x = 3x^2 + 72y + 0 + 18$$

$$= 3x^2 + 72y + 18$$

$$f_y = 0 + 72x + 2y + 0$$

$$= 72x + 2y$$

$$f_{xx} = 6x$$

$$f_{yy} = 2$$

$$f_{xy} = 72$$

$$f_{yx} = 72$$

(iii) $f(x, y) = x^4 - 4x + y^4 - 3x^2y - 3xy^2 + y^3$

Solution:

$$f = x^4 - 4x + y^4 - 3x^2 y - 3xy^2 + y^3$$

$$f_x = 4x^3 - 4 + 0 - 6xy - 3y^2 + 0$$

$$= 4x^3 - 4 - 6xy - 3y^2$$

$$f_y = 4y^3 - 3x^2 - 6xy + 3y^2$$

$$f_{xx} = 12x^2 - 6y$$

$$f_{xy} = - 6x - 6y$$

$$f_{yy} = 12y^2 - 6x + 6y$$

(iv) $f(x, y) = x^2 + 2xy + y^2$

Solution:

$$f = x^2 + 2xy + y^2$$

$$f_x = 2x + 2y$$

$$f_y = 2x + 2y$$

$$f_{xx} = 2$$

$$f_{xy} = 2$$

$$f_{yy} = 2$$

(v) $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$

Solution:

$$f = x^3 - 3x^2 y + 3xy^2 - y^3$$

$$f_x = 3x^2 - 6xy + 3y^2$$

$$f_y = -3x^2 + 6xy - 3y^2$$

$$f_{xx} = 6x - 6y$$

$$f_{xy} = -6x + 6y$$

$$f_{yy} = 6x - 6y$$

Euler's Theorem for Homogeneous Functions
[For two variables *x* and *y*]

- Let u be a homogeneous function of degree n in x and y.

$$\text{Then } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

- Let f be a homogeneous function of degree n in x and y

$$\text{Then } x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

Euler's theorem [For 3 variables x, y and z]

- Let u be a homogeneous function of degree n in x, y and z

$$\text{Then } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = n\, u$$

Euler's extension theorem

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n\,(n-1)\, u \quad \text{(or)}$$

$$x^2\, u_{xx} + 2xy\, u_{xy} + y^2\, u_{yy} = n\,(n-1)\, u$$

Find the degree of the following functions.

1. $f(x, y) = \dfrac{x^2 + y^2}{x + y}$

degree (n) = Numerator degree − Denominator degree

$= 2 - 1$

$n = 1$

2. $f(x, y) = \dfrac{x^3 + y^3}{3x + 4y}$

degree (n) = Numerator degree − Denominator degree

$= 3 - 1$

$n = 2$

3. $u = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$

degree (n) = Numerator degree − Denominator degree

$= 1 - \dfrac{1}{2}$

$n = \dfrac{1}{2}$

4. $u = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$

degree (n) = Numerator degree − Denominator degree

$= 1 - 1$

$n = 0$

3.3.3 Problems on Euler's theorem

Example 1: If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$

Solution:

Given

$$u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $f = \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function

$$\therefore \text{ degree } (n) = 1 - \frac{1}{2}$$

$$n = \frac{1}{2}$$

By Euler's Theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$(-\sin u)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{\cos u}{-\sin u}\right]$$

$$\boxed{\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u}}$$

$\therefore$ Hence proved

Example 2: If $\log u = \dfrac{x^3 + y^3}{3x + 4y}$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \log u$$

Solution:

Given

$$\log u = \frac{x^3 + y^3}{3x + 4y}$$

Let $f = \log u = \dfrac{x^3 + y^3}{3x + 2y}$ is homogeneous function

$\therefore$ degree $(n) = 3 - 1$

$$n = 2$$

By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow\ x\frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) = 2\log u$$

$$x\left(\frac{1}{u}\right)\frac{\partial u}{\partial x} + y\left(\frac{1}{u}\right)\frac{\partial u}{\partial y} = 2\log u$$

$$\frac{1}{u}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\log u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\log u}$$

$\therefore$ Hence proved

Example 3: **If** $u = \tan^{-1}\left[\dfrac{x^3+y^3}{x-y}\right]$**. Then prove that** $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$**.**

Solution:

Given

$$u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$$

$$\tan u = \frac{x^3+y^3}{x-y}$$

$$\text{Let } f = \tan u = \frac{x^3+y^3}{x-y}$$

$$\therefore \text{ degree } (n) = 3 - 1$$

$$n = 2$$

By Euler's Theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x(\sec^2 u)\frac{\partial u}{\partial x} + y(\sec^2 u)\frac{\partial u}{\partial y} = 2\tan u$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$

$$\frac{2\,\dfrac{\sin u}{\cos u}}{\dfrac{1}{\cos^2 u}} = 2\,\frac{\sin u}{\cos u} \times \cos^2 u$$

$$= 2 \sin u \cos u$$

$$\boxed{x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = \sin 2u}$$

Hence proved.

Example 4: If $u = \log\left[\dfrac{x^5 + y^5}{x^3 + y^3}\right]$ then prove that $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = 2$

Solution:

Given

$$u = \log\left[\frac{x^5 + y^5}{x^3 + y^3}\right]$$

Taking exponential on both sides

$$e^u = \frac{x^5 + x^5}{x^3 + y^3}$$

degree $(n) = 5 - 3 = 2$

Let $$f = e^u = \frac{x^5 + y^5}{x^3 + y^3}$$

By Euler's theorem

$$x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} = nf$$

$$x\,\frac{\partial}{\partial x}\,e^u + y\,\frac{\partial}{\partial y}\,e^u = 2e^u$$

$$xe^u \frac{\partial u}{\partial x} + y\, e^u \frac{\partial u}{\partial y} = 2e^u$$

$$e^u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 2e^u$$

$$\boxed{\boldsymbol{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2}}$$

Hence proved.

Example 5: If $u = \sin^{-1}\left(\dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}} \right)$. Then show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} + 3 \tan u = 0.$

Solution:

Given

$$u = \sin^{-1}\left(\frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}} \right)$$

$$\sin u = \frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}$$

Let $f = \sin u = \dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}$ is homogeneous function

$\therefore$ degree $(n) = 1 - 4$

$$n = -3$$

By Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf$$

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) + z \frac{\partial}{\partial z} \sin u = -3 \sin u$$

$$x\cos u\,\frac{\partial u}{\partial x}+y\cos u\,\frac{\partial u}{\partial y}+z\cos u\,\frac{\partial u}{\partial z}=-3\sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=-3\sin u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-3\,\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-3\tan u$$

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\tan u=0}$$

$\therefore$ Hence proved

Example 6: If $u=\sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$ prove by Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u$.

Solution:

Given

$$u=\sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$$

$$\sin u=\frac{x^3-y^3}{x+y}$$

Let $f=\sin u=\dfrac{x^3+y^3}{x+y}$ is homogeneous function

$\therefore$ degree $=3-1$

$n=2$

By Euler's Theorem

$$x\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=nf$$

$$x \frac{\partial}{\partial x} \sin u + y \frac{\partial}{\partial y} \sin u = 2 \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = 2 \sin u$$

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial x} \right) = 2 \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cos u}$$

$$\boxed{\boldsymbol{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u}}$$

$\therefore$ Hence proved

Example 7: If $z = \log (x^2 + xy + y^2)$ prove that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2$.

Solution:

Given

$$z = \log (x^2 + xy + y^2)$$

degree $(n) = 2$

$$z = \log (x^2 + xy + y^2)$$

Taking exponential on both sides

$$e^z = (x^2 + xy + y^2)$$

z is homogeneous function

Let $f = e^z = x^2 + xy + y^2$

By Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

$$x \frac{\partial}{\partial x} e^z + y \frac{\partial}{\partial y} e^z = 2e^z$$

$$xe^z \frac{\partial z}{\partial x} + ye^z \frac{\partial z}{\partial y} = 2e^z$$

$$e^z \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) = 2e^z$$

$$\boxed{\boldsymbol{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2}}$$

$\therefore$ Hence proved

Example 8: If $u = \frac{x^2 + y^2}{\sqrt{x+y}}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$

Solution:

$$u = \frac{x^2 + y^2}{\sqrt{x+y}}$$

$\therefore$ u is homogeneous function

degree $(n) = 2 - \frac{1}{2} = \frac{3}{2}$

By Euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n\, u$$

$$\boxed{\boldsymbol{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u}}$$

Hence proved.

Example 9: If $u = e^{x^3 + y^3}$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u \log u$.

Solution:

Given

$$u = e^{x^3 + y^3}$$

u is homogeneous function

degree $(n) = 3$

$$u = e^{x^3 + y^3}$$

Taking log on both sides

$$\log u = x^3 + y^3$$

Let $f = \log u = x^3 + y^3$

By Euler's theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

$$x \frac{\partial}{\partial x} (\log u) + y \frac{\partial}{\partial y} (\log u) = 3 \log u$$

$$x \left(\frac{1}{u} \right) \frac{\partial u}{\partial x} + y \left(\frac{1}{u} \right) \frac{\partial u}{\partial y} = 3 \log u$$

$$\frac{1}{u} \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 3 \log u$$

$$\boxed{\boldsymbol{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u \log u}}$$

$\therefore$ Hence proved.

Example 10: If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ then prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.$

Solution:

Given

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

degree $(n) = 1 - 1$

$$n = 0$$

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Example 11: If $u = \sec^{-1}\left(\frac{x^3 + y^3}{x + y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.

Solution:

Given

$$u = \sec^{-1}\left(\frac{x^3 + y^3}{x + y}\right)$$

$$\sec u = \frac{x^3 + y^3}{x + y}$$

Let $f = \sec u = \frac{x^3 + y^3}{x + y}$ is homogeneous function

$\therefore$ degree $(n) = 3 - 1$

$$n = 2$$

By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}(\sec u) + y\frac{\partial}{\partial y}(\sec u) = 2\sec u$$

$$x(\sec u \tan u)\frac{\partial u}{\partial x} + y(\sec u \tan u)\frac{\partial u}{\partial y} = 2\sec u$$

$$\sec u \tan u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 2 \sec u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \sec u}{\sec u \tan u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \left(\frac{1}{\tan u} \right)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cot u$$

$\therefore$ Hence proved.

Example 12: If $u = \sin^{-1} \left(\frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$.

Solution:

Given

$$u = \sin^{-1} \left(\frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$$

$$\sin u = \frac{x+y}{\sqrt{x} + \sqrt{y}}$$

Let $f = \sin u = \frac{x+y}{\sqrt{x} + \sqrt{y}}$ is homogeneous function

degree $(n) = 1 - \frac{1}{2}$

$$n = \frac{1}{2}$$

By Euler's theorem $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$

$$x \frac{\partial}{\partial x} \sin u + y \frac{\partial}{\partial y} \sin u = \frac{1}{2} \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = \frac{1}{2} \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \frac{\sin u}{\cos u}$$

$$\boxed{\boldsymbol{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u}}$$

$\therefore$ Hence proved.

3.4 TOTAL DERIVATIVE

If $u = f(x, y)$ where x and y are functions of t, then u can be expressed as a function of t alone when the values of x and y are substituted in $f(x, y)$. Then the ordinary derivative $\frac{du}{dt}$ is called the total derivative of u. Now, we find $\frac{du}{dt}$ without actually substituting the values of x and y in $f(x, y)$. For this let us prove the following formula.

$$\boxed{\boldsymbol{\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}}}$$

3.4.1 Problems on Total Derivatives

Example 1: If $\omega = f(y - z\,;\; z - x\,;\; x - y)$ then find the value of $\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z}$.

Solution:

Given

$$\omega = f(y - z\,;\; z - x\,;\; x - y)$$

$A = y - z$	$B = z - x$	$C = x - y$
$\dfrac{\partial A}{\partial x} = 0$	$\dfrac{\partial B}{\partial x} = -1$	$\dfrac{\partial C}{\partial x} = 1$
$\dfrac{\partial A}{\partial y} = 1$	$\dfrac{\partial B}{\partial x} = 0$	$\dfrac{\partial C}{\partial y} = -1$
$\dfrac{\partial A}{\partial z} = -1$	$\dfrac{\partial B}{\partial z} = 1$	$\dfrac{\partial C}{\partial z} = 0$

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial A}\frac{\partial A}{\partial x} + \frac{\partial \omega}{\partial B}\frac{\partial B}{\partial x} + \frac{\partial \omega}{\partial C}\frac{\partial C}{\partial x}$$

$$= \frac{\partial \omega}{\partial A}(0) + \frac{\partial \omega}{\partial B}(-1) + \frac{\partial \omega}{\partial C}(1) \qquad \text{... (1)}$$

$$\frac{\partial \omega}{\partial y} = \frac{\partial \omega}{\partial A}\frac{\partial A}{\partial y} + \frac{\partial \omega}{\partial B}\frac{\partial B}{\partial y} + \frac{\partial \omega}{\partial C}\frac{\partial C}{\partial y}$$

$$= \frac{\partial \omega}{\partial A}(1) + \frac{\partial \omega}{\partial B}(0) + \frac{\partial \omega}{\partial C}(-1) \qquad \text{... (2)}$$

$$\frac{\partial \omega}{\partial z} = \frac{\partial \omega}{\partial A}\frac{\partial A}{\partial z} + \frac{\partial \omega}{\partial B}\frac{\partial B}{\partial z} + \frac{\partial \omega}{\partial C}\frac{\partial C}{\partial z}$$

$$= \frac{\partial \omega}{\partial A}(-1) + \frac{\partial \omega}{\partial B}(1) + \frac{\partial \omega}{\partial C}(0) \qquad \text{... (3)}$$

Adding (1) (2) and (3)

$$\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z} = -\frac{\partial \omega}{\partial B} + \frac{\partial \omega}{\partial C} + \frac{\partial \omega}{\partial A} - \frac{\partial \omega}{\partial C} - \frac{\partial \omega}{\partial A} + \frac{\partial \omega}{\partial B}$$

$$\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial z} = 0$$

Example 2: **If $u = f(2x - 3y\ ;\ 3y - 4z\ ;\ 4z - 2x)$ then prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$**

Solution:

Given

$$u = f(2x - 3y\ ;\ 3y - 4z\ ;\ 4z - 2x)$$

$A = 2x - 3y$	$B = 3y - 4z$	$C = 4z - 2x$
$\frac{\partial A}{\partial x} = 2$	$\frac{\partial B}{\partial x} = 0$	$\frac{\partial C}{\partial x} = -2$
$\frac{\partial A}{\partial y} = -3$	$\frac{\partial B}{\partial y} = 3$	$\frac{\partial C}{\partial y} = 0$
$\frac{\partial A}{\partial z} = 0$	$\frac{\partial B}{\partial z} = -4$	$\frac{\partial C}{\partial z} = 4$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial x} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial x} + \frac{\partial u}{\partial C}\frac{\partial C}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial y} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial y} + \frac{\partial u}{\partial C}\frac{\partial C}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial z} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial z} + \frac{\partial u}{\partial C}\frac{\partial C}{\partial z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A}(2) + \frac{\partial u}{\partial B}(0) + \frac{\partial u}{\partial C}(-2)$$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial A} - 2\frac{\partial u}{\partial C} \Rightarrow \frac{\partial u}{\partial x} = 2\left(\frac{\partial u}{\partial A} - \frac{\partial u}{\partial C}\right)$$

$$\frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} - \frac{\partial u}{\partial C} \qquad \text{... (1)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial A}(-3)+\frac{\partial u}{\partial B}(3)+\frac{\partial u}{\partial C}(0)$$

$$\frac{\partial u}{\partial y}=+3\left(-\frac{\partial u}{\partial A}+\frac{\partial u}{\partial B}\right)$$

$$\frac{1}{3}\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial A}+\frac{\partial u}{\partial B} \quad \text{... (2)}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial A}(0)+\frac{\partial u}{\partial B}(-4)+\frac{\partial u}{\partial C}(4)$$

$$\frac{\partial u}{\partial z}=4\left(-\frac{\partial u}{\partial B}+\frac{\partial u}{\partial C}\right)$$

$$\frac{1}{4}\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial B}+\frac{\partial u}{\partial C} \quad \text{... (3)}$$

Adding (1) (2) (3)

$$\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial x}=\frac{\partial u}{\partial A}-\frac{\partial u}{\partial C}-\frac{\partial u}{\partial A}+\frac{\partial u}{\partial B}-\frac{\partial u}{\partial B}+\frac{\partial u}{\partial C}$$

$$=0$$

$\therefore$ Hence proved

Example 3: If $u=\left(\frac{x}{y};\ \frac{y}{z};\ \frac{z}{x}\right)$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

Solution:

Given

$$u=\left(\frac{x}{y};\ \frac{y}{z};\ \frac{z}{x}\right)$$

$A = \frac{x}{y}$	$B = \frac{y}{z}$	$C = \frac{z}{x}$
$\frac{\partial A}{\partial x} = \frac{1}{y}$	$\frac{\partial B}{\partial x} = 0$	$\frac{\partial C}{\partial x} = -\frac{z}{x^2}$
$\frac{\partial A}{\partial y} = -\frac{x}{y^2}$	$\frac{\partial B}{\partial y} = \frac{1}{z}$	$\frac{\partial C}{\partial y} = 0$
$\frac{\partial A}{\partial z} = 0$	$\frac{\partial B}{\partial z} = -\frac{y}{z^2}$	$\frac{\partial C}{\partial z} = \frac{1}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial x} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial x} + \frac{\partial u}{\partial C}\frac{\partial C}{\partial x} \quad \text{.... (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial y} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial y} + \frac{\partial u}{\partial C}\frac{\partial C}{\partial y} \quad \text{.... (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial z} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial z} + \frac{\partial u}{\partial C}\frac{\partial C}{\partial z} \quad \text{.... (3)}$$

$$(1) \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial A}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial B}(0) + \frac{\partial u}{\partial C}\left(-\frac{z}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial A} - \frac{z}{x^2}\frac{\partial u}{\partial C}$$

Multiply by x

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial A} - \frac{z}{x}\frac{\partial u}{\partial C}$$

$$(1) \times x \Rightarrow \quad x\frac{du}{dx} = \frac{x}{y}\frac{\partial u}{\partial A} - \frac{z}{x}\frac{\partial u}{\partial C} \quad \text{.... (4)}$$

Similarly

$$(2) \times y \Rightarrow \quad y\frac{\partial u}{\partial y} = y\left(\frac{-x}{y^2}\right)\frac{\partial u}{\partial A} + \frac{y}{z}\frac{\partial u}{\partial B} \quad \text{.... (5)}$$

$$(3) \times z \Rightarrow \quad z\frac{\partial u}{\partial z} = z\left(\frac{-y}{z^2}\right)\frac{\partial u}{\partial B} + \frac{z}{x}\frac{\partial u}{\partial C} \qquad \text{.... (6)}$$

Adding (4), (5) & (6) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Example 4: If $u = f\left(\frac{x-y}{xy}\,;\ \frac{z-x}{zx}\right)$ prove that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution:

Given

$$u = f\left(\frac{x-y}{xy}\,;\ \frac{z-x}{zx}\right)$$

Let

$A = \frac{x-y}{xy}$	$B = \frac{z-x}{zx}$
$A = \frac{x}{xy} - \frac{y}{xy}$	$B = \frac{z}{zx} - \frac{x}{zx}$
$A = \frac{1}{y} - \frac{1}{x}$	$B = \frac{1}{x} - \frac{1}{z}$
$\frac{\partial A}{\partial x} = -\left(\frac{-1}{x^2}\right) = \frac{1}{x^2}$	$\frac{\partial B}{\partial x} = \frac{-1}{x^2}$
$\frac{\partial A}{\partial y} = \frac{-1}{y^2}$	$\frac{\partial B}{\partial y} = 0$
$\frac{\partial A}{\partial z} = 0$	$\frac{\partial B}{\partial z} = \frac{1}{z^2}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A}\frac{\partial A}{\partial x} + \frac{\partial u}{\partial B}\frac{\partial B}{\partial x}$$

$$= \frac{\partial u}{\partial A}\left(\frac{1}{x^2}\right) + \frac{\partial u}{\partial B}\left(\frac{-1}{x^2}\right)$$

$$= \left[\frac{1}{x^2} \frac{\partial u}{\partial A} \frac{-1}{x^2} \frac{\partial u}{\partial B} \right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2} \left[\frac{\partial u}{\partial A} - \frac{\partial u}{\partial B} \right]$$

$$x^2 \frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} - \frac{\partial u}{\partial B} \quad \text{... (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A} \frac{\partial A}{\partial y} + \frac{\partial u}{\partial B} \frac{\partial B}{\partial y}$$

$$= \frac{\partial u}{\partial A} \left(\frac{-1}{y^2} \right) + \frac{\partial u}{\partial B} (0)$$

$$\frac{\partial u}{\partial y} = \frac{-1}{y^2} \frac{\partial u}{\partial A}$$

$$y^2 \frac{\partial u}{\partial A} = -\frac{\partial u}{\partial A} \quad \text{... (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial A} \frac{\partial A}{\partial z} + \frac{\partial u}{\partial B} \frac{\partial B}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial A} (0) + \frac{\partial u}{\partial B} \left(\frac{1}{z^2} \right)$$

$$\frac{\partial u}{\partial z} = \frac{1}{z^2} \frac{\partial u}{\partial B}$$

$$z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial B} \quad \text{... (3)}$$

Add (1), (2) and (3)

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial A} - \frac{\partial u}{\partial B} - \frac{\partial u}{\partial A} + \frac{\partial u}{\partial B}$$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

$\therefore$ Hence proved

Example 5: **If z is the function of x & y, u and v are functions of x and y, such that $u = lx + my$, $v = ly - mx$. Prove that**

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Solution:

$u = lx + my$	$v = 2y - mx$
$\frac{\partial u}{\partial x} = l$	$\frac{\partial v}{\partial x} = -m$
$\frac{\partial u}{\partial y} = m$	$\frac{\partial v}{\partial y} = l$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(l) + \frac{\partial z}{\partial v}(-m)$$

$$\frac{\partial z}{\partial x} = l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v} \qquad \text{.... (1)}$$

$$\frac{\partial}{\partial x} = l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v} \qquad \text{.... (2)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(m) + \frac{\partial z}{\partial v}(l)$$

$$\frac{\partial z}{\partial y} = m\frac{\partial z}{\partial u} + l\frac{\partial z}{\partial v} \qquad \text{.... (3)}$$

$$\frac{\partial}{\partial y} = m\frac{\partial}{\partial v} + l\frac{\partial}{\partial v} \qquad \text{.... (4)}$$

Multiply $(1) \times (2)$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(l\frac{\partial}{\partial u}-m\frac{\partial}{\partial v}\right)\left(l\frac{\partial z}{\partial u}-m\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial x^2}=l^2\frac{\partial^2 z}{\partial u^2}-lm\frac{\partial^2 z}{\partial u\,\partial v}-lm\frac{\partial^2 z}{\partial v\,\partial u}+m^2\frac{\partial^2 z}{\partial v^2} \qquad \text{.... (5)}$$

Multiply (3) and (4)

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(m\frac{\partial}{\partial u}+l\frac{\partial}{\partial v}\right)\left(m\frac{\partial z}{\partial u}+l\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial y^2}=m^2\frac{\partial^2 z}{\partial u^2}+lm\frac{\partial^2 z}{\partial u\,\partial v}+lm\frac{\partial^2 z}{\partial v\partial u}+l^2\frac{\partial^2 z}{\partial v^2} \qquad \text{.... (6)}$$

Adding (5) and (6)

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$$

Example 6: $\phi(u, v)$ is written interms of variables $u = e^x \cos y$, $v = e^x \sin y$ then prove that

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=(u^2+v^2)\left(\frac{\partial^2 \phi}{\partial u^2}+\frac{\partial^2 \phi}{\partial v^2}\right)$$

Solution:

$u = e^x \cos y$	$v = e^x \sin y$
$\frac{\partial u}{\partial x} = e^x \cos y = u$	$\frac{\partial v}{\partial x} = e^x \sin y = v$
$\frac{\partial u}{\partial y} = -e^x \sin y = -v$	$\frac{\partial v}{\partial y} = e^x \cos y = u$

$$\frac{\partial \phi}{\partial x}=\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}$$

$$=\frac{\partial \phi}{\partial u}(u)+\frac{\partial \phi}{\partial v}(v)$$

$$\frac{\partial \phi}{\partial x} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} \qquad \text{... (1)}$$

$$\frac{\partial}{\partial x} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \qquad \text{... (2)}$$

Multiply (1) and (2)

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \left(u\frac{\partial}{\partial u} + v\frac{\partial \phi}{\partial v}\right)\left(u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} = u^2\frac{\partial^2 \phi}{\partial u^2} + uv\frac{\partial^2 \phi}{\partial u\,\partial v} + uv\frac{\partial^2 \phi}{\partial v\,\partial u} + v^2\frac{\partial^2 \phi}{\partial v^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = u^2\frac{\partial^2 \phi}{\partial u^2} + 2\,uv\frac{\partial^2 \phi}{\partial u} + v^2\frac{\partial^2 \phi}{\partial v^2} \qquad \text{... (3)}$$

Similarly

$$\frac{\partial^2 \phi}{\partial y^2} = u^2\frac{\partial^2 \phi}{\partial v^2} - 2uv\frac{\partial^2 \phi}{\partial u\,\partial v} + v^2\frac{\partial^2 \phi}{\partial u^2} \qquad \text{... (4)}$$

Adding (3) and (4)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = u^2\frac{\partial^2 \phi}{\partial u^2} + 2uv\frac{\partial^2 \phi}{\partial u\,\partial v} + v^2\frac{\partial^2 \phi}{\partial v^2}$$

$$+ u^2\frac{\partial^2 \phi}{\partial u^2} - 2uv\frac{\partial^2 \phi}{\partial u\,\partial v} + v^2\frac{\partial^2 \phi}{\partial u^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial^2 v^2}\right)$$

$\therefore$ Hence proved.

3.5 TAYLOR'S SERIES EXPANSION FOR A FUNCTION OF TWO VARIABLES

We know that Taylor's theorem for a function of a single variable

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots$$

Let $f(x, y)$ be a function of two independent variables x and y.

If we keep y as a constant, then by Taylor's theorem for a function of a single variable x, we have

$$f(x+h, y+k) = f(x, y+k) + h \frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!} \frac{\partial^2}{\partial x^2} f(x, y+k) + \ldots$$

Note 1:

Put $x = a$ and $y = b$, in Taylor's series, we get

$$f(a+h, b+k) = f(a, b) + \left\{ h \frac{\partial f}{\partial x}(a, b) + k \frac{\partial f}{\partial y}(a, b) \right\}$$

$$+ \frac{1}{2!} \left\{ h^2 \frac{\partial^2 f}{\partial x^2}(a, b) + 2hk \frac{\partial^2 f(a, b)}{\partial x \partial y} + k^2 \frac{\partial^2 f(a, b)}{\partial y^2} \right\} + \ldots$$

Note 2:

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$ and $k = y - b$, we get

$$f(x, y) = f(a, b) + \left\{ (x-a) \frac{\partial f(a, b)}{\partial x} + (y-b) \frac{\partial f(a, b)}{\partial y} \right\}$$

$$+ \frac{1}{2!} \left\{ (x-a)^2 \frac{\partial^2 f(a, b)}{\partial x^2} + 2(x-a)(y-b) \frac{\partial^2 f(a, b)}{\partial x \partial y} + (y-b)^2 \frac{\partial^2 f(a, b)}{\partial y^2} \right\}$$

$$+ \ldots\ldots$$

This formula is used to expand $f(x, y)$ in the neighbourhood of (a, b)

Note 3:

Putting $a = 0$, $b = 0$ in corollary 2, we get

$$f(x, y) = f(0, 0) + \left\{x \frac{\partial f(0, 0)}{\partial x} + y \frac{\partial f(0, 0)}{\partial y}\right\}$$

$$+ \frac{1}{2!}\left\{x^2 \frac{\partial^2 f(0, 0)}{\partial x^2} + 2xy \frac{\partial^2 f(0, 0)}{\partial x \partial y} + y^2 \frac{\partial^2 f(0\ 0)}{\partial y^2}\right\} + \ldots$$

This formula is used to expand $f(x, y)$ in powers of x and y in the neighbourhood of the origin (0, 0).

Taylor's series for functions of two variables

Case (i): (0, 0) or origin

First degree term

$$f(x, y) = f(0, 0) + \frac{1}{1!}[(x-0) f_x(0, 0) + (y-0) f_y(0, 0)] + \ldots\ldots$$

Second degree term

$$f(x, y) = f(0, 0) + \frac{1}{1!}[(x-0) f_x(0, 0) + (y-0) f_y(0, 0)]$$

$$+ \frac{1}{2!}[(x-0)^2 f_{xx}(0, 0) + 2(x-0)(y-0) f_{xy}(0, 0)$$

$$+ (y-0)^2 f_{yy}(0, 0)] + \ldots\ldots$$

Third degree term of origin

$$f(x, y) = f(0, 0) + \frac{1}{1!}[(x-0) f_x(0, 0) + (y-0) f_y(0, 0)] +$$

$$+ \frac{1}{2!}[(x-0)^2 f_{xx}(0, 0) + 2(x-0)(y-0) f_{xy}(0, 0)$$

$$+ (y-0)^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}[(x-0)^3 + 3(x-0)^2 (y-0) f_{xxy}(0, 0) + 3(x-0)(y-0)^2 f_{xyy}(0, 0)$$

$$+ (y-0)^3 f_{yyy}(0, 0)] + \ldots$$

Case (ii): (a, b) (numbers)

First degree term

$$f(x, y) = f(a, b) + \frac{1}{1!}\Big[(x-a) f_x(a,b) + (y-b) f_y(a, b) \Big]$$

Second degree term

$$f(x, y) = f(a, b) + \frac{1}{1!}[(x-a) f_x(a, b) + (y-b) f_y(a, b)] +$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)] + \dots$$

Third degree term

$$f(x, y) = f(a, b) + \frac{1}{1!}[(x-a) f_x(a, b) + (y-b) f_y(a, b)] +$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)] +$$

$$\frac{1}{3!}[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2 (y-b) f_{xxy}(a, b)$$

$$+ 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)] + \dots$$

WORKED EXAMPLES

Example 1: Expand $f(x, y) = x^2 y^2 + 2x^2 y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ upto second degree terms.

Solution:

Function	Point $(-2, 1)$
$f = x^2 y^2 + 2x^2 y + 3xy^2$	$f(-2, 1) = 6$
$f_x = 2xy^2 + 4xy + 3y^2$	$f_x(-2, 1) = -9$
$f_y = 2x^2 y + 2x^2 + 6xy$	$f_y(-2, 1) = 4$
$f_{xx} = 2y^2 + 4y$	$f_{xx}(-2, 1) = 6$
$f_{xy} = 4xy + 4x + 6y$	$f_{xy}(-2, 1) = -10$
$f_{yy} = 2x^2 + 0 + 6x$	$f_{yy}(-2, 1) = -4$

Formula:

$$f(x, y) = f(a, b) + \frac{1}{1!}[(x-a) f_x(a, b) + (y-b) f_y(a, b)]$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)]$$

$$f(x, y) = f(-2, 1) + \frac{1}{1!}\Big[(x+2) f_x(-2, 1) + (y-1) f_y(-2, 1)\Big]$$

$$+ \frac{1}{2!}\Big[(x+2)^2 f_{xx}(-2, 1) + 2(x+2)(y-1) f_{xy}(-2, 1)\Big]$$

$$+ (y-1)^2 f_{yy}(-2, 1)]$$

Taylor series expansion

$$f(x, y) = 6 + \frac{1}{1!}[(x+2)(-9) + (y-1)(4)] + \frac{1}{2!}[(x+2)^2 (6)$$

$$+ 2(x+2)(y-1)(-10) + (y-1)^2(-4)]$$

Example 2: Obtain the taylor series expansion of $f(x, y) = x^3 + y^3 + xy^2$ in powers of $(x - 1)$ and $(y - 2)$ upto second degree terms.

Solution:

Function	Points (1, 2)
$f = x^3 + y^3 + xy^2$	$f(1, 2) = 13$
$f_x = 3x^2 + 0 + y^2$	$f_x(1, 2) = 7$
$f_y = 0 + 3y^2 + 2xy$	$f_y(1, 2) = 16$
$f_{xx} = 6x$	$f_{xx}(1, 2) = 6$
$f_{yy} = 6y + 2x$	$f_{yy}(1, 2) = 14$

Formula

$$f(x, y) = f(a, b) + \frac{1}{1!}[(x - a) f_x(a, b) + (y - b) f_y(a, b)]$$

$$+ \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b)$$

$$+ (y - b) f_{yy}(a, b)]$$

$$f(x, y) = f(1, 2) + \frac{1}{1!}[(x - 1) f_x(1, 2) + (y - 2) f_y(1, 2)]$$

$$+ \frac{1}{2!}[(x - 1) f_{xx}(1, 2)$$

$$+ 2(x - 1)(y - 2) f_{xy}(1, 2) + (y - 2)^2 f_{yy}(1, 2)]$$

Taylor series expansion

$$f(x, y) = 13 + \frac{1}{1!}[(x - 1)(7) + (y - 2)(16)] +$$

$$\frac{1}{2!}[(x - 1)(6) + 2(x - 1)(y - 2)(4) + (y - 2)^2 (14)]$$

Example 3: Obtain the taylor series expansion of $f(x, y) = x^2 y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ upto 2^{nd} degree terms.

Solution:

Function	Points $(1, -2)$
$f = x^2 y + 3y - 2$	$f(1, -2) = -10$
$f_x = 2xy$	$f_x(1, -2) = -4$
$f_y = x^2 + 3$	$f_y(1, -2) = 4$
$f_{xx} = 2y$	$f_{xx}(1, -2) = -4$
$f_{yy} = 0$	$f_{yy}(1, -2) = 0$
$f_{xy} = 2x$	$f_{xy}(1, -2) = 2$

Formula

$$f(x, y) = f(a, b) + \frac{1}{1!}[(x-a) f_x(a, b) + (y-b) f_y(a, b)] +$$

$$\frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)]$$

$$f(1, -2) = f(1, -2) + \frac{1}{1!}[(x-1) f_x(1, -2) + (y+2) f_y(1, -2)] +$$

$$\frac{1}{2!}[(x-1)^2 f_x(1, -2) + 2(x-1)(y+2) f_{xy}(-1, 2)$$

$$+ (y-2)^2 f_{yy}(-1, 2)]$$

Taylor series expansion

$$f(x, y) = -10 + \frac{1}{1!}[(x-1)(-4) + (y+2)(4)]$$

$$+ \frac{1}{2!}[(x-2)^2(-4) + 2(x-1)(y+2)(2) + (y-2)^2(0)]$$

Example 4: Expand $f(x, y) = e^x \sin y$ near the origin by using Taylor series expansion upto third degree terms.

Solution:

Function	**Point (0, 0)**
$f = e^x \sin y$	$f(0, 0) = 0$
$f_x = e^x \sin y$	$f_x(0, 0) = 0$
$f_y = e^x \cos y$	$f_y(0, 0) = 1$
$f_{xx} = e^x \sin y$	$f_{xx}(0, 0) = 0$
$f_{xy} = e^x \cos y$	$f_{xy}(0, 0) = 1$
$f_{yy} = -e^x \sin y$	$f_{yy}(0, 0) = 0$
$f_{xxx} = e^x \sin y$	$f_{xxx}(0, 0) = 0$
$f_{xxy} = e^x \cos y$	$f_{xxy}(0, 0) = 1$
$f_{yyx} = -e^x \sin y$	$f_{yyx}(0, 0) = 0$
$f_{yyy} = -e^x \cos y$	$f_{yyy}(0, 0) = -1$

Formula

$$f(x, y) = f(0, 0) + \frac{1}{1!}[x - 0)f_x(0, 0) + (y - 0)f_y(0, 0)]$$

$$+ \frac{1}{2!}[(x-0)^2 f_{xx}(0, 0) + 2(x-0)(y-0)f_{xy}(0, 0) + (y-0)^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}[(x-0)^3 f_{xxx}(0, 0) + 3(x-0)^2(y-0)f_{xxy}(0, 0) + 3(x-0)(y-0)^2$$

$$f_{xyy}(0, 0) + (y-0)^3 f_{yyy}(0, 0)] + \ldots$$

Taylor series expansion

$$f(x, y) = 0 + \frac{1}{1!}[(x-0)(0) + (y-0)(1)] + \frac{1}{2!}[(x-0)^2(0)$$

$$+ 2(x-0)(y-0)(1) + (y-0)^2(0)]$$

$$+ \frac{1}{3!}[(x-0)^3(0) + 3(x-0)^2(y-0)(1) + 3(x-0)(y-0)^2(0) + (y-0)^3(-1)]$$

Example 5: Obtain the taylor series expansion of $f(x, y) = e^x \cos y$ above the $\left(0, \frac{\pi}{2}\right)$ upto third degree terms.

Solution:

Function	**Point** $\left(0, \frac{\pi}{2}\right)$
$f = e^x \cos y$	$f\left(0, \frac{\pi}{2}\right) = 0$
$f_x = e^x \cos y$	$f_x\left(0, \frac{\pi}{2}\right) = 0$
$f_y = -e^x \sin y$	$f_y\left(0, \frac{\pi}{2}\right) = -1$
$f_{xx} = e^x \cos y$	$f_{xx}\left(0, \frac{\pi}{2}\right) = 0$
$f_{xy} = -e^x \sin y$	$f_{xy}\left(0, \frac{\pi}{2}\right) = -1$
$f_{yy} = -e^x \cos y$	$f_{yy}\left(0, \frac{\pi}{2}\right) = 0$
$f_{xxx} = e^x \cos y$	$f_{xxx}\left(0, \frac{\pi}{2}\right) = 0$
$f_{xxy} = -e^x \sin y$	$f_{xxy}\left(0, \frac{\pi}{2}\right) = -1$
$f_{yyx} = -e^x \cos y$	$f_{yyx}\left(0, \frac{\pi}{2}\right) = 0$
$f_{yyy} = e^x \sin y$	$f_{yyy}\left(0, \frac{\pi}{2}\right) = 1$

Formula

$$f(x,y) = f\left(0,\frac{\pi}{2}\right) + \frac{1}{1!}\left[(x-0) f_x\left(0,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right) f_y\left(0,\frac{\pi}{2}\right)\right]$$

$$+\frac{1}{2!}\left[(x-0)^2 f_{xx}\left(0,\frac{\pi}{2}\right) + 2(x-0)\left(y-\frac{\pi}{2}\right) f_{xy}\left(0,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2 f_{yy}\left(0,\frac{\pi}{2}\right)\right]$$

$$+\frac{1}{3!}\left[(x-0)^3 f_{xxx}\left(0,\frac{\pi}{2}\right) + 3(x-0)^2\left(y-\frac{\pi}{2}\right) f_{xxy}\left(0,\frac{\pi}{2}\right) + 3(x-0)\right.$$

$$\left.\left(y-\frac{\pi}{2}\right)^2 f_{xyy}\left(0,\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^3 f_{yyy}\left(0,\frac{\pi}{2}\right)\right]$$

$$f\left(0,\frac{\pi}{2}\right) = 0 + \frac{1}{1!}\left[(x-0)(0) + \left(y-\frac{\pi}{2}\right)(-1)\right] + \frac{1}{2!}$$

$$\left[(x-0)^2(0) + 2(x-0)\left(y-\frac{\pi}{2}\right)(-1) + \left(y-\frac{\pi}{2}\right)^2 (0)\right] +$$

$$\frac{1}{3!}\left[(x-0)^3(0) + 3(x-0)^2\left(y-\frac{\pi}{2}\right)(-1) + 3(x-0)\left(y-\frac{\pi}{2}\right)^2(0) + \left(y-\frac{\pi}{2}\right)^3(1)\right]$$

Example 6: Obtain the taylor series expansions of $f(x,y) = e^x \sin y$ above the $\left(0,\frac{\pi}{2}\right)$ upto second degree term

Solution:

Function	**Point** $\left(0,\frac{\pi}{2}\right)$
$f = e^x \sin y$	$\left(0,\frac{\pi}{2}\right) = 1$
$f_x = e^x \sin y$	$f_x\left(0,\frac{\pi}{2}\right) = 1$
$f_y = e^x \cos y$	$f_y\left(0,\frac{\pi}{2}\right) = 0$
$f_{xx} = e^x \sin y$	$f_{xx}\left(0,\frac{\pi}{2}\right) = 1$
$f_{xy} = e^x \cos y$	$f_{xy}\left(0,\frac{\pi}{2}\right) = 0$

Function	**Point $\left(0, \frac{\pi}{2}\right)$**
$f_{yy} = -e^x \sin y$	$f_{yy}\left(0, \frac{\pi}{2}\right) = -1$

$$f(x, y) = \left(0, \frac{\pi}{2}\right) + \frac{1}{1!}\left[(x-0) f_x\left(0, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right) f_y\left(0, \frac{\pi}{2}\right)\right] +$$

$$\frac{1}{2!}\left[(x-0)^2 f_{xx}\left(0, \frac{\pi}{2}\right) + 2(x-0)\left(y-\frac{\pi}{2}\right) f_{xy}\left(0, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\left(0, \frac{\pi}{2}\right)\right]$$

$$f\left(0, \frac{\pi}{2}\right) = 1 + \frac{1}{1!}\left[(x-0)(1) + \left(y - \frac{\pi}{2}\right)(0)\right] + \frac{1}{2!}$$

$$\left[(x-0)^2 (1) + 2(x-0)\left(y - \frac{\pi}{2}\right)(0) + \left(y - \frac{\pi}{2}\right)^2 (-1)\right]$$

$$f(x, y) = 1 + \frac{1}{1!}[(x-0) + 0] + \frac{1}{2!}\left[(x-0)^2 + 0 - \left(y - \frac{\pi}{2}\right)^2\right]$$

Example 7: Obtain the taylor series expansion of $f(x, y) = e^x \cos y$ above the (0, 0) upto third degree term

Solution:

Function	**Point (0, 0)**
$f = e^x \cos y$	$f(0, 0) = 1$
$f_x = e^x \cos y$	$f_x(0, 0) = 1$
$f_y = -e^x \sin y$	$f_y(0, 0) = 0$
$f_{xx} = e^x \cos y$	$f_{xx}(0, 0) = 1$
$f_{xy} = -e^x \sin y$	$f_{xy}(0, 0) = 0$
$f_{yy} = -e^x \cos y$	$f_{yy}(0, 0) = -1$
$f_{xxx} = e^x \cos y$	$f_{xxx}(0, 0) = 1$
$f_{xxy} = -e^x \sin y$	$f_{xxy}(0, 0) = 0$

Function	Point (0, 0)
$f_{xyy} = -e^x \cos y$	$f_{xyy}(0, 0) = -1$
$f_{yyy} = e^x \sin y$	$f_{yyy}(0, 0) = 0$

$$f(x, y) = 0 + \frac{1}{1!}[(x-0)(1) + (y-0)(0)]$$

$$+ \frac{1}{2!}[(x-0)^2(1) + 2(x-0)(y-0)(0) + (y-0)^2(-1)]$$

$$+ \frac{1}{3!}[(x-0)^3(1) + 3(x-0)^2(y-0)(0)$$

$$+ 3(x-0)(y-0)^2(-1) + (y-0)^3(0)]$$

Example 8: Obtain explain using Taylor's series sin (xy) about the point $(x-1)$ and $\left(y - \frac{\pi}{2}\right)$ upto 2^{nd} degree term

Solution:

Function	Point $(1, \pi/2)$
$f = \sin(xy)$	$f\left(1, \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$
$f_x = y\cos(xy)$	$f_x\left(1, \frac{\pi}{2}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$
$f_y = x\cos(xy)$	$f_y\left(1, \frac{\pi}{2}\right) = 1 \times \cos\left(1 \times \frac{\pi}{2}\right) = 0$
$f_{xx} = -y^2\sin(xy)$	$f_{xx}\left(1, \frac{\pi}{2}\right) = -\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right)$ $= -\frac{\pi^2}{4}$
$f_{xy} = \cos(xy)(1) - yx\sin(xy)$	$f_{xy}\left(1, \frac{\pi}{2}\right) = -\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$
$f_{yy} = -x^2\sin(xy)$	$f_{yy}\left(1, \frac{\pi}{2}\right) = -1$

$$f(x, y) = f\left(1, \frac{\pi}{2}\right) + \frac{1}{1!}\left[(x-1) f_x\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right) f_y\left(1, \frac{\pi}{2}\right)\right] +$$

$$\frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1, \frac{\pi}{2}\right) + 2(x-1)\left(y - \frac{\pi}{2}\right) f_{xy}\left(1, \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\left(1, \frac{\pi}{2}\right)\right]$$

$$f(x, y) = 1 + \frac{1}{1!}\left[(x-1)(0) + \left(y - \frac{\pi}{2}\right)(0)\right] +$$

$$\frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)\left(y - \frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 (-1)\right]$$

Example 9: Expand $e^x \log(1 + y)$ in power of x and y upto third degree terms.

Solution:

Function	**Point (0, 0)**
$f = e^x \log(1 + y)$	0
$f_x = e^x \log(1 + y)$	0
$f_y = e^x \frac{1}{(1 + y)}$	1
$f_{xx} = e^x (\log 1 + y)$	0
$f_{xy} = e^x \frac{1}{(1 + y)}$	1
$f_{yy} = e^x \left(\frac{-1}{(1 + y)^2}\right)$	− 1
$f_{xxx} = e^x \log(1 + y)$	0
$f_{xxy} = e^x \left(\frac{1}{1 + y}\right)$	1
$f_{yyx} = e^x \left(\frac{-1}{(1 + y)^2}\right)$	− 1
$f_{yyy} = e^x \left(\frac{2}{(1 + y)^3}\right)$	2

$$f(x, y) = f(0, 0) + \frac{1}{1!}[(x-0) f_x (0, 0) + (y-0) f_y (0, 0)] +$$

$$\frac{1}{2!}[(x-0)^2 f_{xx} (0, 0) + 2 (x-0)(y-0) f_{xy} (0, 0) + (y-0)^2 f_{yy} (0, 0)]$$

$$+ \frac{1}{3!}[(x-0)^3 f_{xxx} (0, 0) + 3 (x-0)^2 (y-0) f_{xxy} (0, 0)$$

$$+ 3 (x-0)(y-0)^2 f_{xyy} (0, 0) + (y-0)^3 f_{yyy} (0, 0)]$$

$$f(x, y) = 0 + \frac{1}{1!}[(x-0)(0) + (y-0)(1)] + \frac{1}{2!}[(x-0)^2 (0)$$

$$+ 2 (x-0)(y-0)(1) + (y-0)^2 (-1)]$$

$$+ \frac{1}{3!}[(x-0)^2 (0) + 3 (x-0)^2 (y-0)(1)$$

$$+ 3 (x-0)(y-0)^2 (-1) + (y-0)^3 (2)]$$

Example 10: Expand e^{xy} in Taylor's series upto 2nd degree terms about the point (1, 1).

Solution:

Function	**Points (1, 1)**
$f = e^{xy}$	$f(1, 1) = e$
$f_x = ye^{xy}$	$f_x (1, 1) = e$
$f_y = xe^{xy}$	$f_y (1, 1) = e$
$f_{xx} = y^2 e^{xy}$	$f_{xx} (1, 1) = e$
$f_{xy} = xye^{xy} + e^{xy}$	$f_{xy} (1, 1) = 2e$
$f_{yy} = x^2 e^{xy}$	$f_{yy} (1, 1) = e$

$$f(x, y) = f(1, 1) + \frac{1}{1!}[(x-1) f_x(1, 1) + (y-1) f_y(1, 1)] +$$

$$\frac{1}{2!}[(x-1)^2 f_{xx}(1, 1) + 2(x-1)(y-1) f_{xy}(1, 1)$$

$$+ (y-1)^2 f_{yy}(1, 1)]$$

$$f(x, y) = e + \frac{1}{1!}[(x-1)(e) + (y-1)e] + \frac{1}{2!}[(x-1)^2 (e)$$

$$+ 2(x-1)(y-1)(e) + (y-1)^2 (2e)]$$

Example 11: Expand $e^x \sin y$ in Taylor's series upto 3rd degree terms about the point $\left(-1, \frac{\pi}{4}\right)$

Solution:

Function	**Point $\left(-1, \frac{\pi}{4}\right)$**
$f = e^x \sin y$	$f\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{1}{e\sqrt{2}}$
$f_x = e^x \sin y$	$f_x\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{1}{e\sqrt{2}}$
$f_y = e^x \cos y$	$f_y\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{1}{e\sqrt{2}}$
$f_{xx} = e^x \sin y$	$f_{xx}\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{1}{e\sqrt{2}}$
$f_{xy} = e^x \cos y$	$f_{xy}\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{1}{e\sqrt{2}}$
$f_{yy} = -e^x \sin y$	$f_{yy}\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{-1}{e\sqrt{2}}$
$f_{xxx} = e^x \sin y$	$f_{xxx}\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{1}{e\sqrt{2}}$
$f_{xyy} = -e^x \sin y$	$f_{xyy}\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{-1}{e\sqrt{2}}$

Function	**Point** $\left(-1, \frac{\pi}{4}\right)$
$f_{xxy} = e^x \cos y$	$f_{xxy}(-1, \pi/4) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{1}{e\sqrt{2}}$
$f_{yyy} = -e^x \cos y$	$f_{yyy}\left(-1, \frac{\pi}{4}\right) = e^{-1} \times \frac{1}{\sqrt{2}} = \frac{-1}{e\sqrt{2}}$

Formula

$$f(x, y) = f\left(-1, \frac{\pi}{4}\right) + \frac{1}{1!}\left[(x+1) f_x\left(-1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) f_y\left(-1, \frac{\pi}{4}\right)\right] +$$

$$\frac{1}{2!}\left[(x+1)^2 f_{xx}\left(-1, \frac{\pi}{4}\right) + 2(x+1)\left(y - \frac{\pi}{4}\right) f_{xy}\left(-1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2\right] +$$

$$\frac{1}{3!}\left[(x+1)^3 f_{xxx}\left(-1, \frac{\pi}{4}\right) + 3(x+1)^2\left(y - \frac{\pi}{4}\right) f_{xxy}\left(-1, \frac{\pi}{4}\right)\right.$$

$$\left. + 3(x+1)\left(y - \frac{\pi}{4}\right)^2 f_{xyy}\left(-1, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^3 f_{yyy}\left(-1, \frac{\pi}{4}\right)\right]$$

Taylor series expansion

$$f(x, y) = e^{-1} \times \frac{1}{\sqrt{2}} + \frac{1}{1!}\left[(x+1)\, e^{-1} \times \frac{1}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right) e^{-1} \times \frac{1}{\sqrt{2}}\right] +$$

$$\frac{1}{2!}\left[(x+1)^2\, e^{-1} \times \frac{1}{\sqrt{2}} + 2(x+1)\left(y - \frac{\pi}{4}\right) e^{-1} \times \frac{1}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right) e^{-1} \times \frac{1}{\sqrt{2}}\right] +$$

$$\frac{1}{3!}\left[(x+1)^3\, e^{-1} \times \frac{1}{\sqrt{2}} + 3(x+1)^2\left(y - \frac{\pi}{4}\right) e^{-1} \times \frac{1}{\sqrt{2}}\right.$$

$$\left. + 3(x+1)\left(y - \frac{\pi}{4}\right) e^{-1} \times \frac{1}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right) e^{-1} \times \frac{1}{\sqrt{2}}\right]$$

Example 12: Find the Taylor's series expansion of x^y near the point (1, 1) upto the 1st degree terms

Solution:

$f = x^y$	$f(1, 1) = 1$
$f_x = yx^{y-1}$	$f_x(1, 1) = 1$
$f_y = x^y \log x$	$f_y(1, 1) = 0$

$$f(x, y) = f(x, y) + \frac{1}{1!}[(x-a) f_x(x, y) + (y-b) f_y(x, y)]$$

$$f(x, y) = f(1, 1) + \frac{1}{1!}[(x-1) f_x(1, 1) + (y-1) f_y(1, 1)]$$

$$f(x, y) = x^y = 1 + \frac{1}{1!}[(x-1)(1) + (y-1)(0)]$$

Example 13: Expand $\sin x \cos y$ in powers of x, y upto the terms of degree 3 by using Taylor's series

Solution:

Function	**Points (0, 0)**
$f = \sin x \cos y$	$f(0, 0) = 0$
$f_x = \cos x \cos y$	$f_x(0, 0) = 1$
$f_y = -\sin x \sin y$	$f_y(0, 0) = 0$
$f_{xx} = -\sin x \cos y$	$f_{xx}(0, 0) = 0$
$f_{xy} = -\cos x \sin y$	$f_{xy}(0, 0) = 0$
$f_{yy} = -\sin x \cos y$	$f_{yy}(0, 0) = 0$
$f_{xxx} = -\cos x \cos y$	$f_{xxx}(0, 0) = -1$
$f_{xxy} = \sin x \sin y$	$f_{xxy}(0, 0) = 0$
$f_{xyy} = -\cos x \cos y$	$f_{xyy}(0, 0) = -1$
$f_{yyy} = \sin x \sin y$	$f_{yyy}(0, 0) = 0$

Formula

$$f(x, y) = f(0, 0) + \frac{1}{1!}[(x-0) f_x(0, 0) + (y-0) f_y(0, 0)]$$

$$+ \frac{1}{2!}[(x-0)^2 f_{xx}(0, 0) + 2(x-0)(y-0) f_{xy}(0, 0) + (y-0)^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}[(x-0)^3 f_{xxx}(0, 0) + 3(x-0)^2 (y-0) f_{xxy}$$

$$+ 3(x-0)(y-0)^2 f_{xyy} + (y-0)^3 f_{yyy}]$$

Example 14: Find the Taylor's series expansion of $f(x, y) = x^2 y + \sin y + e^x$ about the point $(1, \pi)$ upto 2^{nd} degree terms.

Solution:

Function	**Point $(1, \pi)$**
$f = x^2 y + \sin y + e^x$	$f(1, n) = \pi + e$
$f_x = 2xy + e^x$	$f_x(1, \pi) = 2\pi + e$
$f_y = x^2 + \cos y$	$f_y(1, \pi) = 0$
$f_{xx} = 2y + e^x$	$f_{xx}(1, \pi) = 2\pi + e$
$f_{xy} = 2x$	$f_{xy}(1, \pi) = 2$
$f_{yy} = -\sin y$	$f_{yy}(1, \pi) = 0$

Formula

$$f(x, y) = f(1, \pi) + \frac{1}{1!}[(x-1) f_x(1, \pi) + (y - \pi) f_y(1, \pi)] +$$

$$\frac{1}{2!}[(x-1)^2 f_{xx}(1, \pi) + 2(x-1)(y-\pi) f_{xy}(1, \pi) + (y-\pi)^2 f_{yy}(1, \pi)]$$

Taylor series expansion

$$f(x, y) = \pi + e + \frac{1}{1!}[(x-1)(2\pi + e) + (y - 0)(0)]$$

$$+ \frac{1}{2!}[(x-1)^2 (2\pi + e) + 2(x+1)(y-\pi)(2) + (y-\pi)^2 (0)]$$

$$f(x, y) = (\pi + e) + \frac{1}{1!}[(x-1)(2\pi + e)] + \frac{1}{2!}[(x-1)^2 (2\pi + e)$$

$$+ 2(x-1)(y-\pi)] + \ldots$$

Example 15: Using Taylor's series expand e^{x+y} in powers of x and y upto 3rd degree terms.

Solution:

Function	**Point (0, 0)**
$f = e^{x+y}$	$f(0, 0) = 1$
$f_x = e^{x+y}$	$f_x(0, 0) = 1$
$f_y = e^{x+y}$	$f_{xy}(0, 0) = 1$
$f_{xx} = e^{x+y}$	$f_{xx}(0, 0) = 1$
$f_{xy} = e^{x+y}$	$f_{xy}(0, 0) = 1$
$f_{yy} = e^{x+y}$	$f_{yy}(0, 0) = 1$
$f_{xxx} = e^{x+y}$	$f_{xxx}(0, 0) = 1$
$f_{xxy} = e^{x+y}$	$f_{xxy}(0, 0) = 1$
$f_{xyy} = e^{x+y}$	$f_{xyy}(0, 0) = 1$
$f_{yyy} = e^{x+y}$	$f_{yyy}(0, 0) = 1$

Formula

$$f(x, y) = f(0, 0) + \frac{1}{1!}[(x-0) f_x(0, 0) + (y-0) f_y(0, 0)]$$

$$+ \frac{1}{2!}[(x-0)^2 f_{xx}(0, 0) + 2(x-0)(y-0) f_{xy}(0, 0) + (y-0) f_y(0, 0)]$$

$$+ \frac{1}{3!}[(x-0)^3 f_{xxx}(0, 0) + 3(x-0)^2 (y-0) f_{xxy}(0, 0) + 3(x-0)$$

$$+ (y-0)^2 f_{xyy}(0, 0) + (y-0)^3 f_{yyy}(0, 0)] + \ldots$$

Taylor series expansion

$$f(x, y) = 1 + \frac{1}{1!}[(x-0) + (y-0)]$$

$$+ \frac{1}{2!}[(x-0)^2 + 2(x-0)(y-0) + (y-0)^2]$$

$$+ \frac{1}{3!}[(x-0)^3 + 3(x-0)^2 (y-0) + 3(x-0)(y-0) + (y-0)^3]$$

3.6 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

The function $f(x, y)$ has a maximum value at (a, b) if $f(a, b) > f(x, y)$ always when (x, y) is in the neighbourhood of (a, b)

i.e., for small and finite value of h, k,

$$f(a+h,\ b+k) < f(a, b).$$

i.e., $f(a+h,\ b+k) - f(a, b)$ is always negative for sufficiently small values of h and k whatever may be the sign of h and k.

Similarly, the function $f(x, y)$ has a minimum value of (a, b) if $f(a+h,\ b+k) - f(a, b)$ is always positive for sufficiently small values of h and k whatever may be the sign of h and k.

Thus $f(x, y)$ attains a maximum or a minimum at (a, b) according as $f(a+h, b+k) - f(a, b)$ is always negative or always positive for sufficiently small values of h and k. The maximum or minimum value is also termed as external value.

3.6.1 Working rule to find the maximum or minimum value of a function of two variables

Let $f(x, y)$ be the given function

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equal to zero, by solving the equation $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$, find the roots (x_1, y_1), (x_2, y_2) … These points may be maximum points or minimum points.

2. Find the values of $A = \frac{\partial^2 f}{\partial x^2}, C = \frac{\partial^2 f}{\partial x \partial y}.\ B = \frac{\partial^2 f}{\partial y^2}$ at these points.

3. (i) If $AC - B^2 > 0$ and $A < 0$ at a certain point, then the function is maximum at that point.

(ii) If $AC - B^2 > 0$ and $A > 0$ at a certain point, then the function is minimum at that point.

(iii) If $AC - B^2 < 0$ for a certain point, then the function is neither maximum nor minimum at that point. This point is known as saddle point.

(iv) If $AC - B^2 = 0$ at a certain point, then nothing can be said whether the function is maximum or minimum at that point. In this case further investigation are required.

- If $AC - B^2 > 0$ and $A < 0$, the function is maximum
- If $AC - B^2 > 0$ and $A > 0$, the function is minimum
- If $AC - B^2 < 0$ there is a saddle point.
- If $AC - B^2 = 0$ No conclusion

3.6.2 Problems on maxima and minima of two variables

Example 1: Find maximum and minimum values of

$$\boldsymbol{f(x, y) = x^3 + y^3 - 3x - 12y + 20}$$

Solution:

$$f = x^3 + y^3 - 3x - 12y + 20$$

$$f_x = 3x^2 + 0 - 3 - 0 + 0$$

$$f_y = 0 + 3y^2 - 0 - 12 + 0$$

$$A = f_{xx} = 6x$$

$$B = f_{xy} = 0$$

$$C = f_{yy} = 6y$$

To find stationary points

Put $f_x = 0$ and $f_y = 0$

$f_x = 3x^2 - 3$	$f_y = 3y^2 - 12$
$3x^2 - 3 = 0$ (i)	$3y^2 - 12 = 0$ (ii)
Solving (i)	Solving (ii)
$\boxed{x_1 = 1 \quad x_2 = -1}$	$\boxed{y_1 = 2 \quad y_2 = -2}$

The stationary points are

$$(1, 2)\ (1, -2)\ (-1, 2)\ (-1, -2)$$

To find maximum and minimum values

Points	$\boldsymbol{A = 6x}$	$\boldsymbol{C = 6y}$	$\boldsymbol{B = 0}$	$\boldsymbol{AC}$	$\boldsymbol{B^2}$	$\boldsymbol{AC - B^2}$	**Conclu sion**
(1, 2)	6 (+ ve)	12	0	72	0	72 (+ ve)	Minimum point
(1, − 2)	6 (+ ve)	− 12	0	− 72	0	− 72 (− ve)	Saddle
(− 1, 2)	− 6 (− ve)	12	0	− 72	0	− 72 (− ve)	Saddle
(− 1, − 2)	− 6 (− ve)	− 12	0	72	0	72 (+ ve)	Maximum point

Minimum value at $(1, 2) = 2$	Maximum value at $(-1, -2) = 38$

Example 2: Find maximum and minimum values of $f(x, y) = x^2 + y^2 + 6x + 12$

Solution:

$$f(x, y) = x^2 + y^2 + 6x + 12$$

$$f = x^2 + y^2 + 6x + 12$$

$$f_x = 2x + 0 + 6 + 0$$

$$f_y = 0 + 2y + 0 + 0$$

$A = f_{xx} = 2$

$B = f_{xy} = 0$

$C = f_{yy} = 2$

To find stationary points

Put $f_x = 0$ and $f_y = 0$

$f_x = 2x + 6$	$f_y = 2y$
$2x + 6 = 0$	$2y = 0$
$2x = -6$	$y = 0$
$x = -3$	

The Stationary point $(-3, 0)$

Point	$\boldsymbol{A = 2}$	$\boldsymbol{C = 2}$	$\boldsymbol{B = 0}$	$\boldsymbol{AC - B^2}$	**Conclusion**
$(-3, 0)$	2	2	0	4	Minimum point

$$\boxed{\textbf{Minimum value at } (-3, 0) = 3}$$

Example 3: Find the extreme value of the function $\boldsymbol{f(x, y) = 3x^2 - y^2 + x^3}$

Solution:

$$f(x, y) = 3x^2 - y^2 + x^3$$

$$f_x = 6x - 0 + 3x^2$$

$$f_y = 0 - 2y + 0$$

$A = f_{xx} = 6 + 6x$

$B = f_{xy} = 0$

$C = f_{yy} = -2$

To find stationary point

Put $f_x = 0$ and $f_y = 0$

$f_x = 6x + 3x^2$	$f_y = -2y$
$3x^2 + 6x = 0$	$-2y = 0$
$x_1 = 0$	$y = 0$
$x_2 = -2$	

The stationary points are (0, 0) (– 2, 0)

Point	$\boldsymbol{A = 6x + 6}$	$\boldsymbol{C = -2}$	$\boldsymbol{B = 0}$	$\boldsymbol{AC - B^2}$	**conclusion**
(0, 0)	6 (+ ve)	– 2	0	12 (– *ve*)	saddle point
(– 2, 0)	– 6 (– ve)	– 2	0	12 (+ ve)	Maximum point

Maximum value of (– 2, 0)

$$f(x, y) = 3x^2 - y^2 + x^3$$

$$f(-2, 0) = 3(-2)^2 - (0)^2 + (-2)^3$$

$$\boxed{\textbf{Maximum value } = 4}$$

Example 4: Discuss the maxima and minima of $\boldsymbol{x^3 y^2 (6 - x - y)}$

Solution:

$$f = x^3 y^3 (6 - x - y)$$

$$f = 6x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$f_x = 18x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$f_y = 12x^3 y - 2x^4 y - 3x^3 y^3$$

$$A = f_{xx} = 36xy^2 - 12x^2 y^2 - 6xy^2$$

$B = f_{xy} = 36x^2 y - 8x^3 y - 9x^2 y^2$

$C = f_{yy} = 12x^3 - 2x^4 - 6x^3 y$

To find stationary points, put $f_x = 0$ and $f_y = 0$

$f_x = 18x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$	$f_y = 12x^3 y - 2x^4 y - 3x^3 y^2 = 0$
$x^2 y^2 (18 - 4x - 3y) = 0$	$x^3 y (12 - 2x - 3y) = 0$
$x^2 y^2 = 0$	$x^3 y = 0$
$18 - 4x - 3y = 0$	$12 - 2x - 3y = 0$
$-4x - 3y = -18 \ldots (1)$	$-2x - 3y = -12 \ldots (2)$

Solve (1) & (2) we get (1)

$(1) \Rightarrow 4x + 3y = 18$

$(2) \Rightarrow 2x + 3y = 12$

$(1) - (2) \Rightarrow 2x = 6$

Put $x = 3$ in $(2) \Rightarrow 6 + 3y = 12$

$$3y = 12 - 6$$

$$y = 2$$

$$x = 3;\ y = 2$$

The stationary point is (3, 2)

Points	A	C	B	$AC - B^2$	Conclusion
(3, 2)	-144 ($-$ ve)	-162	-108	+ve	Maximum

Maximum value of (3, 2)

$$f(x, y) = 6x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$f(3, 2) = 6\,(3)^3\,(2)^2 - (3)^4\,(2)^2 - (3)^3\,(2)^3$$

$$\boxed{\textbf{Maximum value} = \textbf{108}}$$

Example 5: Discuss the maxima and minima of $x^3 y^2 (12 - x - y)$

Solution:

$$f = x^3 y^2 (12 - x - y)$$

$$f = 12x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$f_x = 36x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$f_y = 24x^3 y - 2x^4 y - 3x^3 y^2$$

$$A = f_{xx} = 72xy^2 - 12x^2 y^2 - 6xy^3$$

$$B = f_{xy} = 72x^2 y - 8x^3 y - 9x^2 y^2$$

To find stationary points, put $f_x = 0$ and $f_y = 0$

$f_x = 36x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$	$f_y = 24x^3 y - 2x^4 y - 3x^3 y^2$
$36x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$	$24x^3 y - 2x^4 y - 3x^2 y^2 = 0$
$x^2 y^2 (36 - 4x - 3y) = 0$	$x^3 y (24 - 2x - 3y) = 0$
$x^2 y^2 = 0$	$x^3 y = 0$
$36 - 4x - 3y = 0$	$24 - 2x - 3y = 0$
$-4x - 3y = -36$	$-2x - 3y = -24$
$4x + 3y = 36$ ... (1)	$2x + 3y = 24$ ... (2)

$(1) - (2) \Rightarrow 2x = 12$ $\therefore x = 6$

$(1) \Rightarrow (4)(6) + 3y = 36$

$$24 + 3y = 36$$

$$3y = 12 \quad \therefore \quad y = 4$$

$$\boxed{\begin{array}{c} \boldsymbol{x = 6} \\ \boldsymbol{y = 4} \end{array}}$$

The stationary point is (6, 4)

Points	A	C	B	AC	B^2	$AC - B^2$	conc
(6, 4)	− 2304 (− ve)	− 2592	− 1728	5971968	2985984	2485984 (+ ve)	Maximum point

Maximum value at (6, 4)

$f(x, y) = 12x^3 y^2 - x^4 y^2 - x^3 y^3$

$f(6, 4) = 12\,(6)^3\,(4)^2 - (6)^4\,(4)^2 - (6)^3\,(4)^3$

$$\boxed{\textbf{Maximum value } = \mathbf{6912}}$$

Example 6: Find the extreme values of

$$\boldsymbol{f = x^2 - xy + y^2 - 2x + y}$$

Solution:

$$f = x^2 - xy + y^2 - 2x + y$$

$$f_x = 2x - y + 0 - 2$$

$$f_y = 0 - x + 2y - 0 + 1$$

$$A = f_{xx} = 2 - 0 - 0 = 2$$

$$B = f_{xy} = 0 - 1 - 0 = -1$$

$$C = f_{yy} = 2 = 2$$

To find stationary points

Put $f_x = 0$ and $f_y = 0$

$f_x = 2x - y - 2$	$f_y = -x + 2y + 1$
$2x - y - 2 = 0$	$-x + 2y + 1 = 0$
$2x - y = 2 \ldots (1)$	$-x + 2y = -1 \ldots (2)$

Solve (1) & (2) we get

$$\boxed{\begin{aligned} \boldsymbol{x} &= \mathbf{1} \\ \boldsymbol{y} &= \mathbf{0} \end{aligned}}$$

The stationary point is (1, 0)

To find maximum and minimum values

Points	$A = 2$	$C = 2$	$B = -1$	AC	B^2	$AC - B^2$	Conclusion
(1, 0)	2 (+ ve)	2	− 1	4	1	3 (+ ve)	Minimum point

Minimum value at (1, 0)

$$f(x, y) = x^2 - xy + y^2 - 2x + y$$

$$f(1, 0) = (1)^2 - (1)(0) + (0)^2 - 2(1) + 0$$

$$\boxed{\textbf{Minimum value } = -1}$$

Example 7: Find maximum and minimum values of $\boldsymbol{f = xy(3 - x - y)}$

Solution:

$$f = 3xy - x^2 y - xy^2$$

$$f_x = 3y - 2xy - y^2$$

$$f_y = 3x - x^2 - 2xy$$

$$A = f_{xx} = -2y$$

$$B = f_{xy} = 3 - 2x - 2y$$

$$C = f_{yy} = -2x$$

To find stationary points

Put $f_x = 0$ and $f_y = 0$

$f_x = 3y - 2xy - y^2$	$f_y = 3x - x^2 - 2xy$
$3y - 2xy - y^2 = 0$	$3x - x^2 - 2xy = 0$
$y\,(3 - 2x - y) = 0$	$x\,(3 - x - 2y) = 0$
$3 - 2x - y = 0$	$-x - 2y = -3 \quad 3 - x - 2y = 0$
$-2x - y = -3 \ldots (1)$	$-x - 2y = -3 \ldots (2)$

Solve (1) & (2) we get

$x = 1\ ;\ y = 1$

The stationary point is (1, 1)

Point	A	C	B	AC	B^2	$AC - B^2$	conclusion
(1, 1)	-2 $-$ ve	-2	-1	4	1	3 + ve	Maximum point

Maximum value at (1, 1)

$$f\,(x, y) = 3xy - x^2\,y - xy^2$$

$$f\,(1, 1) = 3\,(1)\,(1) - (1)^2\,(1) - (1)\,(1)^2$$

$$\boxed{\textbf{Maximum value} = \mathbf{1}}$$

Example 8: Find maximum and minimum values of $x^3\,y^2\,(1 - x - y)$

Solution:

$$f = x^3\,y^2 - x^4\,y^2 - x^3\,y^3$$

$$f_x = 3x^2\,y^2 - 4x^3\,y^2 - 3x^2\,y^3$$

$$f_y = 2x^3\,y - 2x^4\,y - 3x^3\,y^2$$

$$A = f_{xx} = 6xy^2 - 12x^2\,y^2 - 6xy^3$$

$$B = f_{xy} = 6x^2\,y - 8x^3\,y - 9x^2\,y^2$$

$$C = f_{yy} = 2x^3 - 2x^4 - 6x^3\,y$$

To find stationary point

Put $f_x = 0$ and $f_y = 0$

$$f_x = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3$$

$$3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$$

$$x^2 y^2 (3 - 4x - 3y) = 0$$

$$3 - 4x - 3y = 0$$

$$-4x - 3y = -3 \quad \ldots (1)$$

$$f_y = 2x^3 y - 2x^4 y - 3x^3 y^2$$

$$2x^3 y - 2x^4 y - 3x^3 y^2$$

$$x^3 y (2 - 2x - 3y) = 0$$

$$2 - 2x - 3y = 0$$

$$-2x - 3y = -2 \quad \ldots (2)$$

Solve (1) & (2) we get

$$x = 1/2 \; ; \; y = 1/3$$

The stationary point is $\left(\frac{1}{2}, \frac{1}{3}\right)$

Point	A	C	B	AC	B^2	$AC - B^2$	Conclu sion
$\left(\frac{1}{2}, \frac{1}{3}\right)$	-0.11 (– ve)	-0.125	-0.08	0.01375	0.0064	0.00735 + ve	Maximum point

Maximum value at $\left(\frac{1}{2}, \frac{1}{3}\right)$

$$f(x, y) = x^3 y^2 - x^4 y^2 - x^3 y^3$$

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^4 \left(\frac{1}{3}\right)^2 - \left(\frac{1}{2}\right)^2 \left(\frac{1}{3}\right)^2$$

$$\boxed{\textbf{Maximum value} = \frac{1}{432}}$$

Example 9: Find the maximum and minimum values of $f(x, y) = x^3 + y^3 - 12x - 3y + 20$

Solution:

$$f = x^3 + y^3 - 12x - 3y + 20$$

$$f_x = 3x^2 - 12$$

$$f_y = 3y^2 - 3$$

$$A = f_{xx} = 6x$$

$$B = f_{xy} = 0$$

$$C = f_{yy} = 6y$$

To find stationary point

Put $f_x = 0$ and $f_y = 0$

$3x^2 - 12 = 0$	$3y^2 - 3 = 0$
$3x^2 = 12$	$3y^2 = 3$
$x^2 = 4$	$y^2 = 1$
$x_2 = 2,\ x_2 = -2$	$y_2 = -1\ ;\ y_2 = -1$

The stationary points are

$$(2, 1)\ (-2, -1)\ (2, -1)\ (-2, 1)$$

Points	A	C	B	AC	B^2	$AC - B^2$	Conclusion
(2, 1)	12 + ve	6	0	72	0	72 + ve	Minimum point
(− 2, 1)	− 12 − ve	6	0	− 72	0	− 72 − ve	Saddle point
(− 2, − 1)	− 12 − ve	− 6	0	72	0	72 + ve	Maximum point
(2, − 1)	12 +ve	− 6	0	− 72	0	− 72 –ve	Saddle point

Minimum value at (2, 1)

$$f(x, y) = x^3 + y^3 - 12x - 3y + 20$$

$$f(2, 1) = 2^3 + 1^3 - 12\,(2) - 3\,(1) + 20$$

$$= 2$$

$$\boxed{\textbf{Minimum value } = \mathbf{2}}$$

Maximum value at (– 2, – 1)

$$f(-2, -1) = (-2)^3 + (-1)^3 - 12\,(-2) - 3\,(-1) + 20 = 38$$

$$\boxed{\textbf{Maximum value } = \mathbf{38}}$$

Example 10: Find the extreme values of function
$$\boldsymbol{f(x, y) - x^4 + y^4 - 2x^2 + 4xy - 2y^2.}$$

Solution:

$$f = x^4 + y^4 - 2x^2 + 4xy - 2y^2 \qquad (*)$$

$$f_x = 4x^3 - 4x + 4y$$

$$f_y = 4y^3 + 4x - 4y$$

$$A = f_{xx} = 12x^2 - 4$$

$$B = f_{xy} = 4$$

$$C = f_{yy} = 12y^2 - 4$$

To find stationary points

Put $f_x = 0$ and $f_y = 0$

$$f_x = 4x^3 - 4x + 4y$$

$$4x^3 - 4x + 4y = 0 \ \ldots (1)$$

$$f_y = 4y^3 + 4x - 4y$$

$$4y^3 + 4x - 4y = 0 \ \ldots (2)$$

Adding (1) and (2) we get

$4x^3 - 4x + 4y = 0$ and $4y^3 + 4x - 4y = 0$

$$4x^3 + 4y^3 = 0$$

$$\because \begin{array}{r} 4x^3 - 4x + 4y = 0 \\ 4y^3 + 4x + 4y = 0 \\ \hline 4x^3 + 4y^3 = 0 \\ \hline \end{array}$$

$$4x^3 = 4y^3$$

$$x^3 = -y^3$$

$$\boxed{\boldsymbol{x = -y}}$$

Put $\quad x = -y$ in equation (1)

$4(-y)^3 - 4(-y) + 4y = 0$

$\div 4 \qquad (-y)^3 + 4 + y = 0$

$$-y^3 + 2y = 0$$

$$y(-y^2 + 2) = 0$$

$$y = 0, \qquad -y^2 + 2 = 0$$

$$-y^2 = -2$$

$$y^2 = 2$$

$$\boxed{\boldsymbol{y = \pm\sqrt{2};\ y = \sqrt{2},\ y = -\sqrt{2}}}$$

$\Rightarrow x = -y \quad x = \sqrt{2}, x = -\sqrt{2}$

The stationary points are

$(-\sqrt{2}, \sqrt{2}), (-\sqrt{2}, -\sqrt{2}), (\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2})\ (0, 0)$

Point	A	C	B	AC	B^2	$AC - B^2$	Conclusion
(0, 0)	-4	-4	4	16	16	0	Non-conclusive
$(-\sqrt{2}, \sqrt{2})$	20	20	4	400	16	384	Minimum point
$(\sqrt{2}, -\sqrt{2})$	20	20	4	400	16	384	Minimum point

Minimum value

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\boxed{\boldsymbol{f(-\sqrt{2}, \sqrt{2}) = f(\sqrt{2}, -\sqrt{2}) = -8}}$$

Example 11: Find the extreme values of

$$\boldsymbol{f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x}$$

Solution:

$$f = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$f_x = 3x^2 + 3y^2 - 30x + 72$$

$$f_y = 6xy - 30y$$

$$A = f_{xx} = 6x - 30$$

$$B = f_{xy} = 6y$$

$$C = f_{yy} = 6x - 30$$

To find stationary points

Put $f_x = 0$ and $f_y = 0$

$$f_x = 3x^2 + 3y^2 - 30x + 72$$

$$3x^2 + 3y^2 - 30x + 72 = 0 \ \dots (1)$$

Put $y = 0$ in (1)

$$3x^2 + 3(0)^2 - 30x + 72 = 0$$

$$3x^2 - 30x + 72 = 0$$

$$f_y - 6xy - 30y$$

$$6xy - 30y = 0$$

$$6y(x - 5) = 0$$

$$6y = 0 \quad x - 5 = 0$$

$$\boxed{\boldsymbol{y = 0}} \quad \boxed{\boldsymbol{x = 5}}$$

$$x^2 - 10x + 24$$

$$(x - 6)(x - 4) = 0$$

$$\therefore \ \boldsymbol{x_1 = 6, \ x_2 = 4}$$

Put $x = 5$

$$3(5)^2 + 3y^2 + 30(5) + 72 = 0$$

$$75 + 3y^2 - 150 + 72 = 0$$

$$3y^2 - 3 = 0$$

$$3y^2 = 3$$

$$y^2 = 1$$

$$y = \pm 1$$

$\therefore$ The stationary points are $(6, 0)$ $(4, 0)$ $(5, 1)$ $(5, -1)$

Points	A	C	B	AC	B^2	$AC - B^2$	Conclusion
(6, 0)	6	6	0	36	0	36	Minimum point
(4, 0)	– 6	– 6	0	36	0	36	Maximum point
(5, 1)	0	0	6	0	36	– 36	Saddle point
(5, – 1)	0	0	– 6	0	+ 36	– 36	Saddle point

Minimum value at $(6, 0) = 108$	Maximum value at $(4, 0) = 112$

3.7 CONSTRAINED MAXIMA AND MINIMA BY LAGRANGIAN MULTIPLIER METHOD

Let us consider a function of three variables $f(x, y, z)$ where these variables are connected by the number of relations

$$\phi(x, y, z) = 0 \qquad \ldots\ldots (1)$$

where the number of relations is less than the number of independent variables.

For $f(x, y, z)$ to have a maximum or minimum value the necessary condition is

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0, \ \frac{\partial f}{\partial z} = 0$$

$$\therefore \ \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 0 \qquad \ldots\ldots (2)$$

From (1) taking differentials, we get

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \ \ldots\ldots \ (3)$$

Consider, (2) + λ(3), where λ is a parameter

then,

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

This equation will hold good if

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

These equations together with equation $\phi(x, y, z) = 0$ gives the values of x, y, z and λ for a maximum or minimum.

3.7.1 Working Rule For Lagrange's Method of Multipliers

1. Assume $u(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$ where λ is an undetermined constant.
2. Find the partial derivatives of u and equate each of them to zero. *i.e.*, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial u}{\partial z} = 0$
3. Solve the above equations together with $\phi(x, y, z) = 0$ and find the values of the four quantities x, y, z and λ.

[Method of Lagrange's Multipliers]
Constraint Maxima and Minima

- To find Extremum (Maximum or Minimum)
  Put $F_x = 0, F_y = 0, F_z = 0$
- $\boldsymbol{F = f + \lambda g}$

where f is the function

3.7.2 Problems on Constraint maxima and minima

Example 1: Find the extreme value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$

Solution:

$$F = f + \lambda\, g \qquad \text{... (1)}$$

$$f = x^2 + y^2 + z^2$$

$$g = x + y + z - 3a$$

$$F = (x^2 + y^2 + z^2) + \lambda\,(x + y + z - 3a) \qquad \text{... (2)}$$

$$F_x = 2x + \lambda$$

$$F_y = 2y + \lambda$$

$$F_z = 2z + \lambda$$

Put $\quad F_x = 0,\ F_y = 0,\ F_z = 0$

$$2x + \lambda = 0 \;\Rightarrow\; 2x = -\lambda \;\Rightarrow\; x = \frac{-\lambda}{2}$$

$$2y + \lambda = 0 \;\Rightarrow\; y = -\frac{\lambda}{2}$$

$$2z + \lambda = 0 \;\Rightarrow\; z = \frac{-\lambda}{2}$$

$$x = y = z = \frac{-\lambda}{2}$$

$\Rightarrow$ $\boldsymbol{x = y = z}$

$$g = x + y + z - 3a$$

$$g = x + x + x - 3a$$

$$g = 3x - 3a$$

Put $g = 0$

$$3x - 3a = 0$$

$$3x = 3a$$

$$\boxed{\boldsymbol{x = a}}$$

$$f = a^2 + a^2 + a^2$$

$$\boxed{\boldsymbol{f = 3a^2}}$$

$$\boxed{\therefore \textbf{ The Extremum value is } \boldsymbol{3a^2}}$$

Example 2: Find the minimum value $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Solution:

$$F = f + \lambda g \qquad \text{... (1)}$$

$$f = x^2 + y^2 + z^2$$

$$g = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

$$F = (x^2 + y^2 + z^2) + \lambda \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 \right)$$

$$F_x = 2x + \lambda \left(-\frac{1}{x^2} \right) = 2x - \frac{\lambda}{x^2}$$

$$F_y = 2y + \lambda \left(-\frac{1}{y^2} \right) = 2y - \frac{\lambda}{y^2}$$

$$F_z = 2z + \lambda \left(-\frac{1}{z^2} \right) = 2z - \frac{\lambda}{z^2}$$

To find minimum values

Put $F_x = 0, F_y = 0, F_z = 0$

$$2x-\frac{\lambda}{x^2}=0 \Rightarrow 2x=\frac{\lambda}{x^2} \Rightarrow x^3=\frac{\lambda}{2}$$

$$2y-\frac{\lambda}{y^2}=0 \Rightarrow y^3=\frac{\lambda}{2}$$

$$2z-\frac{\lambda}{z^2}=0 \Rightarrow z^3=\frac{\lambda}{2}$$

$$x^3=y^3=z^3=\frac{\lambda}{2}$$

$$\Rightarrow x=y=z$$

$$g=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$$

$$g=\frac{1}{x}+\frac{1}{x}+\frac{1}{x}-1$$

Put $g=0$

$$\frac{1}{x}+\frac{1}{x}+\frac{1}{x}-1=0$$

$$\frac{3}{x}-1=0$$

$$\frac{3}{x}=1$$

$$\boxed{\boldsymbol{x=3}}$$

$$f=3^2+3^2+3^2$$

$$f=9+9+9$$

$$\boxed{\boldsymbol{f=27}}$$

$$\boxed{\therefore\ \textbf{The minimum value is 27}}$$

Example 3: Find the maximum and minimum value of $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$

Solution:

$$F = f + \lambda g \qquad \text{... (1)}$$

$$f = 3x + 4y$$

$$g = x^2 + y^2 - 1$$

$$F = 3x + 4y + \lambda (x^2 + y^2 - 1)$$

$$F_x = 3 + \lambda (2x) = 3 + 2\lambda x$$

$$F_y = 4 + \lambda (2y) = 4 + 2\lambda y$$

To find maximum and minimum value

Put $F_x = 0,\ F_y = 0$

$$\Rightarrow F_x = 3 + 2\lambda x$$

$$3 + 2\lambda x = 0$$

$$2\lambda x = -3$$

$$x = \frac{-3}{2\lambda}$$

$$\Rightarrow 2\lambda = \frac{-3}{x}$$

$$\Rightarrow F_y = 4 + 2\lambda y$$

$$4 + 2\lambda y = 0$$

$$2\lambda y = -4$$

$$y = \frac{-4}{2\lambda}$$

$$\Rightarrow 2\lambda = \frac{-4}{y}$$

$$\frac{-3}{x} = \frac{-4}{y} = 2\lambda$$

$$3y = 4x$$

$$\Rightarrow \boldsymbol{y = \frac{4x}{3}}$$

$$g = x^2 + y^2 - 1$$

$$g = x^2 + \left(\frac{4x}{3}\right)^2 - 1$$

$$g = x^2 + \frac{16x^2}{9} - 1$$

Put $g = 0$

$$x^2 + \frac{16x^2}{9} - 1 = 0$$

$$\frac{9x^2 + 16x^2 - 9}{9} = 0$$

$$9x^2 + 16x^2 - 9 = 0$$
$$25x^2 - 9 = 0$$
$$25x^2 = 9$$
$$x^2 = \frac{9}{25}$$
$$\Rightarrow \boldsymbol{x = \pm \frac{3}{5}}$$

$$\Rightarrow y = \frac{4x}{3}$$
$$y = \frac{4\left(\frac{3}{5}\right)}{3}$$
$$y = \frac{12}{5} \times \frac{1}{3}$$
$$y = \frac{4}{5}$$

$$f = 3x + 4y$$
$$= 3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right)$$
$$f = \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5$$

Example 4: Find the extreme value of the function $\boldsymbol{x^3 + y^3 + z^3}$ subject to the condition $\boldsymbol{x + y + z = 24}$

Solution:

$$F = f + \lambda\, g \qquad \text{... (1)}$$

$$f = x^3 + y^3 + z^3$$

$$g = x + y + z - 24$$

$$\Rightarrow\ F = x^3 + y^3 + z^3 + \lambda\,(x + y + z - 24)$$

$$F_x = 3x^2 + \lambda$$

$$F_y = 3y^2 + \lambda$$

$$F_z = 3z^2 + \lambda$$

To find extreme value

Put $F_x = 0,\ F_y = 0,\ F_z = 0$

$$3x^2 + \lambda = 0 \ \Rightarrow\ 3x^2 = -\lambda \ \Rightarrow\ x^2 = \frac{-\lambda}{3}$$

$$3y^2 + \lambda = 0 \ \Rightarrow\ 3y^2 = -\lambda \ \Rightarrow\ y^2 = -\frac{\lambda}{3}$$

$$3z^2 + \lambda = 0 \ \Rightarrow\ 3z^2 = -\lambda \ \Rightarrow\ z^2 = -\frac{\lambda}{3}$$

$x^2 = y^2 = z^2 = \frac{-\lambda}{3}$	$3x^2 = 3y^2 = 3z^2 = -\lambda$
$x = y = z = -\frac{\lambda}{3}$	$x = y = z = -\lambda$
$\Rightarrow\ x = y = z$	

$$\Rightarrow\ g = x + y + z - 24$$

$$= x + x + x - 24$$

Put $g = 0$, $3x - 24 = 0$

$$3x = 24$$

$$x = 8$$

$$\Rightarrow\ f = x^3 + y^3 + z^3$$

$$= 8^3 + 8^3 + 8^3$$

$$\boxed{\boldsymbol{f = 1536}}$$

Example 5: Find the minimum values of $x^2 yz^3$ subject to the condition $2x + y + 3z = a$.

Solution:

$$F = f + \lambda g$$

$$f = x^2 yz^3$$

$$g = 2x + y + 3z - a$$

$$F = x^2 yz^3 + \lambda (2x + y + 3z - a)$$

$$F_x = 2xyz^3 + \lambda (2)$$

$$F_y = x^2 z^3 + \lambda (1)$$

$$F_z = 3x^2 yz^2 + 3 \lambda$$

To find the minimum value

Put $F_x = 0, F_y = 0, F_z = 0$

$$2xyz^3 + 2\lambda = 0 \Rightarrow 2\lambda = -2xyz^3 \Rightarrow \boldsymbol{\lambda = -xyz^3}$$

$$x^2 z^3 + \lambda = 0 \Rightarrow \boldsymbol{\lambda = -x^2 z^3}$$

$$3x^2 y z^2 + 3\lambda = 0 \Rightarrow 3\lambda = -3 x^2 y z^2$$

$$\lambda = -x^2 yz^2$$

$$-xyz^3 = -x^2 z^3 = -x^2 yz^2 = \lambda$$

$$xyz^3 = x^2 z^3$$

$$y = \frac{x^2 z^3}{xz^3}$$

$$\boldsymbol{y = x}$$

$$x^2 z^3 = x^2 yz^2$$

$$x^2 yz^2 = x^2 z^3$$

$$y = \frac{x^2 z^3}{x^2 z^2} \Rightarrow \boldsymbol{y = z}$$

$$\Rightarrow x = y = z$$

$$g = 2x + y + 3z - a$$

$$g = 2x + x + 3x - a$$

$$g = 6x - a$$

Put $g = 0$

$$6x - a = 0$$

$$6x = a$$

$$\boxed{\boldsymbol{x = \frac{a}{6}}}$$

$\Rightarrow f = x^2 yz^3$

$$f = \left(\frac{a}{6}\right)^2 \left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3 = \left(\frac{a}{6}\right)^6$$

Example 6: A rectangular box open at the top is to have a volume of 32 cc. Find the dimension of the box requiring the least material for the construction.

Solution:

$$F = f + \lambda\, g \qquad \text{... (1)}$$

$$f = xy + 2yz + 2zx$$

$$g = xyz - 32$$

$$F = xy + 2yz + 2zx + \lambda\,(xyz - 32)$$

$$F_x = y + 2z + \lambda\,(yz)$$

$$F_y = x + 2z + \lambda\,(xz)$$

$$F_x = 2y + 2x + \lambda\,(xy)$$

Put $F_x = 0,\ F_y = 0,\ F_z = 0$

$$y + 2z + \lambda\, yz = 0 \qquad \text{... (i)}$$

$$x + 2z + \lambda\, xz = 0 \qquad \text{... (ii)}$$

$$2y + 2x + \lambda\, xy = 0 \qquad \text{... (iii)}$$

(i) $\times x \Rightarrow xy + 2xz + \lambda\, xyz = 0$

(ii) $\times y \Rightarrow xy + 2yz + \lambda\, xyz = 0$

(iii) $\times z \Rightarrow 2yz + 2xz + \lambda\, xyz = 0$

(i) − (ii)

$$xy + 2xz + \lambda\, xyz - (xy + 2yz + \lambda\, xyz)$$

$$2xz - 2zy = 0$$

$$2zx = 2zy$$

$$\boxed{x = y}$$

(i) − (iii)

$$xy + 2xz + \lambda\, xyz - (2yz + 2xz + \lambda\, xyz)$$

$$xy - 2yz = 0$$

$$y\,(x - 2z) = 0$$

$$y = 0 \quad ; \qquad x - 2z = 0$$

$$\boxed{x = 2z}$$

$\Rightarrow g = xyz - 32$

$g = (2z)\,(2z)\,(z) - 32$

$g = 4z^3 - 32$

Put $g = 0$

$4z^3 - 32 = 0$

$4z^3 = 32$

$z^3 = \frac{32}{4}$

$z^3 = 8 \quad z^3 = 2^3$

$$\boxed{z = 2}$$

$\Rightarrow \; x = 2z$

$x = 2\,(2)$

$x = 4$

$y = 4$

$\therefore$ The dimensions are

$$\boldsymbol{x = 4,\; y = 4,\; z = 2}$$

Example 7: Find the dimension of the rectangular box without top of with maximum capacity whose surface area is 432 sq.m.

Solution:

$$F = f + \lambda\, g \qquad \text{... (1)}$$

$$f = xyz$$

$$g = xy + 2yz + 2zx - 432$$

$$F = xyz + \lambda\,(xy + 2yz + 2zx - 432)$$

$$F_x = yz + \lambda\,(y + 2z)$$

$$F_y = xz + \lambda\,(x + 2z)$$

$$F_z = xy + \lambda\,(2y + 2x)$$

Put$F_x = 0,\;\; F_y = 0,\;\; F_z = 0$

$$yz + \lambda\, y + 2\lambda\, z = 0 \qquad \text{... (1)}$$

$$xz + \lambda\, x + 2\lambda\, z = 0 \qquad \text{... (ii)}$$

$$xy + 2y\,\lambda + 2\lambda\, x = 0 \qquad \text{... (iii)}$$

(i) $\times\; x \;\Rightarrow\; xyz + \lambda\, xy + 2\,\lambda\, xz = 0$

(ii) $\times\, y \;\Rightarrow\; xyz + \lambda\, xy + 2\lambda\, yz = 0$

(iii) $\times\, z \;\Rightarrow\; xyz + 2\,\lambda\, yz + 2x\,\lambda\, z = 0$

(i) − (ii)

$$xy\cancel{z} + \lambda\, x\cancel{y} + 2\,\lambda\, xz = 0$$

$$(-)\ \cancel{x}yz + \cancel{\lambda}\, xy + 2\,\lambda\, yz = 0$$

$$2\,\lambda\, xz - 2\,\lambda\, yz = 0$$

$$2\,\lambda\, z\,(x - y) = 0$$

$$2\,\lambda\, z = 0 \quad x - y = 0$$

$$\boxed{\boldsymbol{x = y}}$$

(i) − (iii)

$$xyz + \lambda\, xy + 2\,\lambda\, xz = 0$$

$$(-)\ \ xyz + 2\,\lambda\, yz + 2x\lambda z = 0$$

$$\lambda\, xy - 2\,\lambda\, yz = 0$$

$$\lambda\, y\,(x - 2z) = 0$$

$$\lambda\, y = 0\,; \qquad x - 2z = 0$$

$$\boxed{\boldsymbol{x = 2z}}$$

$\Rightarrow\ x = y = 2z$

$g = xy + 2yz + 2zx - 432$

$g = (2z)\,(2z) + 2\,(2z)\,(z) + 2\,(z)\,(2z) - 432$

Put $\qquad g = 0$

$$4z^2 + 4z^2 + 4z^2 - 432 = 0$$

$$12z^2 - 43z = 0$$

$$12z^2 = 432$$

$$z^2 = \frac{432}{12}$$

$$z^2 = 36 \quad z^2 = 6^2$$

$$\boxed{z = 6}$$

$\therefore$ **The dimensions are**

lenth, $x = 12$

breadth, $y = 12$

height, $z = 6$

Example 8: The temperature at any point (x, y, z) in space is given by $T = 400xyz^2$. Find the maximum temperature on the surface of the units sphere $x^2 + y^2 + z^2 = 1$.

Solution:

$$F = f + \lambda\, g$$

$$f = 400xyz^2 \qquad \therefore \quad g = x^2 + y^2 + z^2 - 1$$

$$F = 400xyz^2 + \lambda\,(x^2 + y^2 + z^2 - 1)$$

$$F_x = 400yz^2 + 2\,\lambda\,x$$

$$F_y = 400xz^2 + 2\,\lambda\,y$$

$$F_z = 800xyz + 2\,\lambda\,z$$

Put $F_x = 0,\; F_y = 0,\; F_z = 0$

$$400yz^2 + 2\,\lambda\,x = 0 \qquad \ldots \text{(i)}$$

$$400xz^2 + 2\,\lambda\,y = 0 \qquad \ldots \text{(ii)}$$

$$800\,xyz + 2\,\lambda\,z = 0 \qquad \ldots \text{(iii)}$$

Adding (i), (ii), (iii)

(i) $\times x \;\Rightarrow\; 400xyz^2 + 2\,\lambda\,x^2 = 0$

(ii) $\times y \;\Rightarrow\; 400xyz^2 + 2\,\lambda\,y^2 = 0$

(iii) $\times z \;\Rightarrow\; 800xyz^2 + 2\,\lambda\,z^2 = 0$

$$1600xyz^2 + 2\lambda(x^2 + y^2 + z^2) = 0$$

$$1600xyz^2 + 2\lambda(1) = 0$$

$$2\lambda = -1600xyz^2$$

$$\boxed{\lambda = -800xyz^2}$$

(i) $\Rightarrow 2\lambda x = -400yz^2$

$$x = \frac{-400yz^2}{2\lambda}$$

$$x = \frac{-400yz^2}{2(-800xyz^2)} = \frac{-400yz^2}{-1600xyz^2}$$

$$x = \frac{1}{4x}$$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4} \Rightarrow \boldsymbol{x = \frac{1}{2}}$$

(ii) $2\lambda y = -400xz^2$

$$2y = \frac{-400xz^2}{\lambda} \Rightarrow 2y = \frac{-400xz^2}{-800xyz^2}$$

$$2y = \frac{1}{2y}$$

$$4y^2 = 1 \Rightarrow y^2 = \frac{1}{4} \Rightarrow \boldsymbol{y = \frac{1}{2}}$$

(iii) $\Rightarrow 2\lambda z^2 = -800xyz$

$$2z = \frac{-800xyz}{-800xyz^2}$$

$$2z = \frac{1}{z}$$

$$2z^2 = 1 \Rightarrow z^2 = \frac{1}{2} \Rightarrow \boldsymbol{z = \frac{1}{\sqrt{2}}}$$

$$\Rightarrow f = 400xyz^2$$

$$= 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{400}{2 \times 2 \times 2}$$

Maximum temperature, $\boxed{f = 50}$ **units**

Example 9: Show that maximum volume of a rectangular parallelopiped enclosed in the ellipse side $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$

Solution:

$$F = f + \lambda\, g \qquad \text{... (1)}$$

Volume of rectangular parallelopiped

$$f = 8xyz$$

$$g = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$F_x = 8yz + \lambda\left(\frac{2x}{a^2}\right)$$

$$F_y = 8xz + \lambda\left(\frac{2y}{b^2}\right)$$

$$F_z = 8xy + \lambda\left(\frac{2z}{c^2}\right)$$

Put $\quad F_x = 0,\ F_y = 0,\ F_z = 0$

$$8yz + \frac{2\lambda x}{a^2} = 0 \quad \text{... (i)}$$

$$8xz + \frac{2\lambda y}{b^2} = 0 \quad \text{... (ii)}$$

$$8xy + \frac{2\lambda z}{c^2} = 0 \quad \text{... (iii)}$$

(i) $\times x \Rightarrow 8xyz + \frac{2\lambda x^2}{a^2} = 0$

(ii) $\times y \Rightarrow 8xyz + \frac{2\lambda y^2}{b^2} = 0$

(iii) $\times z \Rightarrow 8xyz + \frac{2\lambda z^2}{c^2} = 0$

Adding (1), (ii), (iii) we get,

$$8xyz + \frac{2\lambda x^2}{a^2} = 0$$

$$8xyz + \frac{2\lambda y^2}{b^2} = 0$$

$$8xyz + \frac{2\lambda z^2}{c^2} = 0$$

$$\Rightarrow 24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$24xyz + 2\lambda\,(1) = 0$$

$$24xyz = -2\lambda$$

$$12xyz = -\lambda$$

$$\lambda = -12xyz$$

Sub λ value in eqn (i)

$$\frac{2\lambda x}{a^2} = -8yz$$

$$2x = \frac{-8yza^2}{\lambda}$$

$$2x = \frac{-8yza^2}{-12xyz}$$

$$2x = \frac{2a^2}{3x}$$

$$6x^2 = 2a^2$$

$$3x^2 = a^2$$

$$x^2 = \frac{a^2}{3}$$

$$\boxed{\boldsymbol{x = \pm \frac{a}{\sqrt{3}}}}$$

(ii) $\dfrac{2\lambda y}{b^2} = -8xz$

$$2y = \frac{-8xzb^2}{\lambda}$$

$$2y = \frac{-8xzb^2}{-12xyz}$$

$$2y = \frac{+2b^2}{3y}$$

$$6y^2 = 2b^2$$

$$y^2 = \frac{b^2}{3}$$

$$\boxed{\boldsymbol{y = \pm \frac{b}{\sqrt{3}}}}$$

Similarly

$$\Rightarrow \; z = \pm \frac{c}{\sqrt{3}}$$

$$f = 8xyz$$

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$\boxed{f = \frac{8abc}{3\sqrt{3}} \text{ cubic units}}$$

Example 10: Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Solution:

$$F = f + \lambda\, g \qquad \text{... (i)}$$

$$f = 8xyz$$

$$g = x^2 + y^2 + z^2 - R^2$$

$$F = 8xyz + \lambda\,(x^2 + y^2 + z^2 - R^2)$$

$$F_x = 8yz + \lambda\,(2x)$$

$$F_y = 8xz + \lambda\,(2y)$$

$$F_z = 8xy + \lambda\,(2z)$$

Put $\quad F_x = 0,\; F_y = 0,\; F_z = 0$

$$8yz + 2\,\lambda\, x = 0 \qquad \text{... (i)}$$

$$8xz + 2\,\lambda\, y = 0 \qquad \text{... (ii)}$$

$$8\, xy + 2\,\lambda\, z = 0 \qquad \text{... (iii)}$$

(i) $\times x \;\Rightarrow\; 8xyz + 2\,\lambda\, x^2 = 0$

(ii) $\times y \;\Rightarrow\; 8xyz + 2\,\lambda\, y^2 = 0$

(iii) $\times z \;\Rightarrow\; 8xyz + 2\,\lambda\, z^2 = 0$

$$2\lambda x^2 = -8xyz$$

$$2\lambda y^2 = -8xyz$$

$$2\lambda z^2 = -8xyz$$

$$\Rightarrow 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2 = -8xyz$$

$$x^2 = y^2 = z^2$$

$$\boxed{\boldsymbol{x = y = z}}$$

$$\Rightarrow g = x^2 + y^2 + z^2 - R^2$$

Put $g = 0$

$$x^2 + x^2 + x^2 - R^2 = 0$$

$$3x^2 - R^2 = 0$$

$$3x^2 = R^2$$

$$x^2 = \frac{R^2}{3}$$

$$\boxed{\boldsymbol{x = \pm \frac{R}{\sqrt{3}}}}$$

Similarly,

$$y = \pm \frac{R}{\sqrt{3}}, \quad z = \pm \frac{R}{\sqrt{3}}$$

$$\Rightarrow f = 8xyz$$

$$f = 8\left(\frac{R}{\sqrt{3}}\right)\left(\frac{R}{\sqrt{3}}\right)\left(\frac{R}{\sqrt{3}}\right)$$

$$\boxed{\boldsymbol{f = \frac{8R^3}{3\sqrt{3}} \text{ units}}}$$

Example 11: Find the maximum value of $x^m y^n z^p$ subject to the condition $x + y + z = a$

Solution:

$$F = f + \lambda\, g \qquad \text{... (i)}$$

$$f = x^m y^n z^p$$

$$g = x + y + z - a$$

$$F = x^m y^n z^p + \lambda\,(x + y + z - a)$$

$$F_x = mx^{m-1} y^n z^p + \lambda\,(1)$$

$$F_y = x^m ny^{n-1} z^p + \lambda\,(1)$$

$$F_z = x^m y^m pz^{p-1} + \lambda\,(1)$$

To find maximum value

Put $F_x = 0,\ F_y = 0,\ F_z = 0$

$$m\frac{x^m}{x} y^n z^p + \lambda = 0$$

$$x^m n\frac{y^n}{y} z^p + \lambda = 0$$

$$x^m y^n p\frac{z^p}{z} + \lambda = 0$$

$$m\frac{x^m}{x} y^n z^p = -\lambda$$

$$x^m n\frac{y^n}{y} z^p = -\lambda$$

$$x^m y^n p\frac{z^p}{z} = -\lambda$$

$$m\frac{x^m}{x} y^n z^p = x^m n\frac{y^n}{y} z^p = x^m y^n p\frac{z^p}{z} = -\lambda$$

$$\frac{m}{x}=\frac{n}{y}=\frac{p}{z}=\frac{m+n+p}{x+y+z}$$

$$\frac{m}{x}=\frac{n}{y}=\frac{p}{z}=\frac{m+n+p}{a}$$

$\frac{m}{x}=\frac{m+n+p}{a}$	$\frac{n}{y}=\frac{m+n+p}{a}$	$\frac{p}{z}=\frac{m+n+p}{a}$
$ma=(m+n+p)\,x$	$na=(m+n+p)\,y$	$pa=(m+n+p)\,z$
$x=\frac{ma}{m+n+p}$	$y=\frac{na}{m+n+p}$	$z=\frac{pa}{m+n+p}$

$$\text{Maximum value }=\left(\frac{ma}{m+n+p}\right)^m\left(\frac{na}{m+n+p}\right)^n\left(\frac{pa}{m+n+p}\right)^p$$

3.8 JACOBIANS

Definition: If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called by Jacobian of u and v with respect to x and y and it is denoted by $\frac{\partial\,(u\,,v)}{\partial\,(x\,,y)}$ or $\mathrm{T}\left(\frac{u\,,v}{x\,,y}\right)$.

Similarly, if $u\,,v\,,w$ are functions of three independent variables $x\,,y$ and z, then the Jacobian of $u\,,v\,,w$ with respect to $x\,,y\,,z$ is defined by

$$\mathrm{J}\left(\frac{u\,,v\,,w}{x\,,y\,,z}\right)=\frac{\partial\,(u\,,v\,,w)}{\partial\,(x\,,y\,,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

- Jacobian is denoted by the J
- If $J=0$, then functionally dependent

- If $J \neq 0$, then functionally independent
- $J \times J' = 1$

$$\text{(ie)} \quad J \times \frac{1}{J} = 1$$

3.8.1 Problems on Jacobians

Example 1: If $x = r\cos\theta, y = r\sin\theta$. Evaluate $\frac{\partial(x, y)}{\partial(r, \theta)}$.

Solution:

$$J = \frac{\partial(x, y)}{\partial(r, \theta)}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$	$y = r\sin\theta$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = -r\sin\theta$	$\frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r[\cos^2\theta + \sin^2\theta]$$

$$= r(1)$$

$$\boxed{J = r}$$

Example 2: **If $x = u(1-v)$ and $y = 2uv$ find $\dfrac{\partial(x, y)}{\partial(u, v)}$.**

Solution:

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = u(1-v) \qquad\qquad y = 2uv$$

$$\frac{\partial x}{\partial u} = 1 - v \qquad\qquad \frac{\partial y}{\partial u} = 2v$$

$$\frac{\partial x}{\partial v} = u(0-1) \qquad\qquad \frac{\partial y}{\partial v} = 2u$$

$$= -u$$

$$J = \begin{vmatrix} 1-v & -u \\ 2v & 2u \end{vmatrix}$$

$$= 1 - v(2u) + 2uv$$

$$= 2u - 2uv + 2uv$$

$$\boxed{J = 2u}$$

Example 3: **Find $u = \dfrac{y^2}{x}, v = \dfrac{x^2}{y}$ find $\dfrac{\partial(u, v)}{\partial(x, y)}$ and $\dfrac{\partial(x, y)}{\partial(u, v)}$.**

Solution:

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$u = \frac{y^2}{x} = y^2 \cdot \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = y^2 \cdot \left(\frac{-1}{x^2} \right) = -\frac{y^2}{x^2}$$

$$\frac{\partial u}{\partial y} = 2y \cdot \frac{1}{x} = \frac{2y}{x}$$

$$v = \frac{x^2}{y} = x^2 \cdot \frac{1}{y}$$

$$\frac{\partial v}{\partial x} = 2x \cdot \frac{1}{y} = \frac{2x}{y}$$

$$\frac{\partial v}{\partial y} = x^2 \cdot \left[\frac{-1}{y^2} \right] = -\frac{x^2}{y^2}$$

$$J = \begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix}$$

$$J = \frac{x^2 y^2}{x^2 y^2} - \frac{4xy}{xy}$$

$$= 1 - 4$$

$$\boxed{\boldsymbol{J = -3}}$$

$$J' = \frac{1}{J} = -\frac{1}{3}$$

Example 4: Find the Jacobian of y_1, y_2, y_3 with respect x_1, x_2, x_3 if $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_1 x_3}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$.

Solution:

$$J = \frac{\partial (y_1, y_2, y_3)}{\partial (x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$y_1 = \frac{x_2\,x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_1} = \frac{-\,x_2\,x_3}{x_1^2}$$

$$\frac{\partial y_1}{\partial x_3} = \frac{x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$y_2 = \frac{x_1\,x_3}{x_2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$$

$$\frac{\partial y_2}{\partial x_2} = \frac{-\,x_1\,x_3}{x_2^2}$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$y_3 = \frac{x_1\,x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$$

$$\frac{\partial y_3}{\partial x_3} = \frac{-\,x_1\,x_2}{x_3^2}$$

$$J = \begin{vmatrix} \dfrac{-\,x_2\,x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & \dfrac{-\,x_1\,x_3}{x_2} & \dfrac{x_1}{x_2} \\ \dfrac{x_1}{x_3} & \dfrac{x_1}{x_3} & \dfrac{-\,x_1\,x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2\,x_2^2\,x_3^2} \begin{vmatrix} -\,x_2\,x_3 & x_3\,x_1 & x_2\,x_1 \\ x_3\,x_2 & -\,x_1\,x_3 & x_1\,x_2 \\ x_2\,x_3 & x_1\,x_3 & -\,x_1\,x_2 \end{vmatrix}$$

$$J = \frac{x_2\,x_3 \cdot x_3\,x_1 \cdot x_2\,x_1}{x_1^2\,x_2^2\,x_3^2} \begin{vmatrix} -\,1 & 1 & 1 \\ 1 & -\,1 & 1 \\ 1 & 1 & -\,1 \end{vmatrix}$$

$$J = \begin{vmatrix} -\,1 & 1 & 1 \\ 1 & -\,1 & 1 \\ 1 & 1 & -\,1 \end{vmatrix}$$

$$= -\,1\,(1-1) - 1\,(-\,1-1) + 1\,(1+1)$$

$$= 2 + 2$$

$$\boxed{\boldsymbol{J = 4}}$$

Example 5: **If $u = x^2 + y^2, v = x^2 - y^2$ find J.**

Solution:

$$J = \frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = x^2 + y^2$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2x$	$\frac{\partial v}{\partial x} = 2x$
$\frac{\partial u}{\partial y} = 2y$	$\frac{\partial v}{\partial y} = - 2y$

$$J = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix}$$

$$= - 4xy - 4xy$$

$$\boxed{J = - 8xy}$$

Example 6: **If $u = x - y, v = y - z, w = z - x$, find $\frac{\partial (u, v, w)}{\partial (x, y, z)}$**

Solution:

$$J = \frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = x - y$	$v - y - z$	$w = z - x$
$\frac{\partial u}{\partial x} = 1$	$\frac{\partial v}{\partial x} = 0$	$\frac{\partial w}{\partial x} = -1$
$\frac{\partial u}{\partial y} = -1$	$\frac{\partial v}{\partial y} = 1$	$\frac{\partial w}{\partial y} = 0$
$\frac{\partial u}{\partial z} = 0$	$\frac{\partial v}{\partial z} = -1$	$\frac{\partial w}{\partial z} = 1$

$$J = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 1\,(1 - 0) + 1\,(0 - 1) + 0 = 1 - 1 = 0$$

$$\boxed{\boldsymbol{J = 0}}$$

$\therefore$ Functionally dependent

Example 7: Find the Jacobian of u, v, w with respect to x, y, z if $u = \frac{yz}{x}, v = \frac{xz}{y}, w = \frac{xy}{z}$

Solution:

$$J = \frac{\partial\,(u, v, w)}{\partial\,(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = \frac{yz}{x}$	$v = \frac{xz}{y}$	$w = \frac{xy}{z}$
$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$	$\frac{\partial v}{\partial x} = \frac{z}{y}$	$\frac{\partial w}{\partial x} = \frac{y}{z}$
$\frac{\partial u}{\partial y} = \frac{x}{z}$	$\frac{\partial v}{\partial y} = -\frac{xz}{y^2}$	$\frac{\partial w}{\partial y} = \frac{x}{z}$
$\frac{\partial u}{\partial z} = \frac{y}{x}$	$\frac{\partial v}{\partial z} = \frac{x}{y}$	$\frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$J = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$J = \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -yz & zx & yx \\ zy & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$J = \frac{(yz)(zx)(yx)}{x^2 y^2 z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$J = -1(1-1) - 1(-1-1) + 1(1+1)$$

$$\boxed{\mathbf{J = 0 + 2 + 2 = 4}}$$

Example 8: If $x + y + z = u$; $y + z = uv$; $z = uvw$ then prove that $\frac{\partial (x, y, z)}{\partial (u, v, w)} = u^2 v$

Solution:

$x + y + z = u$	$y + z = uv$	$\boldsymbol{z = uvw}$
$x = u - y - z$	$y = uv - z$	
$x = u - (y + z)$	$\boldsymbol{y = uv - uvw}$	
$\boldsymbol{x = u - uv}$		

$$J = \frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = u - uv$	$y = uv - uvw$	$z = uvw$
$\dfrac{\partial x}{\partial u} = 1 - v$	$\dfrac{\partial y}{\partial u} = v - vw$	$\dfrac{\partial z}{\partial u} = vw$
$\dfrac{\partial x}{\partial v} = - u$	$\dfrac{\partial y}{\partial v} = u - uw$	$\dfrac{\partial z}{\partial v} = uw$
$\dfrac{\partial x}{\partial w} = 0$	$\dfrac{\partial y}{\partial w} = - uv$	$\dfrac{\partial z}{\partial w} = uv$

$$J = \begin{vmatrix} 1 - v & - u & 0 \\ v - vw & u - uw & - uv \\ vw & uw & uv \end{vmatrix}$$

$$J = 1 - v\,[(u - uw)\,(uv) + uvuw] + u\,[(v - vw)\,uv + vwuv]$$

$$= (1 - v)\,[u^2\,v - u^2\,vw + u^2\,vw] + [uv^2 - uv^2\,w + uv^2\,w]$$

$$= (1 - v)\,u^2\,v + u\,(uv^2)$$

$$= u^2\,v - u^2\,v^2 + u^2\,v^2$$

$$\boxed{J = u^2\,v}$$

$\therefore$ Hence proved

Example 9: $x = r \sin \theta \cos \phi$ **;** $y = r \sin \theta \sin \phi$ **;** $z = r \cos \theta$ **then find** $\dfrac{\partial\,(x, y, z)}{\partial\,(r, \theta, \phi)}$.

Solution:

$$J = \frac{\partial\,(x, y, z)}{\partial\,(r, \theta, \phi)} \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r \sin\theta \cos\phi$	$y = r \sin\theta \sin\phi$	$z = r\cos\theta$
$\dfrac{\partial x}{\partial r} = \sin\theta\cos\phi$	$\dfrac{\partial y}{\partial r} = \sin\theta\sin\phi$	$\dfrac{\partial z}{\partial r} = \cos\theta$
$\dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$	$\dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$	$\dfrac{\partial z}{\partial \theta} = -r\sin\theta$
$\dfrac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$	$\dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$	$\dfrac{\partial z}{\partial \phi} = 0$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$J = \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r\cos\theta\sin\theta\cos\phi)$$
$$-\,r\sin\theta\sin\phi\,[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$$

$$J = r^2\sin^3\theta\cos^2\phi + r^2\cos^2\theta\sin\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi$$
$$+\,r^2\cos^2\theta\sin\theta\sin^2\phi$$
$$= r^2\sin^3\theta\,[\cos^2\phi + \sin^2\phi] + r^2\cos^2\theta\sin\theta\,[\cos^2\phi + \sin^2\phi]$$
$$= r^2\sin^3\theta + r^2\cos^2\theta\sin\theta$$
$$= r^2\sin\theta\,(\sin^2\theta + \cos^2\theta)$$

$$\boxed{J = r^2\sin\theta}$$

Example 10: Find the Jacobian if
$u = x + y + z,\ v = xy + yz + zx,\ w = x^2 + y^2 + z^2$

Solution:

$$J = \frac{\partial\,(u, v, w)}{\partial\,(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$u = x + y + z \qquad v = xy + yz + zx \qquad w = x^2 + y^2 + z^2$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = y + z \qquad \frac{\partial w}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = x + z \qquad \frac{\partial w}{\partial y} = 2y$$

$$\frac{\partial u}{\partial z} = 1 \qquad \frac{\partial v}{\partial z} = y + x \qquad \frac{\partial w}{\partial z} = 2z$$

$$J = \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & y+x \\ 2x & 2y & 2z \end{vmatrix}$$

$$J = 1\,[(x+z)\,2z - 2y\,(y+x)] - 1\,[(y+z)\,2z - 2x\,(y+x)] + 1\,[(y+z)\,2y - 2x\,(x+z)]$$

$$J = 2zx + 2z^2 - y^2 - 2xy - 1\,[2yz + 2z^2 - 2xy - 2x^2] + 2y^2 + 2yz - 2x^2 - 2xz$$

$$J = 0$$

$\therefore$ Functionally dependent.

Example 11: **If $x = uv$ and $y = \frac{u}{v}$ find $J = \frac{\partial\,(x, y)}{\partial\,(u, v)}$ and $J' = \frac{\partial\,(u, v)}{\partial\,(x, y)}$. Also verify $JJ' = 1$**

Solution:

$$J = \frac{\partial\,(x, y)}{\partial\,(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = uv \qquad\qquad y = \frac{u}{v}$$

$$\frac{\partial x}{\partial u} = v \qquad\qquad \frac{\partial y}{\partial u} = \frac{1}{v}$$

$$\frac{\partial x}{\partial v} = u \qquad\qquad \frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$J = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= v\left(\frac{-u}{v^2}\right) - u\left(\frac{1}{v}\right)$$

$$= \left(-\frac{u}{v}\right)\left(-\frac{u}{v}\right)$$

$$J = \frac{-2u}{v}$$

$$J' = \frac{-v}{2u}$$

$$\Rightarrow JJ' = 1$$

$$\frac{-2u}{v} \times \frac{-v}{2u} = 1$$

$$\boxed{\boldsymbol{JJ' = 1}}$$

Example 12: If $x = u \cos v$ and $y = u \sin v$, prove that

$$\frac{\partial (x, y)}{\partial (u, v)} \times \frac{\partial (u, v)}{\partial (x, y)} = 1$$

Solution:

$$J = \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = u\cos v \qquad\qquad y = u\sin v$$

$$\frac{\partial x}{\partial u} = \cos v \qquad\qquad \frac{\partial y}{\partial u} = \sin v$$

$$\frac{\partial x}{\partial v} = -u\sin v \qquad\qquad \frac{\partial y}{\partial v} = u\cos v$$

$$J = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix}$$

$$= u\cos^2 v + u\sin^2 v$$

$$= u(\cos^2 v + \sin^2 v)$$

$$J = u$$

$$J' = \frac{1}{u}$$

$$J \times J' = 1$$

$$u \times \frac{1}{u} = 1$$

$1 = 1 \Rightarrow$ LHS = RHS

$\therefore$ Hence proved

Example 13: Let $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$. Are u, v and w functionally related? If so find this relationship.

Solution:

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = 3x + 2y - z$	$v = x - 2y + z$	$w = x^2 + 2xy - xz$
$\frac{\partial u}{\partial x} = 3$	$\frac{\partial v}{\partial x} = 1$	$\frac{\partial w}{\partial x} = 2x$
$\frac{\partial u}{\partial y} = 2$	$\frac{\partial v}{\partial y} = -2$	$\frac{\partial w}{\partial y} = 2x$
$\frac{\partial u}{\partial z} = -1$	$\frac{\partial v}{\partial z} = 1$	$\frac{\partial w}{\partial z} = -x$

$$J = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x & 2x & -x \end{vmatrix}$$

$$= 3\,(2x - 2x) - 2\,(-x - 2x) - 1\,(2x + 4x)$$

$$= 0 - 2\,(-3x) - 1\,(6x)$$

$$= 6x - 6x$$

$$\boxed{J = 0}$$

$\therefore$ Functionally dependent

Example 14: Find $\frac{\partial (x, y, z)}{\partial (u, v, w)}$ if $x = \frac{u^2}{v}, y = \frac{v^2}{w}, z = \frac{w^2}{u}$

Solution:

$$J = \frac{\partial (x, y, z)}{\partial (u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial v}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = \dfrac{u^2}{v}$	$y = \dfrac{v^2}{w}$	$z = \dfrac{w^2}{u}$
$\dfrac{\partial x}{\partial u} = \dfrac{2u}{v}$	$\dfrac{\partial y}{\partial u} = 0$	$\dfrac{\partial z}{\partial u} = -\dfrac{w^2}{u^2}$
$\dfrac{\partial x}{\partial v} = -\dfrac{u^2}{v^2}$	$\dfrac{\partial y}{\partial v} = \dfrac{2v}{w}$	$\dfrac{\partial z}{\partial v} = 0$
$\dfrac{\partial x}{\partial w} = 0$	$\dfrac{\partial y}{\partial w} = -\dfrac{v^2}{w^2}$	$\dfrac{\partial z}{\partial w} = \dfrac{2w}{u}$

$$J = \begin{vmatrix} \dfrac{2u}{v} & \dfrac{-u^2}{v^2} & 0 \\ 0 & \dfrac{2v}{w} & \dfrac{-v^2}{w^2} \\ \dfrac{-w^2}{u^2} & 0 & \dfrac{2w}{u} \end{vmatrix}$$

$$= \frac{2u}{v}\left[\left(\frac{2v}{w}\right)\left(\frac{2w}{u}\right) + 0\right] + \frac{u^2}{v^2}\left[0 - \left(\frac{w^2}{u^2}\right)\left(\frac{v^2}{w^2}\right)\right] + 0$$

$$= \frac{2u}{v}\left(\frac{4vw}{wu}\right) + \frac{u^2}{v^2}\left(-\frac{w^2\,v^2}{u^2\,w^2}\right)$$

$$= 8 - 1$$

$$\boxed{\boldsymbol{J = 7}}$$

TWO MARKS QUESTIONS AND ANSWERS

1. Define stationary points? [critical points]

Solution:

Stationary points are the points at which function attains its maximum (or) minimum value.

2. Define saddle points?

Solution:

A point at which the function neither maximum nor minimum.

For this $AC - B^2 < 0$ (Negative)

3. Condition for maximum and minimum $\dfrac{\partial^2 f}{\partial x^2} = A$, $\dfrac{\partial^2 f}{\partial x\, \partial y} = B$, $\dfrac{\partial^2 f}{\partial y^2} = C$

Solution:

If $AC - B^2 > 0$ and $A < 0$ then function is maximum.

If $AC - B^2 > 0$ and $A > 0$ then function is minimum.

If $AC - B^2 < 0$ saddle point exists there is no maximum nor minimum.

If $AC - B^2 = 0$ No conclusion.

4. If $x^2 + y^2 = 1$ find $\dfrac{dy}{dx}$.

Solution:

$$x^2 + y^2 = 1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{dy}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

5. If $x^y + y^x = c$ find $\frac{dy}{dx}$

Solution:

$$x^y + y^x = c$$

$$x^y + y^x - c = f$$

$$f_x = \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$f_y = \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-[yx^{y-1} + y^x \log y]}{x^y \log x + xy^{x-1}}$$

6. Find $\frac{du}{dt}$ for $u = x^2 + y^2$; $x = at^2$; $y = 2at$

Solution:

$$u = x^2 + y^2 \qquad x = at^2 \qquad y = 2at$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 2x\,(2at) + 2y\,(2a)$$

$$\frac{du}{dt} = 4xat + 4ay$$

$$= 4\,(at^2)\,at + 4a\,(2at)$$

$$\boxed{\therefore \frac{du}{dt} = 4a^2 t^3 + 8a^2 t}$$

7. Find $\frac{du}{dt}$ given $u = y^2 - 4ax$; $x = at^2$; $y = 2at$.

Solution:

$u = y^2 - 4ax \qquad x = at^2 \qquad y = 2at$

$\frac{\partial u}{\partial x} = -\,4a \qquad \frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$

$\frac{\partial u}{\partial y} = 2y$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = -\,4a\,(2at) + 2y\,(2a)$$

$$\frac{du}{dt} = -\,8a^2\,t + 2\,(2at)\,(2a)$$

$$\therefore \frac{du}{dt} = -\,8a^2\,t + 8a^2\,t$$

$$\boxed{\therefore \frac{du}{dt} = 0}$$

8. If $x = u\,(1 - v)$ and $y = 2uv$ find $\frac{\partial\,(x, y)}{\partial\,(u, v)}$

Solution:

$$\frac{\partial\,(x, y)}{\partial\,(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ \partial v & 2u \end{vmatrix}$$

$$= 2u\,(1-v) - (-\,2uv)$$

$$= 2u - 2uv + 2uv$$

$$\boxed{\frac{\partial\,(x, y)}{\partial\,(u, v)} = 2u}$$

9. If $x = r\cos\theta$; $y = r\sin\theta$ find $\dfrac{\partial\,(x, y)}{\partial\,(r, \theta)}$ and $\dfrac{\partial\,(r, \theta)}{\partial\,(x, y)}$

Solution:

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad\qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -\,r\sin\theta \qquad\qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial\,(x, y)}{\partial\,(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\,r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\,(\cos^2\theta + \sin^2\theta)$$

$$\frac{\partial\,(x, y)}{\partial\,(r, \theta)} = r$$

$$\therefore\ \frac{\partial\,(x, y)}{\partial\,(r, \theta)} = r\ ;\ \frac{\partial\,(r, \theta)}{\partial\,(x, y)} = \frac{1}{r}$$

10. If $u = \dfrac{y^2}{x}, v = \dfrac{x^2}{y}$ find $\dfrac{\partial (x, v)}{\partial (u, v)}$.

Solution:

$$u = \frac{y^2}{x} \qquad v = \frac{x^2}{y}$$

$$\frac{\partial u}{\partial x} = \frac{-y^2}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{\partial x}{y}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x} \qquad \frac{\partial v}{\partial y} = \frac{-x^2}{y^2}$$

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{-y^2}{x^2} & \dfrac{2y}{x} \\ \dfrac{2x}{y} & \dfrac{-x^2}{y^2} \end{vmatrix}$$

$$= \frac{y^2}{x^2} \times \frac{x^2}{y^2} - \frac{4xy}{xy}$$

$$= 1 - 4$$

$$\boxed{\boldsymbol{J = -3}}$$

$$\therefore \frac{\partial (u, v)}{\partial (x, y)} = -3 \; ; \; \frac{\partial (x, y)}{\partial (u, v)} = \frac{-1}{3}$$

11. If $x = r\cos\theta$; $y = r\sin\theta$ then find $\dfrac{\partial r}{\partial x}$

Solution:

$$x = r\cos\theta \; ; \; y = r\sin\theta \qquad x^2 + y^2 = r^2$$

$$r = \sqrt{x^2 + y^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{r^2}}$$

$$= \frac{x}{r}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

12. If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = ?$

Solution:

$$u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$$

$$n = \text{N.D} - \text{D.D} = 1 - 1 = 0$$

$$n = 0$$

By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

13. Find the Taylor series expansion of x^y mean the point (1, 1) upto the first degree term.

Solution:

$$f = x^y$$

$$f_x = yx^{y-1}$$

$$f_y = x^y \log x$$

$$f_x(1, 1) = 1$$

$$f_y(1, 1) = 0$$

$$f(x, y) = f(1, 1) + \frac{1}{1!}\,[(x-1)\,f_x\,(1, 1), + (y-1)\,f_y\,(1, 1)] + -$$

$$= 1 + \frac{1}{1!}\,[(x-1)\,(1) + (y-1)\,(0)]$$

$$= 1 + x - 1$$

$$= x$$

$$\boxed{\therefore\, f(x, y) = x}$$

14. Verify Euler's Theorem for the function $u = x^3 + y^3 + z^3 + 3xyz$

Solution:

Given: $u = x^3 + y^3 + z^3 + 3xyz$

degree of u, $n = 3$

Euler's Theorem;

$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = nu$$

LHS = RHS

RHS

$nu = 3u$

$$\text{RHS} = 3\,(x^3 + y^3 + z^3 + 3xyz) \qquad \ldots (1)$$

LHS

$\frac{\partial u}{\partial x} = 3x^2 + 3yz$	$\frac{\partial u}{\partial y} = 3y^2 + 3xz$	$\frac{\partial u}{\partial z} = 3z^2 + 3xy$
$x\,\frac{\partial u}{\partial x} = 3x^3 + 3xyz$	$y\,\frac{\partial u}{\partial y} = 3y^3 + 3xyz$	$z\,\frac{\partial u}{\partial z} = 3z^3 + 3xyz$
… (A)	… (B)	… (C)

And A, B, C

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3x^3+3xyz$$

$$+3y^3+3xyz+3z^3+3xyz$$

$$=3x^3+3y^3+3z^3+9xyz$$

$$=3\,(x^3+y^3+z^3+3xyz)$$

$$\text{LHS}=3\,u \qquad \ldots (2)$$

from (1) and (2)

$$(1)=(2)$$

Euler's Theorem is verified

15. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z=x^3y-x\sin(xy)$.

Solution:

$$z=x^3\,y-x\sin(xy)$$

$$\frac{\partial z}{\partial x}=2xy-[xy\,(\cos xy)+\sin xy\,(1)]$$

$$\frac{\partial z}{\partial y}=x^2-[x^2\cos(xy)]$$

16. If $u=x^2, v=y^2$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution:

$$J=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$u = x^2 \qquad v = y^2$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 2y$$

$$J = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix}$$

$$\therefore\ J = 4xy$$

17. Verify Euler's theorem for the function $\boldsymbol{u = x^2 + y^2 + 2xy}$

Solution:

$$u = x^2 + y^2 + 2xy$$

degree of u, $n = 2$

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\text{LHS} = \text{RHS}$$

RHS

$nu = 2u$

$$\text{RHS} = 2\,(x^2 + y^2 + 2xy) \qquad \dots (1)$$

LHS

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$\frac{\partial u}{\partial x} = 2x + 2y$	$\frac{\partial u}{\partial y} = 2y + 2x$
$x\frac{\partial u}{\partial x} = 2x^2 + 2xy \ \dots \ (A)$	$y\frac{\partial u}{\partial y} = 2y^2 + 2xy \ \dots \ (B)$

Add (A) and (B)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2x^2+2xy+2y^2+2xy$$

$$=2x^2+2y^2+4xy$$

$$\text{LHS}=2\,(x^2+y^2+2xy) \qquad \dots (2)$$

from (1) and (2)

$$(1)=(2)$$

Euler's Theorem is verified.

18. Prove that $\dfrac{\partial^2 f}{\partial x\,\partial y}=\dfrac{\partial^2 f}{\partial y\,\partial x}$ if $x^3+y^3+z^3-3xyz$.

Solution:

$$f=x^3+y^3+z^3-3xyz$$

$$f_x=\frac{\partial f}{\partial x}=3x^2-3yz$$

$$f_y=\frac{\partial f}{\partial y}=3y^2-3xz$$

$$f_{xy}=\frac{\partial^2 f}{\partial x\,\partial y}=-3z \qquad \dots (1)$$

$$f_{yx}=\frac{\partial^2 f}{\partial y\,\partial x}=-3z \qquad \dots (2)$$

From (1) and (2) $\dfrac{\partial^2 f}{\partial x\,\partial y}=\dfrac{\partial^2 f}{\partial y\,\partial x}=-3z$

19. If $u = \dfrac{2x - y}{2}$ and $v = \dfrac{y}{2}$ then find $\dfrac{\partial (u, v)}{\partial (x, y)}$

Solution:

$$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial (u, v)}{\partial (x, y)}$$

$$u = \frac{2x - y}{2} \qquad v = \frac{y}{2}$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = \frac{-1}{2} \qquad \frac{\partial v}{\partial y} = \frac{1}{2}$$

$$J = \begin{vmatrix} 1 & \dfrac{-1}{2} \\ 0 & \dfrac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2} - 0$$

$$\boxed{\therefore J = \frac{1}{2}}$$

20. If $x = uv, y = \dfrac{u}{v}$, then find the Jacobian.

Solution:

$$\frac{\partial (x, v)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$x = uv$	$y = \frac{u}{v}$
$\frac{\partial x}{\partial u} = v$	$\frac{\partial y}{\partial u} = \frac{1}{v}$
$\frac{\partial x}{\partial v} = u$	$\frac{\partial y}{\partial v} = \frac{-u}{v^2}$

$$J = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-uv}{v^2} - \frac{u}{v}$$

$$\boxed{\therefore J = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v}}$$

21. If $u = \frac{y}{z} + \frac{z}{x}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Solution:

$$u = \frac{y}{z} + \frac{z}{x}$$

n degree of u = degree of Numerator − degree of denominator

$= 1 - 1$

$n = 0$

$\Rightarrow$ u is a homogeneous function of x, y and z in degree 0.

By Euler's Theorem;

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

22. State Euler's theorem for homogeneous function.

Solution:

If u is a homogeneous function of degree n in x and y then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

23. State any two properties of Jacobian.

Solution:

(i) If u and v are the functions of x and y then

$$\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)} = 1$$

(ii) If u and v are the functions of x and y then $\frac{\partial (u, v)}{\partial (x, y)} = 0$, then it is functionally dependent.

24. If $x = u^2 - v^2$ and $y = 2uv$, then find the Jacobian of x and y with respect to u and v.

Solution:

$x = u^2 - v^2$	$y = 2uv$
$\frac{\partial x}{\partial u} = 2u$	$\frac{\partial y}{\partial u} = 2v$
$\frac{\partial x}{\partial v} = -2v$	$\frac{\partial y}{\partial v} = 2u$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4u^2 + 4v^2$$

$$\therefore J = 4 (u^2 + v^2)$$

EXERCISE

Euler's Theorem on Homogeneous Functions

1. If $u = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2. If $u = \tan^{-1}\left(\frac{x^3 + y^3)}{x - y}\right)$, prove that
$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cos 3u \sin u$

3. If $u = \tan^{-1}\left[\frac{x + y}{\sqrt{x} + \sqrt{y}}\right]$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{4} \sin 2u$

4. If $u = \sin^{-1}\left[\frac{x^3 + y^3}{\sqrt{x} + \sqrt{y}}\right]$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{5}{2} \tan u$

5. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$.

Total Derivatives

1. Find $\frac{dy}{dx}$, if $x^3 + y^3 = 3axy$

2. If $z = f(x, y)$ and $x = u - v, y = uv$ then show that
$(u + v) \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial u} - v \frac{\partial z}{\partial v}$

3. If $u = x^2 + y^3, x = 1 + \sqrt{t}, y = 1 - \sqrt{t}$, then find $\frac{du}{dt}$.

4. If $u = (x^2 + y^2 + z^2)^{-1/2}$ then find the value of
$\frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} + z \frac{\partial^2 u}{\partial z^2}$

5. If $u = f(r, s, t),\ r = \frac{x}{y},\ s = \frac{y}{z},\ t = \frac{z}{x}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Find the extreme points of the following functions

1. $x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$
2. $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
3. $3x^2 - y^2 + x^3$
4. $x^3 y^2 (a - x - y)$
5. $x^3 + 3xy^2 - 15x - 12y$

Taylor's series

1. Find the Taylor's series expansion of x^y near the point (1, 1) upto the first degree terms.
2. Find the Taylor's series expansion of $x^2 y^2 + 2x^2 y + 3xy^2$ in powers of $(x + 2)$ and $(y - 1)$ upto third degree terms.
3. Expand $e^x \sin y$ by Taylor's theorem in powers of x and y as far as the terms of third degree.
4. Use Taylor's formula to expand the function f defined by $f(x, y) = x^2 + xy + y^2$ in powers of $(x - 1)$ and $(y - 2)$.

Jacobians

1. (a) If $x = u^2 - v^2,\ y = 2uv$, find $\frac{\partial (x, y)}{\partial (u, v)}$

   (b) If $x = uv,\ y = \frac{u}{v}$, find $\frac{\partial (x, y)}{\partial (u, v)}$

2. Find the Jacobian of the transformation
   (i) $x = u - 2v$; $y = 2u - v$ (ii) $x = 2u, y = 3v^2, z = 4w^3$
   (iii) $x = e^{2u} \cos v, y = e^{2v} \sin v$

3. Are $u(x, y) = \frac{x}{y}$ and $v(x, y) = \frac{(x+y)}{x-y}$ functionally dependent? If it so, find the functional relation between them.

4. If $u = x + 3y^2 - z^3, v = 4x^2 yz, w = 2z^2 - xy,$ find $\frac{\partial (u, v, w)}{\partial (x, y, z)}$ at $(1, -1, 0)$

5. If $u = xyz, v = x^2 + y^2 + z^2, w = x + y + z,$ find $\frac{\partial (x, y, z)}{\partial (u, v, w)}$

6. State any two properties of Jacobian.

Lagrange's method or undetermined multipliers

1. Determine the greatest and the smallest values of xy on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1.$

2. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

3. Find the dimensions of the rectangular box without a top of maximum capacity, whose surface area is 108 sq.cm.

4. Find the minimum distance from the point (1, 2, 0) to the cone $x^2 = x^2 + y^2$

5. Find the maximum and minimum values of the function $f(x, y) = 3x + dy$ on the circle $x^2 + y^2 = 1.$

UNIT - 4

MULTIPLE INTEGRALS AND THEIR APPLICATIONS

4.1 INTRODUCTION

- Integration symbol $\int$ was identified by great Mathematician Leibnitz
- Integration means area under the curve [summing]
- There are two types of integral
  1. Definite Integral [limit is given]
  2. Indefinite Integral [limit is not given]

4.1.1 Integration formula

- $\int dx = x + c$
- $\int x dx = \frac{x^2}{2} + c$
- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$
- $\int \frac{dx}{x} = \log x$ (or) $\int \frac{1}{x} dx = \log x$
- $\int \frac{dy}{y} = \log y$ (or) $\int \frac{1}{y} dy = \log y$
- $\int \frac{dz}{z} = \log z$ (or) $\int \frac{1}{z} dz = \log z$
- $\frac{1}{x^2} dx = \int x^{-2} \, dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x} + c$

- $\int x dy = xy$
- $\int y dx = yx$
- $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} + c$
- $\int \sin x \, dx = -\cos x + c$
- $\int \cos x \, dx = \sin x + c$
- $\int \tan x \, dx = \log (\sec x) + c$
- $\int \sec x \, dx = \log (\sec x + \tan x) + c$
- $\int \cot x \, dx = \log (\sin x) + c$

$$\int \cos \theta \, d\theta = \sin \theta$$

$$\int \sin \theta \, d\theta = -\cos \theta$$

$$\sin \pi = 0, \ \sin n\pi = 0$$

$$\left.\begin{array}{l} \cos \pi \\ \cos 3\pi \\ \cos 5\pi \end{array}\right\} = -1 \ \text{(odd)}$$

$$\left.\begin{array}{l} \cos 2\pi \\ \cos 4\pi \end{array}\right\} = 1 \ \text{(even)}$$

4.1.2 Problems on Single Integral

Example 1: Evaluate $\displaystyle\int_0^1 x^2\,dx$

Solution:

$$I = \int_0^1 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= \left[\frac{1}{3} - 0\right]$$

$$\boxed{I = \frac{1}{3}}$$

Example 2: Evaluate $\displaystyle\int_0^2 (x^2 + x)\,dx$

Solution:

$$I = \int_0^2 (x^2 + x)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2$$

$$= \left[\left(\frac{2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{0^3}{3} + \frac{0^2}{2}\right)\right]$$

$$= \frac{8}{3} + \frac{4}{2} = \frac{16 + 12}{6}$$

$$\boxed{\boldsymbol{I = \frac{28}{6}}}$$

Example 3: Evaluate $\displaystyle\int_{0}^{\pi/2} (\sin x + \cos x)\, dx$

Solution:

$$\boxed{\begin{array}{l} \cos \frac{\pi}{2} = 0 \\ \cos 0 = 1 \\ \sin \frac{\pi}{2} = 1 \\ \sin 0 = 0 \end{array}}$$

$$I = \int_{0}^{\pi/2} (\sin x + \cos x)\, dx$$

$$= [\, -\cos x + \sin x\,]_{0}^{\pi/2}$$

$$= \left[\left(-\cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right) - (-\cos 0 + \sin 0) \right]$$

$$= [(-0 + 1) - (-1 + 0)]$$

$$= 1 + 1$$

$$\boxed{\boldsymbol{I = 2}}$$

Example 4: Evaluate $\displaystyle\int_{0}^{\pi} (\sin x - \cos x)\, dx$

Solution:

$$\boxed{\begin{array}{l} \cos \pi = -1 \\ \sin \pi = 0 \end{array}}$$

$$I = \int_{0}^{\pi} (\sin x - \cos x)\, dx$$

$$= [\, -\cos x - \sin x\,]_{0}^{\pi}$$

$$= [(-\cos \pi - \sin \pi) - (-\cos 0 - \sin 0)]$$

$= [(-(-1)-0)-(-1-0)]$

$= 1+1$

$$\boxed{I = 2}$$

Example 5: Evaluate $\displaystyle\int_0^{\pi} (x+x^3)\,dx$

Solution:

$$I = \int_0^1 (x+x^3)\,dx$$

$$= \left[\frac{x^2}{2}+\frac{x^4}{4}\right]_0^1$$

$$= \left[\left(\frac{1^2}{2}+\frac{1^4}{4}\right)-\left(\frac{0^2}{2}+\frac{0^4}{4}\right)\right]$$

$$= \frac{1}{2}+\frac{1}{4}$$

$$\boxed{I = \frac{3}{4}}$$

4.2 DOUBLE INTEGRALS

- The definite integral $\displaystyle\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_i)\,\delta x_1 + f(x_2)\,\delta x_2 + \ldots + f(x_n)\,\delta x_n$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2 \ldots$ tends to zero.

A **double integral** is its counterpart in two dimensions.

- Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into 'n' elementary areas $\delta A_1, \delta A_2 \dots \delta A_n$. Let (x_r, y_r) be any point within the r^{th} elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \dots + f(x_n, y_n)\,\delta A_n$$

$$\text{i.e} \qquad \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if its exists as the number of sub-division increases indefinitely and area of each sub-division decreases to zero, is defined as the **double integral** of $f(x, y)$ over the region R.

Thus

$$\iint f(x, y)\,\delta A = \underset{\substack{n \to \infty \\ \delta A \to 0}}{Lt} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The utility of double integrals would be limited if it were required to take limit of sums of evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

- For purposes of evaluation (1) is expressed at the repeated integral $\displaystyle\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dx\,dy$

Its value is found as follows:

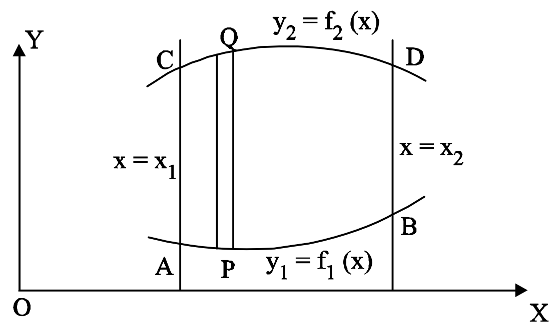
Case (i): When y_1, y_2 are functions of x and x_1, x_2 are constants

- $f(x, y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t x within the limits x_1, x_2 i.e

$$\boxed{I_1 = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\, dy\, dx}$$

where integration is carried from the inner to the outer rectangle.

Diagram illustration



AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

- The inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant). While the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

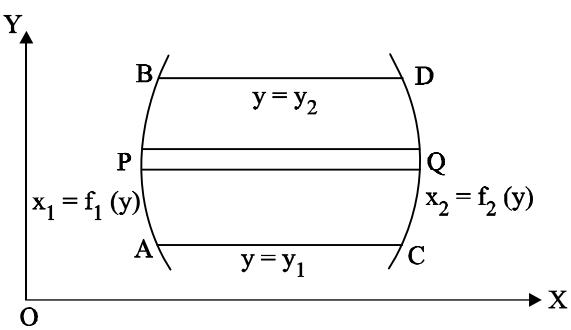
- Thus the whole region of integration is the area $ABCD$.

Case (ii): When x_1, x_2 are functions of y and y_1, y_2 are constants

$f(x, y)$ is first integrated w.r.t x keeping 'y' fixed within the limits x_1, x_2 and the resulting expression is integrated w.r.t y between the limits y_1, y_2 ie.

$$\boxed{I_2 = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y)\, dx\, dy}$$

Diagram illustration



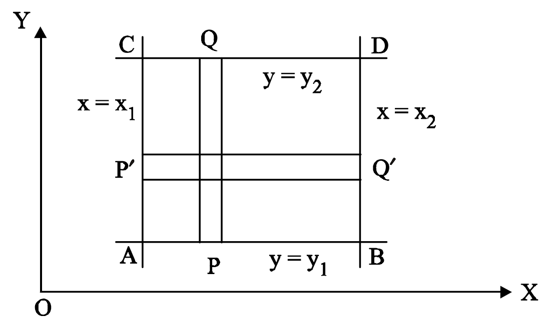
- AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.
- Inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.
- Thus the whole region of integration is the area $ABDC$.

Case (iii): When both pairs of limits are constants the region of integration is the rectangle $ABDC$

- In I_1, we integrate along the **vertical strip** $\boldsymbol{PQ}$ and then slide it from AC to BD.
- In I_2, we integrate along the **horizontal strip '$\boldsymbol{PQ}$'** and the slide it from AB to CD.

Here obviously $I_1 = I_2$

Thus for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t y or vice versa.



Type I: Problems Based on Double Integration in Cartesian Co-ordinates

Example 1: **Evaluate** $\displaystyle\int_0^1 \int_1^2 x\,(x+y)\,dy\,dx$

Solution:

$$= \int_0^1 \int_1^2 x\,(x+y)\,dy\,dx$$

$$= \int_0^1 \int_1^2 (x^2 + xy)\,dy\,dx$$

$$= \int_0^1 \int_1^2 (x^2 + xy)\,dy\,dx$$

$$= \int_0^1 \int_1^2 (x^2\,dy + xy\,dy)\,dx$$

$$= \int_0^1 \left[x^2\,y + \frac{xy^2}{2} \right]_{y=1}^{y=2} dx$$

$$= \int_0^1 \left[[2x^2 + 2x] - \left(x^2 + \frac{x}{2} \right) \right] dx$$

$$= \int_0^1 \left[2x^2 + 2x - x^2 - \frac{x}{2} \right] dx$$

$$= \int_0^1 \left(x^2 + \frac{3}{2} x \right) dx$$

$$= \left[\frac{x^3}{3} + \frac{3}{2} \frac{x^2}{2} \right]_0^1$$

$$= \left[\frac{1}{3} + \frac{3}{4} \right] - (0 + 0) = \frac{13}{12}$$

Example 2: Evaluate $\displaystyle\int_0^1 \int_0^x dy\, dx$

Solution:

$$= \int_0^1 \int_0^x dy\, dx$$

$$= \int_0^1 [\, y \,]_0^x \, dx$$

$$= \int_0^1 x dx$$

$$= \left[\frac{x^2}{2} \right]_0^1$$

$$\boxed{= \frac{1}{2}}$$

Example 3: Evaluate $\int_0^1 \int_1^2 (y^2 + x^2)\, dx\, dy$

Solution:

$$= \int_0^1 \int_1^2 (y^2 + x^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{y^3}{3} + x^2 y \right]_1^2 dx$$

$$= \int_0^1 \left[\frac{8}{3} + 2x^2 \right] - \left[\frac{1}{3} + x^2 \right] dx$$

$$= \int_0^1 \left[\frac{8}{3} + 2x^2 - \frac{1}{3} - x^2 \right] dx$$

$$= \int_0^1 \left[\frac{7}{3} + x^2 \right] dx$$

$$= \left[\frac{7}{3} x + \frac{x^3}{3} \right]_0^1$$

$$= \left(\frac{7}{3} + \frac{1}{3} \right) - (0 + 0)$$

$$\boxed{= \frac{8}{3}}$$

Example 4: Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$

Solution:

$$= \int_0^a y \left(\frac{x^2}{2} \right)_0^{\sqrt{ay}} dy$$

$$= \int_0^a \left(y \cdot \frac{ay}{2} \right) dy$$

$$= \frac{a}{2} \int_0^a y^2 \, dy$$

$$= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a$$

$$= \frac{a}{2} \times \frac{a^3}{3} = \mathbf{\frac{a^4}{6}}$$

4.2.1 Double Integral Type I (Constant limit)

Example 1: Evaluate $\displaystyle \int_0^1 \int_0^2 (x + y) \, dy \, dx$

Solution:

$$I = \int_0^1 \int_0^2 (x + y) \, dy \, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^2 dx$$

$$= \int_0^1 \left[\left(2x + \frac{4}{2} \right) - \left(0x + \frac{0^2}{2} \right) \right] dx$$

$$I = \int_0^1 (2x + 2) \, dx = \left[\frac{2x^2}{2} + 2x \right]_0^1$$

$$= \left[x^2 + 2x \right]_0^1 = (1 + 2)$$

$$\boxed{I = 3}$$

Example 2: Evaluate $\int\limits_0^1 \int\limits_0^3 (x^2 + y^2)\, dx\, dy$

Solution:

$$I = \int\limits_0^1 \int\limits_0^3 (x^2 + y^2)\, dx\, dy$$

$$= \int\limits_0^1 \left[\frac{x^3}{3} + y^2 x\right]_0^3 dy$$

$$= \int\limits_0^1 \left[\left(\frac{3^3}{3} + 3y^2\right) - (0)\right] dy$$

$$I = \int\limits_0^1 \left(\frac{27}{3} + 3y^2\right) dy$$

$$= \left[\frac{27}{3} y + \frac{y^3}{3}\right]_0^1$$

$$= \left[9y + y^3\right]_0^1$$

$$= [9 + 1]$$

$$\boxed{I = 10}$$

Example 3: Evaluate $\int\limits_0^a \int\limits_0^b (x^3 + y^3)\, dy\, dx$

Solution:

$$I = \int\limits_0^a \int\limits_0^b (x^3 + y^3)\, dy\, dx$$

$$= \int_0^a \left[x^3 y + \frac{y^4}{4} \right]_0^b dx$$

$$= \int_0^a \left[\left(x^3 (b) + \frac{b^4}{4} \right) - 0 \right] dx$$

$$I = \int_0^a \left[bx^3 + \frac{b^4}{4} \right] dx$$

$$= \int \left[b \frac{x^4}{4} + \frac{b^4}{4} x \right]_0^a$$

$$\boxed{\boldsymbol{I = b \frac{a^4}{4} + \frac{b^4}{4} a}}$$

Example 4: Evaluate $\displaystyle\int_0^1 \int_0^2 xy\, dx\, dy$

Solution:

$$I = \int_0^1 \int_0^2 xy\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y \right]_0^2 dy$$

$$= \int_0^1 \left[\frac{4}{2} y - 0 \right] dy$$

$$I = 2 \int_0^1 y\, dy = 2 \left[\frac{y^2}{2} \right]_0^1$$

$$\boxed{\boldsymbol{I = 1}}$$

Example 6: Evaluate $\displaystyle\int_2^a \int_2^b \frac{dx}{x}\frac{dy}{y}$

Solution:

$$I = \int_2^a \int_2^b \frac{dx}{x}\frac{dy}{y}$$

$$= \int_2^a [\log x]_2^b \frac{dy}{y}$$

$$= \int_2^a (\log b - \log 2)\frac{dy}{y}$$

$$= \log b - \log 2 \int_2^a \frac{dy}{y}$$

$$= (\log b - \log 2)\,[\log y]_2^a$$

$$\Rightarrow I = (\log b - \log 2)(\log a - \log 2)$$

4.2.2 Type 2 [Variable limit]

Example 1: Evaluate $\displaystyle\int_0^1 \int_0^x dx\,dy$

Solution:

$$I = \int_0^1 \int_0^x dx\,dy \qquad \text{[not in correct form]}$$

$$I = \int_0^1 \int_0^x dy\,dx \quad \text{[correct form]}$$

$$= \int_0^1 [\,y\,]_0^x\,dx$$

$$I = \int_0^1 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1^2}{2} - \frac{0^2}{2}$$

$$\boxed{I = \frac{1}{2}}$$

Example 2: Evaluate $\displaystyle\int_1^2 \int_2^x xy\,dx\,dy$

Solution:

$$I = \int_1^2 \int_2^x xy\,dx\,dy \quad \text{[not in correct form]}$$

$$I = \int_1^2 \int_2^x xy\,dy\,dx \quad \text{[correct form]}$$

$$= \int_1^2 \left[x\,\frac{y^2}{2}\right]_2^x dx$$

$$= \int_{1}^{2} \left[\left(\frac{x^3}{2}\right) - \left(\frac{4x}{2}\right)\right] dx$$

$$I = \int_{1}^{2} \left(\frac{x^3}{2} - 2x\right) dx$$

$$= \left[\frac{x^4}{2 \times 4} - \frac{2x^2}{2}\right]_1^2$$

$$I = \left[\left(\frac{2^4}{8} - 2^2\right) - \left(\frac{1^4}{8} - 1^2\right)\right]$$

$$I = \left(\frac{16}{8} - 4\right) - \left(\frac{1}{8} - 1\right) = (2-4) - \left(\frac{-7}{8}\right)$$

$$= -2 + \frac{7}{8}$$

$$\boxed{\boldsymbol{I = \frac{-9}{8}}}$$

Example 3: Evaluate $\displaystyle\int_{0}^{1} \int_{0}^{x} xy \, dy \, dx$

Solution:

$$I = \int_{0}^{1} \int_{0}^{x} xy \, dy \, dx$$

$$= \int_{0}^{1} \left[x \frac{y^2}{2}\right]_0^x dx$$

$$I = \int_{0}^{1} \left(x \frac{x^2}{2}\right) dx$$

$$\Rightarrow I = \int_0^1 \frac{x^3}{2}\,dx$$

$$= \left[\frac{x^4}{8}\right]_0^1$$

$$= \left(\frac{1}{8} - 0\right)$$

$$\boxed{I = \frac{1}{8}}$$

4.2.3 Reduction formula

Case (i): *n* is even

$$\left.\begin{aligned} I &= \int_0^{\pi/2} \sin^n x\,dx \\ I &= \int_0^{\pi/2} \cos^n x\,dx \end{aligned}\right\} = \frac{n-1}{n-0}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

Case (ii): *m* is odd

$$\left.\begin{aligned} I &= \int_0^{\pi/2} \sin^n x\,dx \\ I &= \int_0^{\pi/2} \cos^n x\,dx \end{aligned}\right\} = \frac{n-1}{n-0}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\frac{2}{3}\cdot 1$$

Example 1: Evaluate $\displaystyle\int_0^{\pi/2} \sin^4 x\, dx$

Solution:

$$I = \int_0^{\pi/2} \sin^4 x\, dx$$

$$\boxed{n = 4 \text{ (even)}}$$

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n-0} \cdot \frac{n-3}{n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^4 x\, dx = \frac{4-1}{4-0} \cdot \frac{4-3}{4-2} \dots \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\boxed{I = \frac{3\pi}{16}}$$

Example 2: Evaluate $\displaystyle\int_0^{\pi/2} \cos^7 x\, dx$

Solution:

$$I = \int_0^{\pi/2} \cos^7 x\, dx$$

$$\boxed{n = 7 \text{ (odd)}}$$

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n-0} \cdot \frac{n-3}{n-2} \dots \frac{2}{3} \cdot 1$$

$$\int_0^{\pi/2} \cos^7 x \, dx = \frac{7-1}{7-0} \cdot \frac{7-3}{7-2} \cdot \frac{7-5}{7-4} \cdots \frac{2}{3} \cdot 1$$

$$= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1$$

$$\boxed{\boldsymbol{I = \frac{48}{105}}}$$

Example 3: Evaluate $\displaystyle\int_0^{\pi/2} \sin^6 x \, dx$

Solution:

$$I = \int_0^{\pi/2} \sin^6 x \, dx$$

$$\boxed{\boldsymbol{n = 6 \text{ (even)}}}$$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n-0} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^6 x \, dx = \frac{6-1}{6-0} \cdot \frac{6-3}{6-2} \cdot \frac{6-5}{6-4} \cdot \frac{1}{2} \cdot \frac{n}{2}$$

$$= \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\boxed{\boldsymbol{I = \frac{15\pi}{96}}}$$

Example 4: Evaluate $\int_{0}^{\pi/2} \cos^{10} x\, dx$

Solution:

$$I = \int_{0}^{\pi/2} \cos^{10} x\, dx$$

$n = 10$ (even)

$$\int_{0}^{\pi/2} \cos^{n} x\, dx = \frac{n-1}{n-0} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \dots \frac{1}{2}$$

$$\int_{0}^{\pi/2} \cos^{10} x\, dx = \frac{10-1}{10-0} \cdot \frac{10-3}{10-2} \cdot \frac{10-5}{10-4} \cdot \frac{10-7}{10-8} \cdot \frac{10-9}{10-8} \dots \frac{1}{2} \cdot \frac{9}{2}$$

$$I = \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$I = \frac{945\pi}{7689} = \frac{184\pi}{1536}$$

$$\boxed{\boldsymbol{I = \frac{184\pi}{1536}}}$$

4.2.4 Double integration in polar form

Example 1: Evaluate $\int_{0}^{\pi/2} \int_{0}^{\cos\theta} r dr\, d\theta$

Solution:

$$I = \int_{0}^{\pi/2} \int_{0}^{\cos\theta} r dr\, d\theta$$

Limits
$r \Rightarrow 0$ to $\cos\theta$
$\theta \Rightarrow 0$ to $\dfrac{\pi}{2}$

$$I = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{\cos^2\theta}{2} - 0\right] d\theta$$

$$= \int_0^{\pi/2} \frac{\cos^2\theta}{2}\, d\theta$$

$$\boxed{\Rightarrow \boldsymbol{n = 2,\ \text{even}}}$$

$$\int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{n-1}{n-0}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\times\frac{\pi}{2}$$

$$= \frac{1}{2}\left[\frac{2-1}{2-0}\times\frac{\pi}{2}\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\times\frac{\pi}{2}\right)$$

$$\boxed{\boldsymbol{I = \frac{\pi}{8}}}$$

Example 2: **Evaluate** $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r^2\,dr\,d\theta$

Solution:

$$I = \int_0^{\pi/2}\int_0^{\sin\theta} r^2\,dr\,d\theta$$

$$\boxed{\begin{array}{l}\text{Limits}\\ r \Rightarrow 0 \text{ to } \sin\theta\\ \theta \Rightarrow 0 \text{ to } \dfrac{\pi}{2}\end{array}}$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin^3\theta}{3} - \frac{\theta^3}{3}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^3\theta}{3}\, d\theta$$

$$I = \frac{1}{3}\int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$\boxed{\Rightarrow\ n = 3\ \text{(odd)}}$$

$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{n-1}{n-0} \times \frac{n-3}{n-2} \times \frac{2}{3} \times 1$$

$$= \frac{1}{3}\left[\frac{3-1}{3-0} \times 1\right]$$

$$= \frac{1}{3} \times \frac{2}{3} \times 1$$

$$\boxed{\boldsymbol{I = \frac{2}{9}}}$$

Example 3: Evaluate $\displaystyle\int_{0}^{\pi/2} \int_{0}^{\cos\theta} r^2 \, dr \, d\theta$

Solution:

$$I = \int_{0}^{\pi/2} \int_{0}^{\cos\theta} r^2 \, dr \, d\theta$$

Limits
$r \Rightarrow$ 0 to $\cos\theta$
$\theta \Rightarrow$ 0 to $\frac{\pi}{2}$

$$= \int_{0}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{\cos\theta} d\theta$$

$$= \int_{0}^{\pi/2} \left(\frac{\cos^3\theta}{3} - 0\right) d\theta$$

$$= \int_{0}^{\pi/2} \frac{\cos^3\theta}{3} \, d\theta$$

$$I = \frac{1}{3} \int_{0}^{\pi/2} \cos^3\theta \, d\theta$$

$$\boxed{\Rightarrow \boldsymbol{n = 3 \ \text{(odd)}}}$$

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{n-1}{n-0} \times \frac{n-3}{n-2} \times \frac{2}{3} \times 1$$

$$= \frac{1}{3}\left[\frac{3-1}{3-0} \times 1\right]$$

$$= \frac{1}{3} \times \frac{2}{3} \times 1$$

$$\boxed{\boldsymbol{I = \frac{2}{9}}}$$

Example 4: $\displaystyle\int_0^{\pi/2} \int_0^{\sin\theta} r dr \, d\theta$

Solution:

$$I = \int_0^{\pi/2} \int_0^{\sin\theta} r dr \, d\theta$$

Limits
$r \Rightarrow$ 0 to $\sin\theta$
$\theta \Rightarrow$ 0 to $\dfrac{\pi}{2}$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin^2\theta}{2} - 0\right) d\theta$$

$$I = \frac{1}{2} \int_{0}^{\pi/2} \sin^2 \theta \, d\,\theta$$

$$\boxed{\boldsymbol{n = 2} \textbf{ (even)}}$$

$$\int_{0}^{\pi/2} \sin^n d\,\theta = \frac{n-1}{n-0} \times \frac{n-3}{n-2} \cdots \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{1}{2} \times \frac{2-1}{2-0} \times \frac{\pi}{2}$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\boxed{\boldsymbol{I = \frac{\pi}{8}}}$$

Example 5: Evaluate $\displaystyle\int_{\mathbf{0}}^{\boldsymbol{\pi/2}} \int_{\mathbf{0}}^{\mathbf{sin}\,\boldsymbol{\theta}} \boldsymbol{rd\,\theta\, dr}$

Solution:

$$I = \int_{0}^{\pi/2} \int_{0}^{\sin\theta} r^2 \, d\,\theta \, dr$$

$$= \int_{0}^{\pi/2} \int_{0}^{\sin\theta} r^2 \, d\,\theta \, dr$$

$$I = \int_{0}^{\pi/2} \int_{0}^{\sin\theta} r^2 \, dr \, d\,\theta$$

$$= \int_{0}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{\sin\theta} d\theta$$

$$= \frac{1}{3}\int_{0}^{\pi/2} \sin^3\theta \, d\theta$$

$$I = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

Example 6: Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 \, d\theta \, dr$

Solution:

$$I = \int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 \, d\theta \, dr$$

$$I = \int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2 \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\theta}{3} d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3\theta \, d\theta$$

$$= \frac{8}{3}(2)\int_0^{\pi/3} \cos^3\theta\, d\theta$$

$$= \frac{16}{3} \times \frac{2}{3}$$

$$\boxed{\mathbf{I = \frac{32}{9}}}$$

Note

$$\int_0^{\pi/2} \cos^n\theta\, d\theta$$

(i) $\dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \ldots \dfrac{\pi}{2}$ if n is even

(ii) $\dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \ldots \dfrac{2}{3} \times 1$ if n is odd

4.3 TRANSFORMING INTO POLAR COORDINATES [CHANGING INTO POLAR COORDINATES]

$r, \theta \longrightarrow$ Polar co-ordinates

$x, y \longrightarrow$ Cartesian co-ordinates

Transformation

$$\boxed{\begin{aligned} &x = r\cos\theta \\ &y = r\sin\theta \\ &x^2 + y^2 = r^2 \\ &dx\,dy = dy\,dx = r\,dr\,d\theta \end{aligned}}$$

Example 1: Evaluate by converting into polar co-ordinates $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Solution:

$$\text{Let } I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx \quad \ldots (1)$$

Transformation

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \\ dx\,dy = dy\,dx &= r\,dr\,d\theta \end{aligned}}$$

Limits in cartesian form

x	0	a
y	0	$\sqrt{a^2-x^2}$

Limits in polar form

r	0	a
θ	0	$\frac{\pi}{2}$

$$y = \sqrt{a^2-x^2}$$

Squaring on both sides

$$y^2 = a^2 - x^2$$

$$y^2 + x^2 = a^2$$

$$r^2 = a^2$$

$$\boxed{\boldsymbol{r = a}}$$

Equation (1) becomes

$$I = \int_0^{\pi/2} \int_0^{a} \sqrt{r^2}\, rdr\, d\,\theta$$

$$= \int_0^{\pi/2} \int_0^{a} r \times rdr\, d\,\theta$$

$$= \int_0^{\pi/2} \int_0^{a} \left[\frac{r^3}{3}\right] d\,\theta$$

$$I = \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^a d\,\theta$$

$$I = \int_0^{\pi/2} \frac{a^3}{3}\, d\,\theta$$

$$I = \frac{a^3}{3} \int_0^{\pi/2} d\,\theta = \frac{a^3}{3}\, [\,\theta\,]_0^{\pi/2} \Rightarrow I = \frac{a^3}{3}\left[\frac{\pi}{2}\right]$$

$$\boxed{\Rightarrow \boldsymbol{I = \frac{a^3\,\pi}{6}}}$$

Example 2: By changing into polar coordinates evaluate

$$\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)\, dy\, dx$$

Solution:

Let $$I = \int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2)\, dy\, dx \qquad \text{... (1)}$$

Transformation

$\Rightarrow x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$dx\, dy = dy\, dx = rdr\, d\,\theta$

Limits in cartesian form

x	0	$2a$
y	0	$\sqrt{2ax - x^2}$

Limits in polar form

r	0	$2a\cos\theta$
θ	0	$\frac{\pi}{2}$

$\Rightarrow y = \sqrt{2ax - x^2}$

Squaring on both sides

$y^2 = 2ax - x^2$

$y^2 + x^2 = 2ax$

$r^2 = 2ax$

$r^2 = 2ar\cos\theta$

$$\boxed{\boldsymbol{r = 2a\cos\theta}}$$

Equation (1) becomes

$$I = \int_0^{\pi/2} \left(\int_0^{2a\cos\theta} r^2\, r dr \right) d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2a\cos\theta} d\theta$$

$$I = \int_0^{\pi/2} \frac{(2a\cos\theta)^4}{4}\, d\theta$$

$$I = \int_0^{\pi/2} \frac{2^4 a^4 \cos^4\theta}{4}\, d\theta$$

$$= \frac{16a^4}{4} \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$I = 4a^4 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

Hint

$n = 4$ (even)

$$\int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{n-1}{n-0} \times \frac{n-3}{n-2} \dots \frac{1}{2} \times \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{4-1}{4-0} \times \frac{4-3}{4-2} \dots \frac{1}{2} \times \frac{\pi}{2}$$

$$I = 4a^4 \left[\frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \right]$$

$$= 4a^4 \left(\frac{3\pi}{16} \right)$$

$$\boxed{I = \frac{3\pi a^4}{4}}$$

Example 3: By transforming into polar co-ordinates evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x - x^2}} (x^2 + y^2)\, dy\, dx$

Solution:

$$\text{Let} \quad I = \int_0^2 \int_0^{\sqrt{2x - x^2}} (x^2 + y^2)\, dy\, dx \qquad \ldots (1)$$

Transformation

$$\boxed{\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ x^2 + y^2 &= r^2 \\ dx\, dy = dy\, dx &= r dr\, d\theta \end{aligned}}$$

Limits in cartesian form

x	0	2
y	0	$\sqrt{2x - x^2}$

Limits in polar form

r	0	$2 \cos \theta$
θ	0	$\frac{\pi}{2}$

$$\Rightarrow \quad y = \sqrt{2x - x^2}$$

Squaring on both sides

$$y^2 = 2x - x^2 \Rightarrow x^2 + y^2 = 2x$$

$$r^2 = 2x$$

$$r^2 = 2r \cos \theta$$

$$\boxed{\boldsymbol{r = 2 \cos \theta}}$$

Equation (1) becomes

$$I = \int_0^{\pi/2} \int_0^{2 \cos \theta} r^2 \, r dr \, d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2 \cos \theta} d\theta$$

$$I = \frac{1}{4} \int_0^{\pi/2} (2 \cos \theta)^4 \, d\theta$$

$$I = \frac{1}{4} \int_0^{\pi/2} 2^4 \cos^4 \theta \, d\theta$$

$$I = \frac{16}{4} \int_0^{\pi/2} \cos^4 \theta \, d\theta$$

$$I = 4 \int_0^{\pi/2} \cos^4 \theta \, d\theta$$

$$\int_0^{\pi/2} \cos^n \theta \, d\theta = \frac{n-1}{n-0} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^4 \theta \, d\theta = \frac{4-1}{4-0} \cdot \frac{4-3}{4-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{3\pi}{16}$$

$$I = 4\left[\frac{3\pi}{16}\right]$$

$$\boxed{I = \frac{3\pi}{4}}$$

Example 4: By transforming into polar coordinates evaluate $\displaystyle\int_0^a \int_y^a \frac{x^2 \, dx \, dy}{(x^2 + y^2)^{3/2}}$

Solution:

$$\text{Let } I = \int_0^a \int_y^a \frac{x^2 \, dx \, dy}{(x^2 + y^2)^{3/2}} \quad \ldots (1)$$

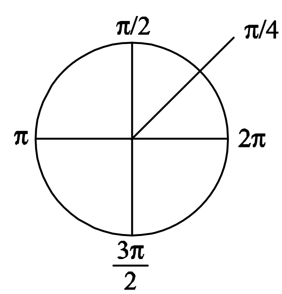
Transformation

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

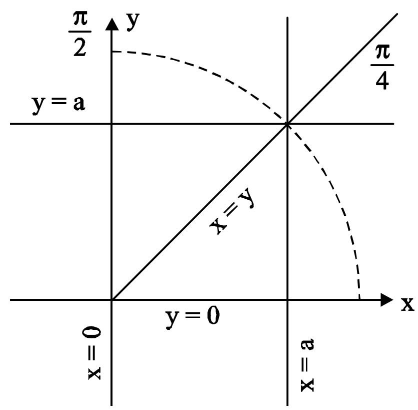
$$dx\,dy = dy\,dx = r\,dr\,d\theta$$



Limits in cartesian form

x	y	a
y	0	a

$x = y$ $y = 0$	$x = a$ $y = a$



Limits in polar form

r	0	$\frac{a}{\cos\theta}$
θ	0	$\frac{\pi}{4}$

$$\Rightarrow \; x = a$$

$$r\cos\theta = a$$

$$r = \frac{a}{\cos\theta}$$

Equation (1) becomes

$$I = \int_0^{\pi/4} \int_0^{\frac{a}{\cos\theta}} \frac{r^2\cos^2\theta}{(r^2)^{3/2}}\, r dr\, d\theta$$

$$I = \int_0^{\pi/4} \int_0^{\frac{a}{\cos\theta}} \frac{r^3\cos^2 d\theta}{r^3}\, r dr\, d\theta$$

$$= \int_0^{\pi/4} \int_0^{\frac{a}{\cos\theta}} \cos^2\theta\, dr\, d\theta$$

$$= \int_{0}^{\pi/4} \cos^2\theta \, [\, r \,]_{0}^{\frac{a}{\cos\theta}} \, d\theta$$

$$= \int_{0}^{\pi/4} \cos^2\theta \left(\frac{a}{\cos\theta} - 0 \right) d\theta$$

$$I = \int_{0}^{\pi/4} a \cos\theta \, d\theta$$

$$\boxed{\begin{array}{l} \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ \sin 0 = 0 \end{array}}$$

$$= a \int_{0}^{\pi/4} \cos\theta \, d\theta$$

$$= a \, [\, \sin\theta \,]_{0}^{\pi/4}$$

$$= a \left[\sin\frac{\pi}{4} - \sin 0 \right]$$

$$= a \frac{1}{\sqrt{2}} - 0$$

$$\boxed{\boldsymbol{I = \frac{a}{\sqrt{2}}}}$$

Example 5: Evaluate by changing $\displaystyle\boldsymbol{\int_{0}^{a} \int_{y}^{a} \frac{x}{x^2 + y^2} \, dx \, dy}$

Solution:

$$\text{Let } I = \int_{0}^{a} \int_{y}^{a} \frac{x}{x^2 + y^2} \, dx \, dy \qquad \text{... (1)}$$

Transformation

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2 + y^2 &= r^2 \\ dx\,dy = dy\,dx &= r dr\,d\theta \end{aligned}}$$

Limits in cartesian form

x	y	a
y	0	a

Limits in polar form

r	0	$\frac{a}{\cos\theta}$
θ	0	$\frac{\pi}{4}$

Equation (1) becomes

$$I = \int_0^{\pi/4}\int_0^{\frac{a}{\cos\theta}} \frac{r\cos\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{\frac{a}{\cos\theta}} \cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta\,[\,r\,]_0^{\frac{a}{\cos\theta}}\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta\left[\frac{a}{\cos\theta} - 0\right]d\theta$$

$$= a \int_0^{\pi/4} d\theta$$

$$= a [\,\theta\,]_0^{\pi/4}$$

$$= a\left[\frac{\pi}{4} - 0\right]$$

$$\boxed{I = \frac{a\pi}{4}}$$

Example 6: Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ **by converting into polar co-ordinates.**

Solution:

$$\text{Let } I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy \qquad \dots (1)$$

Transformation

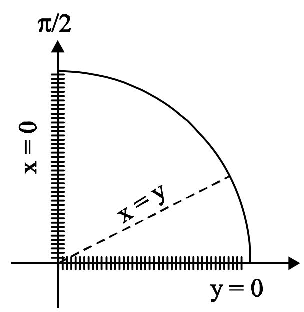
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \\ dx\,dy = dy\,dx &= r\,dr\,d\theta \end{aligned}}$$

Limits in cartesian form

x	0	∞
y	0	∞

Limits in polar form

r	0	∞
0	0	$\pi/2$



Eqn (1) becomes

$$I = \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-r^2} r dr \, d\theta$$

Substitution method
Put $r^2 = t$
$2rdr = dt$
$rdr = \dfrac{dt}{2}$

$$I = \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-t} \frac{dt}{2} \, d\theta$$

$e^{-\infty} = 0$
$e^{-0} = 1$

$$= \frac{1}{2} \int_{0}^{\pi/2} \int_{0}^{\infty} e^{-t} dt \, d\theta$$

$$= \frac{1}{2} \int_{0}^{\pi/2} \left[\frac{e^{-t}}{-1} \right]_0^{\infty} d\theta$$

$$= -\frac{1}{2} \int_{0}^{\pi/2} (e^{-\infty} - e^{-0}) \, d\theta$$

$$= -\frac{1}{2} \int_{0}^{\pi/2} (0 - 1) \, d\theta$$

$$= \frac{1}{2} \int_{0}^{\pi/2} d\theta$$

$$= \frac{1}{2} [\,\theta\,]_0^{\pi/2}$$

$$I = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\boxed{I = \frac{\pi}{4}}$$

Example 7: Transform the double integral into polar coordinates and hence evaluate

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

Solution:

Given: $I = \displaystyle\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$ (not in correct form)

Let $I = \displaystyle\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-(x^2+y^2)}}$ (correct form) ... (1)

Trasformation

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \\ dx\,dy = dy\,dr = rdr\,d\theta \end{array}}$$

Limits in cartesian form

x	0	a
y	$\sqrt{ax-x^2}$	$\sqrt{a^2-x^2}$

Limits in polar form

r	$a\cos\theta$	a
θ	0	$\frac{\pi}{2}$

$y = \sqrt{ax - x^2}$	$y = \sqrt{a^2 - x^2}$
$y^2 = ax - x^2$	$y^2 = a^2 - x^2$
$y^2 + x^2 = ax$	$x^2 + y^2 = a^2$
$r^2 = ar\cos\theta$	$r^2 = a^2$
$\boxed{\boldsymbol{r = a\cos\theta}}$	$\boxed{\boldsymbol{r = a}}$

Equation (1) becomes

$$I = \int_0^{\pi/2} \int_{a\cos\theta}^{a} \frac{rdr\, d\theta}{\sqrt{a^2 - r^2}}$$

Substitution method

Put $\quad a^2 - r^2 = t^2$

$$0 - 2rdr = 2tdt$$

$$-rdr = tdt$$

$$I = \int_0^{\pi/2} \int_{a\sin\theta}^{0} \frac{-tdt}{\sqrt{t^2}}\, d\theta$$

$$= -\int_0^{\pi/2} \int_{a\sin\theta}^{0} dt\, d\theta$$

$$= - \int_0^{\pi/2} [\, t \,]_{a \sin\theta}^{0} \, d\theta$$

$$= - \int_0^{\pi/2} - a \sin\theta \, d\theta$$

$$I = a \int_0^{\pi/2} \sin\theta \, d\theta$$

$$= a \, [\, -\cos\theta \,]_0^{\pi/2}$$

$$= - a \left(\cos\frac{\pi}{2} - \cos 0 \right)$$

$$= - a \, [0 - 1]$$

$$\boxed{I = a}$$

Example 8: By transforming into polar coordinates evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{x^2 + y^2} \, dy \, dx$

Solution:

$$I = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{x^2 + y^2} \, dy \, dx \qquad \ldots (1)$$

Transformation

$$\boxed{\begin{array}{c} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \\ dx\,dy = dy\,dx \;\; rdr\,d\theta \end{array}}$$

Limits in cartesian form

x	0	a
y	0	$\sqrt{a^2 - x^2}$

Limits in polar form

r	0	a
θ	0	$\frac{\pi}{2}$

$$y = \sqrt{a^2 - x^2}$$

$$y^2 = a^2 - x^2$$

$$y^2 + x^2 = a^2$$

$$r^2 = a^2$$

$$\boxed{\boldsymbol{r = a}}$$

Equation (1) becomes

$$I = \int_0^{\pi/2} \int_0^a r^2 \, r dr \, d\theta$$

$$I = \int_0^{\pi/2} \int_0^a r^3 \, dr \, d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^a d\theta$$

$$I = \int_0^{\pi/2} \frac{a^4}{4} \, d\theta$$

$$I = \frac{a^4}{4} \int_0^{\pi/2} d\theta$$

$$I = \frac{a^4}{4} [\,\theta\,]_0^{\pi/2}$$

$$I = \frac{a^4}{4} \left[\frac{\pi}{2} \right]$$

$$\boxed{I = \frac{a^4 \pi}{8}}$$

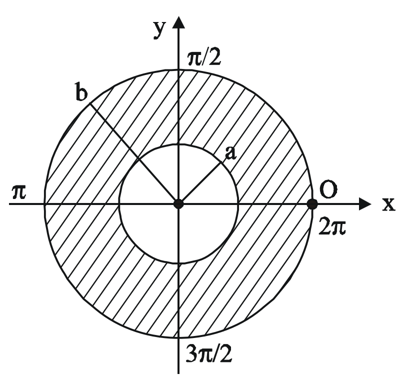
Example 9: By transforming into polar co-ordinates evaluate $\iint \frac{x^2 y^2}{x^2 + y^2} dx\, dy$ over the annular region between the concentric circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$ $(b > a)$.

Solution:

$$I = \iint \frac{x^2 y^2}{x^2 + y^2} dx\, dy \qquad \text{... (1)}$$

Transformation

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2 + y^2 &= r^2 \\ dx\,dy = dy\,dx &= r dr\, d\theta \end{aligned}}$$



Limits in polar form

r	a	b
θ	0	2π

Now (1) becomes

$$I = \int_0^{2\pi} \int_0^{b} \frac{r^2 \cos^2 \theta \, r^2 \sin^2 \theta \, r dr \, d\theta}{r^2}$$

$$= \int_0^{2\pi} \int_0^{b} r^3 \cos^2 \theta \sin^2 \theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} \right]_a^b \cos^2 \theta \sin^2 \theta \, d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} [b^4 - a^4] \cos^2 \theta \sin^2 \theta \, d\theta$$

$$= \frac{b^4 - a^4}{4} \int_0^{2\pi} \cos^2 \theta \, (1 - \cos^2 \theta) \, d\theta$$

$$= \frac{b^4 - a^4}{4} \int_0^{2\pi} (\cos^2 \theta - \cos^4 \theta) \, d\theta$$

$$= \frac{b^4 - a^4}{4} \times 4 \int_0^{\pi/2} (\cos^2 \theta - \cos^4 \theta) \, d\theta$$

$$I = b^4 - a^4 \left[\int_0^{\pi/2} \cos^2 \theta \, d\theta - \int_0^{\pi/2} \cos^4 \theta \, d\theta \right]$$

$$= b^4 - a^4 \left[\left(\frac{1}{2} \times \frac{\pi}{2} \right) - \left(\frac{4-1}{4-0} \times \frac{1}{2} \times \frac{\pi}{2} \right) \right]$$

$$= b^4 - a^4 \left[\frac{\pi}{4} - \left(\frac{3}{4} \times \frac{\pi}{4} \right) \right]$$

$$= b^4 - a^4 \cdot \frac{\pi}{4} \left(1 - \frac{3}{4} \right)$$

$$= (b^4 - a^4)\,\frac{\pi}{4}\left(\frac{1}{4}\right)$$

$$\boxed{\Rightarrow \boldsymbol{I = (b^4 - a^4)\,\frac{\pi}{16}}}$$

Example 10: By transforming into polar co-ordinates evaluate $\displaystyle\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$

Solution:

$$I = \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy \qquad \text{... (1)}$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \\ dx\,dy = dy\,dx &= r\,dr\,d\theta \end{aligned}}$$

Limits in cartesian form

x	y	a
y	0	a

Limits in polar form

r	0	$\dfrac{a}{\cos\theta}$
θ	0	$\dfrac{\pi}{4}$

$$\Rightarrow x = a$$

$$r\cos\theta = a$$

$$r = \frac{a}{\cos\theta}$$

Now (1) becomes

$$I = \int_0^{\pi/4} \int_a^{\frac{a}{\cos\theta}} \frac{r^2 \cos^2\theta}{\sqrt{r^2}}\, r dr\, d\theta$$

$$= \int_0^{\pi/4} \int_0^{\frac{a}{\cos\theta}} r^2 \cos^2\theta\, dr\, d\theta$$

$$= \int_0^{\pi/4} \left[\frac{r^3}{3}\right]_0^{\frac{a}{\cos\theta}} \cos^2\theta\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \left[\left(\frac{a}{\cos\theta}\right)^3 - 0\right] \cos^2\theta\, d\theta$$

$$\boxed{\begin{aligned} &\sec\frac{\pi}{4} = \sqrt{2} \\ &\tan\frac{\pi}{4} = 1 \\ &\sec 0 = 1 \\ &\tan 0 = 0 \end{aligned}}$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{a^3}{\cos^3\theta} \cos^2\theta\, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \frac{1}{\cos\theta}\, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \sec\theta\, d\theta$$

$$= \frac{a^3}{3}\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log(\sec 0 + \tan 0)\right]$$

$$\boxed{\boldsymbol{I = \frac{a^3}{3}\log(\sqrt{2}+1)}}$$

4.4 AREA IN POLAR COORDINATES

$$\boxed{\textbf{Area} = \iint r\,dr\,d\theta}$$

Example 1: Find the area of the cardioid $r = a(1 + \cos\theta)$

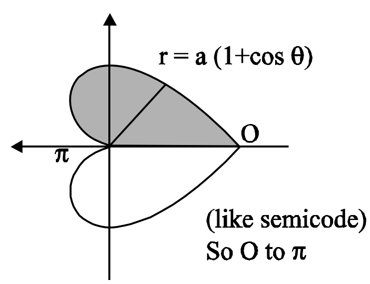
Solution:

Area $A = \iint r\,dr\,d\theta$

$$\boxed{\Rightarrow \textbf{Area of the cardioid } r = a(1+\cos\theta)}$$

$$A = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$A = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$



$$= \frac{2}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$\boxed{(1+\cos\theta)^2 = 1^2 + \cos^2\theta + 2\cos\theta}$$

$$= a^2\int_0^{\pi}(1^2 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$A = a^2\left[\int_0^{\pi} 1^2\,d\theta + \int_0^{\pi} 2\cos\theta\,d\theta + 2\int_0^{\pi/2}\cos^2\theta\,d\theta\right]$$

$$= a^2\left[[\theta]_0^{\pi} + 2[\sin\theta]_0^{\pi} + 2\left(\frac{2-1}{2-0}\times\frac{\pi}{2}\right)\right]$$

[By reduction formula]

$$= a^2 \left[(\pi - 0) + 2 [\sin \pi - \sin 0] + \left(\frac{\pi}{2} \right) \right]$$

$$= a^2 \left[\pi + \frac{\pi}{2} \right] = a^2 \left[\frac{2\pi + \pi}{2} \right]$$

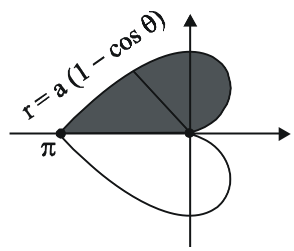
$$\boxed{\mathbf{A = a^2 \left[\frac{3\pi}{2} \right] sq \cdot units}}$$

Example 2: Find the area of cardioid $r = a\,(1 - \cos\theta)$.

Solution:

Area, $A = \iint r dr\, d\theta$

$\Rightarrow$ Given area of the cardioid $r = a\,(1 - \cos\theta)$



$$A = 2 \int_0^{\pi} \int_0^{a(1-\cos\theta)} r dr\, d\theta$$

$$A = 2 \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{2}{2} \int_0^{\pi} a^2 (1 - \cos\theta)^2\, d\theta$$

$$\boxed{(1 - \cos\theta)^2 = 1^2 + \cos^2\theta - 2\cos\theta}$$

$$A = a^2 \int_0^{\pi} (1^2 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$A = a^2 \left[\int_0^{\pi} 1^2\, d\theta - \int_0^{\pi} 2\cos\theta\, d\theta + 2 \int_0^{\pi/2} \cos^2\theta\, d\theta \right]$$

$$= a^2\left[[\,\theta\,]_0^{\pi} - 2\,[\,\sin\theta\,]_0^{\pi} + 2\left(\frac{2-1}{2-0}\times\frac{\pi}{2}\right)\right]$$

(by Reduction formula)

$$= a^2\left[(\pi - 0) - + 2\left(\frac{\pi}{4}\right)\right]$$

$$= a^2\left[\pi + \frac{\pi}{2}\right]$$

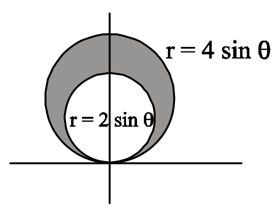
$$= a^2\left(\frac{2\pi + \pi}{2}\right)$$

$$\boxed{\boldsymbol{A = a^2\left(\frac{3\pi}{2}\right)\text{sq}\cdot\text{units}}}$$

Example 3: Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Solution:

$$A = \iint r dr\, d\theta \qquad \text{... (1)}$$



Circles

$$\boxed{\begin{array}{c} r = 2\sin\theta \\ r = 4\sin\theta \end{array}}$$

Limits in polar form

r	$2\sin\theta$	$4\sin\theta$
θ	0	π

(1) becomes

$$A = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4} \right]_{2 \sin \theta}^{4 \sin \theta} d\,\theta$$

$$= \frac{1}{4} \int_0^{\pi} (4 \sin \theta)^4 - (2 \sin \theta)^4 \, d\,\theta$$

$$A = \frac{1}{4} \int_0^{\pi} (4^4 \sin^4 \theta - 2^4 \sin^4 \theta) \, d\,\theta$$

$$= \frac{1}{4} \left[\int_0^{\pi} 256 \sin^4 \theta \, d\,\theta - \int_0^{\pi} 16 \sin^4 \theta \, d\,\theta \right]$$

$$= \frac{1}{4} \left[2 \int_0^{\pi/2} 256 \sin^4 \theta \, d\,\theta - 2 \int_0^{\pi/2} 16 \sin^4 \theta \, d\,\theta \right]$$

[using Reduction formula]

$$= \frac{1}{4} \left[512 \int_0^{\pi/2} \sin^4 \theta \, d\,\theta - 32 \int_0^{\pi/2} \sin^4 \theta \, d\,\theta \right]$$

$$= \frac{1}{4} \left[512 \left(\frac{3\pi}{16} \right) - 32 \left(\frac{3\pi}{16} \right) \right]$$

$$= \frac{512\,(3\pi)}{64} - \frac{32\,(3\pi)}{64} = \frac{(1536 - 96)\,\pi}{64} = \frac{1440}{64}\,\pi$$

$$\boxed{\Rightarrow \boldsymbol{A = 22.5\,\pi} \textbf{ sq.units}}$$

Example 4: Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

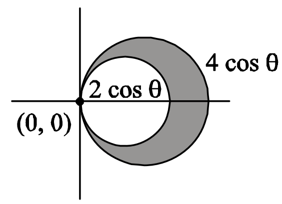
Solution:

$$A = \iint r^3\, dr\, d\theta \qquad \text{... (1)}$$

Circles

$r = 2\cos\theta$

$r = 4\cos\theta$



Limits in polar form

r	$2\cos\theta$	$4\cos\theta$
θ	0	π

Eqn (1) becomes

$$A = \int_0^{\pi} \int_{2\cos\theta}^{4\cos\theta} r^3\, dr\, d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} (4\cos\theta)^4 - (2\cos\theta)^4\, d\theta$$

$$A = \frac{1}{4}\int_0^{\pi} (4^4\cos^4\theta - 2^4\cos^4\theta)\, d\theta$$

$$= \frac{1}{4}\left[\int_0^{\pi} 256\cos^4\theta\, d\theta - \int_0^{\pi} 16\cos^4\theta\, d\theta\right]$$

$$= \frac{1}{4}\left[2 \int_0^{\pi/2} 256 \cos^4 \theta \, d\theta - 2 \int_0^{\pi/2} 16 \cos^4 \theta \, d\theta \right]$$

$$= \frac{1}{4}\left[512 \int_0^{\pi/2} \cos^4 \theta \, d\theta - 32 \int_0^{\pi/2} \cos^4 \theta \, d\theta \right]$$

$$= \frac{1}{4}\left[512 \left(\frac{3\pi}{16} \right) - 32 \left(\frac{3\pi}{16} \right) \right]$$

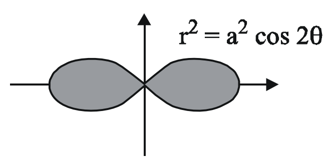
$$= \frac{512\,(3\pi)}{64} - \frac{32\,(3\pi)}{64} = \frac{(1536 - 96)\,\pi}{64} = \frac{1440}{64}\pi$$

$$\boxed{\Rightarrow \boldsymbol{A = 22.5\,\pi} \textbf{ sq.units}}$$

Example 5: Find the Area of the Leminiscate $r^2 = a^2 \cos 2\theta$.

Solution:

Area, $A = \int\int r dr\, d\theta$



Leminiscate

$$r^2 = a^2 \cos 2\theta$$

Polar form limits

r	0	$a\sqrt{\cos 2\theta}$
θ	0	$\frac{\pi}{4}$

$$\text{Area, } A = 4 \int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r dr\, d\theta$$

$$= 4 \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{4^2}{2} \int_0^{\pi/4} (a \sqrt{\cos 2\,\theta})^2 \, d\,\theta$$

$$A = 2 \int_0^{\pi/4} a^2 \cos 2\theta \, d\,\theta$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta \, d\,\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{2a^2}{2} \left[\sin 2 \left(\frac{\pi}{4} \right) - \sin 2\,(0) \right]$$

$$A = a^2 \left[\sin \frac{\pi}{2} - \sin 0 \right]$$

$$A = a^2\,(1 - 0)$$

$$\boxed{\Rightarrow\; \boldsymbol{A = a^2} \textbf{ sq.units}}$$

4.5 AREA IN CARTESIAN COORDINATES

$$A = \int\int dxdy \text{ or } \int\int dy\,dx$$

Example 1: Find the area of ellipse $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

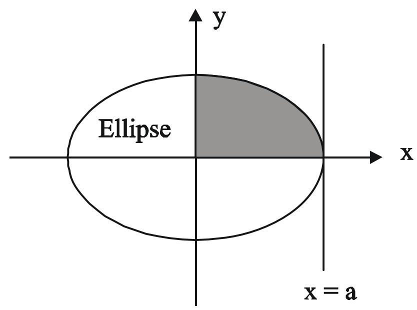
Solution:

$$\boxed{\textbf{Area, } \boldsymbol{A = \int\int dx\,dy}}$$

Ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (1)

Limits in cartesian form

x	0	a
y	0	$b\sqrt{1-\frac{x^2}{a^2}}$



Put $y=0$ in (1)	from (1)
$\frac{x^2}{a^2}=1$	$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$
$x^2=a^2$	$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$
$\boldsymbol{x=a}$	$\left[y=b\sqrt{1-\frac{x^2}{a^2}}\right]$

$$A = 4 \times \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\, dx$$

$$= 4 \times \int_0^a [\, y\,]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 4 \times \int_0^a \left(b\sqrt{1-\frac{x^2}{a^2}} \right) dx$$

$$A = 4b \int_0^a \frac{\sqrt{a^2 - x^2}}{\sqrt{a^2}}\, dx$$

$$= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$\boxed{\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

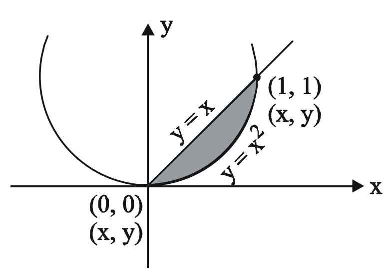
$$A = \frac{4b}{a}\left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right) - \left(\frac{0}{2}\sqrt{a^2 - 0} + \frac{a^2}{2}\sin^{-1}\left(\frac{0}{a}\right)\right)\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)\right] = \frac{4b}{a}\left[\frac{a^2}{2} \times \frac{\pi}{2}\right]$$

$$\boxed{A = ba\pi \text{ sq.units}}$$

Example 2: Find by double integration the area enclosed by curves $y = x$ and $y = x^2$

Solution:



Given $y = x$

$y = x^2$

Area, $A = \iint dy\, dx$

$$\boxed{\begin{array}{l} x \Rightarrow 0 \text{ to } 1 \\ y \Rightarrow x^2 \text{ to } x \end{array}}$$

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 [\,y\,]_{x^2}^{x}\, dx$$

$$= \int_0^1 [x - x^2]\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right]$$

$$\boxed{A = \frac{1}{6}\ \text{sq} \cdot \text{units}}$$

Hint:

$$y = x^2$$

$$x = x^2$$

$$x^2 - x = 0$$

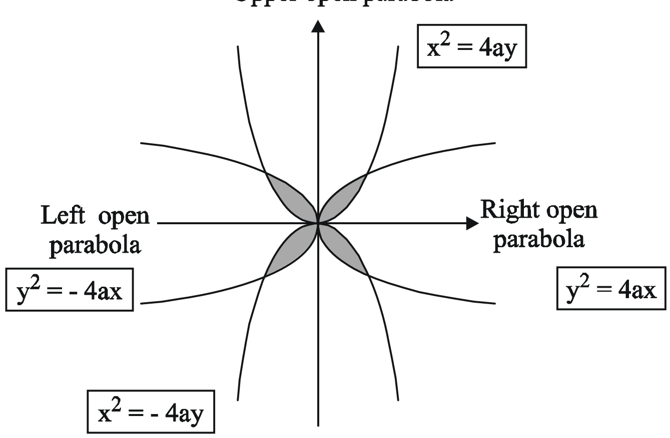
$$x\,(x - 1) = 0$$

$$x = 0,\ x = 1$$

$$x = 0 \Rightarrow y = 0$$

$$x = 1 \Rightarrow y = 1$$

Note



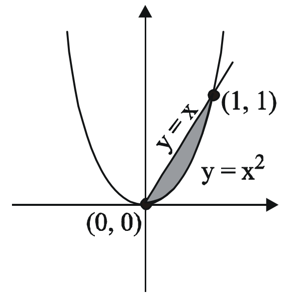
Example 3: Evaluate $\int\int (x - y)\, dx\, dy$ over the region between the line $y = x$ and the parabola $y = x^2$.

Solution:

$$A = \int\int (x - y)\, dx\, dy \qquad \text{... (1)}$$

Given $y = x$

$y = x^2$



x	0	1
y	x^2	x

$$A = \int_0^1 \int_{x^2}^{x} (x - y)\, dy\, dx$$

$$= \int_0^1 \left(xy - \frac{y^2}{2} \right)_{x^2}^{x} dx$$

$$= \int_0^1 \left[\left(x \times x - \frac{(x)^2}{2} \right) - \left(x \times x^2 - \frac{(x^2)^2}{2} \right) \right] dx$$

$$= \int_0^1 \left(x^2 - \frac{x^2}{2} - x^3 + \frac{x^4}{2} \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{5 \times 2} \right]_0^1$$

$$= \left[\left(\frac{1}{3} - \frac{1}{6} - \frac{1}{4} + \frac{1}{10} \right) - 0 \right]$$

$$\boxed{A = \frac{1}{60} \text{ sq} \cdot \text{units}}$$

Example 4: Evaluate double integral $\int\int (x^2 y + xy^2)\, dy\, dx$ over the region between line $y = x$ and $y = x^2$.

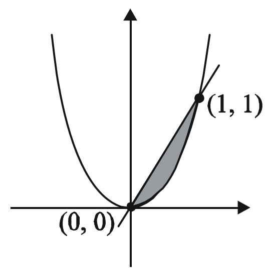
Solution:

$$A = \int\int (x^2 y + xy^2)\, dy\, dx$$

Given: $y = x$

$y = x^2$

x	0	1
y	x^2	x



$$A = \int_0^1 \int_{x^2}^{x} (x^2 y + xy^2)\, dy\, dx$$

$$= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) \right] dx$$

$$= \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$= \left(\frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24} \right)$$

$$\boxed{\mathbf{A = \frac{3}{56}\ sq \cdot units}}$$

Example 5: Evaluate double integral $\iint xy\, dx\, dy$ over the quadrant of the circle $x^2 + y^2 = a^2$.

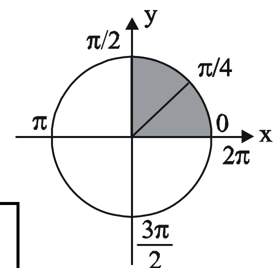
Solution:

$$A = \iint xy\, dx\, dy \qquad \text{... (1)}$$

Given circle

$$\boxed{x^2 + y^2 = a^2}$$

x	0	a
y	0	$\sqrt{a^2-x^2}$



$$\Rightarrow x^2+y^2=a^2 \qquad \ldots (1)$$

Put $y=0$

$$x^2=a^2$$

$$\boxed{x=a}$$

$$(1) \Rightarrow y^2=a^2-x^2$$

Taking square root on both sides

$$y=\sqrt{a^2-x^2}$$

$$A=4\times\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$=4\int_0^a\left[x\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}dx$$

$$=\frac{4}{2}\int_0^a x\,[(\sqrt{a^2-x^2})^2-0]\,dx$$

$$=2\int_0^a x\,(a^2-x^2)\,dx$$

$$=2\int_0^a (a^2x-x^3)\,dx$$

$$=2\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$= 2\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= 2\left(\frac{4a^4 - 2a^4}{8}\right) = \frac{2\,(a^4)}{8}$$

$$\boxed{\mathbf{A = \frac{a^4}{2}\ sq \cdot units}}$$

Example 6: Find the area of the circle $x^2 + y^2 = a^2$

Solution:

$$A = \int\int dx\, dy$$

$$x^2 + y^2 = a^2$$

$$\Rightarrow x^2 + y^2 = a^2 \qquad \text{... (1)}$$

Put $y = 0$ in (1)

$$x^2 = a^2$$

$$\boxed{\boldsymbol{x = a}}$$

$$(1) \Rightarrow y^2 = a^2 - x^2$$

Taking square root on both sides

$$\boxed{\boldsymbol{y = \sqrt{a^2 - x^2}}}$$

Limits

x	0	a
y	0	$\sqrt{a^2 - x^2}$

$$A = 4 \int_0^a \int_0^{\sqrt{a^2 - x^2}} dy\, dx$$

$$= 4 \int_{0}^{y} [\, y\,]_0^{\sqrt{a^2 - x^2}} \; dx$$

$$= 4 \int_{0}^{a} [(\sqrt{a^2 - x^2}) - 0] \; dx$$

$$\boxed{\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right]_0^a$$

$$= 4\left[\frac{a}{2}(\sqrt{a^2 - a^2}) + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) - (0)\right]$$

$$= 4\left[\frac{a^2}{2}\sin^{-1}(1)\right] \qquad \boxed{\because \; \sin^{-1}(1) = \frac{\pi}{2}}$$

$$= 4\left[\frac{a^2}{2} \times \frac{\pi}{2}\right]$$

$$\boxed{\mathbf{A = \pi\, a^2\, sq \cdot units}}$$

Example 7: Find by double integration the area enclosed by the curves $y^2 = 4ax$ and $x^2 = 4ay$

Solution:

$$\text{Area } = \int\int dy\, dx$$

Given curves

$$y^2 = 4ax \qquad \text{... (1)}$$

$$x^2 = 4ay \qquad \text{... (2)}$$

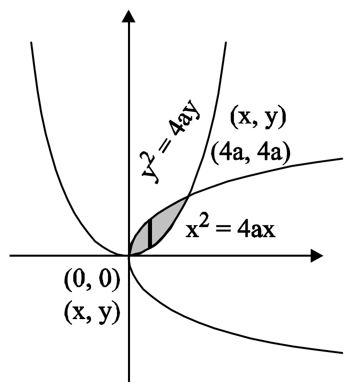
Solving (1) & (2) we get (0, 4a)

x	0	$4a$
y	$\frac{x^2}{4a}$	$2\sqrt{ax}$

Hint

Lower limit	**Upper limit**
$x^2 = 4ay$ $4ay = x^2$ $y = \frac{x^2}{4a}$	$y^2 = 4ax$ $y = \sqrt{4ax}$ $= 2\sqrt{ax}$

$$\Rightarrow A = \int_0^4 \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$$



$$= \int_0^{4a} [\, y\,]_{\frac{x^2}{4a}}^{2\sqrt{ax}}\, dx$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$x^{3/2} = x \cdot \sqrt{x}$

$$= \int_0^{4a} \left(2\sqrt{a}\, x^{1/2} - \frac{x^2}{4a}\right) dx = \left[2\sqrt{a}\,\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3 \times 4a}\right]_0^{4a}$$

$$= \left[\frac{4\sqrt{a}\, x^{3/2}}{3} - \frac{x^3}{12a}\right]_0^{4a} = \left[\left(\frac{4\sqrt{a}\,(4a)\sqrt{4a}}{3} - \frac{(4a)^3}{12a}\right) - (0)\right]$$

$$= \left[\frac{32a^2}{3} - \frac{64a^3}{12}\right] = \left[\frac{32a^2}{3} - \frac{16}{3}a^2\right]$$

$$\boxed{\Rightarrow \mathbf{A = \frac{16a^2}{3}\ \text{sq. units}}}$$

Example 8: Find double integration area enclosed by the curves $x^2 = 4y$ and $y^2 = 4x$

Solution:

$$\text{Area } A = \int\int dy\, dx$$

Given curves

$$\boxed{\begin{array}{c} x^2 = 4y \\ y^2 = 4x \end{array}}$$

x	0	4
y	$\frac{x^2}{4}$	$2\sqrt{x}$

Lower limit	**Upper limit**
$x^2 = 4y$	$y^2 = 4x$
$4y = x^2$	$y = \sqrt{4x}$
$y = \frac{x^2}{4}$	$y = 2\sqrt{x}$

$$\Rightarrow A = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\, dx$$

$$= \int_0^4 [\, y \,]_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx$$

$$= \int_0^4 \left(2.x^{1/2} - \frac{x^2}{4}\right) dx = \left[2\,\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3 \times 4}\right]_0^4$$

$$= \left[\frac{4\,(x^{1/2} \cdot x)}{3} - \frac{x^3}{12}\right]_0^4 = \left[\frac{4\,\sqrt{4}\,4}{3} - \frac{(4)^3}{12} - (0)\right]$$

$$= \left[\frac{32}{3} - \frac{64}{12}\right] = \frac{32 - 16}{3} = \frac{16}{3}$$

$$\boxed{\Rightarrow \boldsymbol{A = \frac{16}{3}\ \mathbf{sq \cdot units}}}$$

4.6 TRIPLE INTEGRAL

4.6.1 Type I [Constant limit]

Example 1: Evaluate $\displaystyle\int_0^a \int_0^b \int_0^c xyz\, dz\, dy\, dx$

Solution:

$$\text{Let } I = \int_0^a \int_0^b \int_0^c xyz\, dz\, dy\, dx$$

$$= \left[\int_0^a xdx\right]\left[\int_0^b ydy\right]\left[\int_0^c zdz\right]$$

$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{a^2}{2}\,\frac{b^2}{2}\,\frac{c^2}{2}$$

$$\Rightarrow I = \frac{(abc)^2}{8}$$

Example 2: Evaluate $\int_0^a \int_0^b \int_0^c e^{x+y+z}\, dz\, dy\, dx$

Solution:

$$\text{Let } I = \int_0^a \int_0^b \int_0^c e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b \int_0^c e^x e^y e^z\, dz\, dy\, dx$$

$$= \left[\int_0^a e^x\, dx\right]\left[\int_0^b e^y\, dy\right]\left[\int_0^c e^z\, dz\right]$$

$$= \left[e^x\right]_0^a \left[e^y\right]_0^b \left[e^z\right]_0^c$$

$$= [e^a - e^0]\,[e^b - e^0]\,[e^c - e^0]$$

$$\Rightarrow I = (e^a - 1)\,(e^b - 1)\,(e^c - 1)$$

EXERCISE

1. Evaluate $\int_0^3 \int_0^2 \int_0^1 x^2 y^2 z^2\, dz\, dy\, dx$

$$I = \frac{216}{27}$$

2. Evaluate $\int_0^e \int_0^f \int_0^h x^3 y^3 z^3\, dz\, dy\, dx$

$$I = \frac{(efb)^4}{64}$$

3. Evaluate $\int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx$

$$I = 6$$

4. Evaluate $\int_0^a \int_0^b \int_0^c dx\, dy\, dz$

$$I = abc$$

4.6.2 Type II (Variable limit)

Example 1: Evaluate $\int_0^1 \int_0^x \int_0^{\sqrt{x+y}} z\, dz\, dy\, dx$ **[correct form]**

Solution:

$$\text{Let } I = \int_0^1 \int_0^x \int_0^{\sqrt{x+y}} z\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\, dx$$

$$= \int_0^1 \int_0^x \left(\frac{x+y}{2}\right) dy\, dx$$

$$= \int_0^1 \int_0^x \left(\frac{x}{2} + \frac{y}{2}\right) dy\, dx$$

$$= \int_0^1 \left[\frac{xy}{2} + \frac{y^2}{4}\right]_0^x dx = \int_0^1 \left(\frac{x^2}{2} + \frac{x^2}{4}\right) dx$$

$$= \left[\frac{x^3}{6} + \frac{x^3}{12}\right]_0^1 = \left[\frac{1}{6} + \frac{1}{12}\right]$$

$$\boxed{\Rightarrow \boldsymbol{I = \frac{1}{4}}}$$

Example 2: $\displaystyle\int_0^1 \int_{y^2}^1 \int_0^{1-x} \boldsymbol{xdz\, dy\, dx}$ [not in correct form]

Solution:

[correct form]

$$I = \int_0^1 \int_{y^2}^1 \int_0^{1-x} xdz\, dx\, dy$$

$$I = \int_0^1 \int_{y^2}^1 x\, [\, z\,]_0^{1-x}\; dx\, dy$$

$$= \int_0^1 \int_{y^2}^1 x\,(1-x)\, dx\, dy = \int_0^1 \int_{y^2}^1 (x - x^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^1 dy = \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right] dy$$

$$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy$$

$$= \left[\frac{1}{6}y - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$= \left[\frac{1}{6} - \frac{1}{10} + \frac{1}{21} - (0)\right]$$

$$\boxed{\Rightarrow \boldsymbol{I = \frac{4}{35}}}$$

Example 3: Evaluate $\displaystyle\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{x+y} x\,dz\,dy\,dx$ [correct form]

Solution:

$$I = \int_0^1 \int_0^{1-x} \int_0^{x+y} x\,dz\,dy\,dx$$

$$I = \int_0^1 \int_0^{1-x} x\,[\,z\,]_0^{x+y}\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} x\,(x+y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (x^2 + xy)\,dy\,dx$$

$$= \int_0^1 \left[x^2 y + x\frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left(x^2 (1-x) + \frac{x\,(1-x)^2}{2} \right) dx$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{x\,(1 + x^2 - 2x)}{2} \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^2}{4} + \frac{x^4}{8} - \frac{2x^3}{6} \right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{4} + \frac{1}{8} - \frac{2\,(1)}{6} - (0) \right] = \left(\frac{1}{3} + \frac{1}{8} - \frac{2}{6} \right) = \frac{1}{8}$$

$$\boxed{\Rightarrow \boldsymbol{I = \frac{1}{8}}}$$

Example 4: Evaluate $\displaystyle\int_0^a \int_0^b \int_0^c (x + y + z)\, dz\, dy\, dx$

Solution:

$$I = \int_0^a \int_0^b \int_0^c (x + y + z)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2} \right]_0^c dy\, dx$$

$$= \int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2} \right) dy\, dx$$

$$= \int_0^a \left[xcy + \frac{y^2}{2}c + \frac{c^2 y}{2} \right]_0^b dx$$

$$= \int_0^a \left(xbc + \frac{b^2}{2}c + \frac{c^2 b}{2} \right) dx$$

$$= \left[\frac{x^2}{2}bc + \frac{b^2}{2}cx + \frac{c^2 b}{2}x \right]_0^a$$

$$= \frac{a^2}{2}bc + \frac{b^2}{2}ca + \frac{c^2}{2}ba$$

$$\boxed{\Rightarrow \boldsymbol{I = \frac{abc}{2}[a + b + c]}}$$

Example 5: Evaluate $\displaystyle\int_0^4 \int_0^1 \int_0^1 (x + y + z)\, dz\, dy\, dx$

Solution:

$$I = \int_0^4 \int_0^1 \int_0^1 (x + y + z)\, dz\, dy\, dx$$

$$= \int_0^4 \int_0^1 \left[xz + yz + \frac{z^2}{2} \right]_0^1 dy\, dx$$

$$= \int_0^4 \int_0^1 \left(x + y + \frac{1}{2} \right) dy\, dx$$

$$= \int_0^4 \left(xy + \frac{y^2}{2} + \frac{1}{2} y \right)_0^1 dx$$

$$= \int_0^4 \left(x + \frac{1}{2} + \frac{1}{2} \right) dx$$

$$= \left[\frac{x^2}{2} + \frac{1}{2} x + \frac{1}{2} x \right]_0^4$$

$$= \left(\frac{16}{2} + \frac{4}{2} + \frac{4}{2} \right)$$

$$\boxed{\Rightarrow \boldsymbol{I = 12}}$$

Example 6: **Evaluate** $\displaystyle\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dz\, dy\, dx$

Solution:

$$I = \int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dz\, dy\, dx$$

$$I = \int_0^a \int_0^b \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_0^c dy\, dx$$

$$= \int_0^a \int_a^b \left(x^2 c + y^2 c + \frac{c^2}{3} \right) dy\, dx$$

$$= \int_0^a \left[x^2 cy + \frac{y^3}{3} + \frac{c^3}{3} y \right]_a^b dx$$

$$= \int_0^a \left(x^2 bc + \frac{b^3}{3} c + \frac{c^3}{3} b \right) dx$$

$$= \left[\frac{x^3}{3} bc + \frac{b^3 c}{3} x + \frac{c^3 b}{3} x \right]_0^a$$

$$= \frac{a^3 b}{3} c + \frac{b^3 c}{3} a + \frac{c^3 b}{3} a$$

$$I = \frac{abc\,(a^2 + b^2 + c^2)}{3}$$

Example 7: Evaluate $\displaystyle\int_{0}^{2a}\int_{0}^{x}\int_{y}^{x} xyz\,dz\,dy\,dx$

Solution:

$$I = \int_{0}^{2a}\int_{0}^{x}\int_{y}^{x} xyz\,dz\,dy\,dx \qquad \text{... (1)}$$

$$= \int_{0}^{2a}\int_{0}^{x} xy\left[\frac{z^2}{2}\right]_y^x dy\,dz$$

$$= \frac{1}{2}\int_{0}^{2a}\int_{0}^{x} xy\,(x^2 - y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{2a}\int_{0}^{x} (x^3\,y - xy^3)\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{2a}\left[\frac{x^3\,y^2}{2} - \frac{xy^4}{4}\right]_0^x dx$$

$$= \frac{1}{2}\int_{0}^{2a}\left(\frac{x^5}{2} - \frac{x^5}{4}\right)dx$$

$$= \frac{1}{2}\left[\frac{x^6}{12} - \frac{x^6}{24}\right]_0^{2a}$$

$$= \frac{1}{2}\left(\frac{(2a)^6}{12} - \frac{(2a)^6}{24}\right)$$

$$I = \frac{1}{2}\left(\frac{64a^6}{12} - \frac{64a^6}{24\times 1}\right)$$

$$= \frac{1}{2}\left(\frac{128a^6 - 64a^6}{24}\right)$$

$$= \frac{1}{2}\left(\frac{64}{24}a^6\right)$$

$$\boxed{I = \frac{4a^6}{3}}$$

Example 8: Evaluate $\displaystyle\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}(x^2 + y^2 + z^2)\,dx\,dy\,dx$

Solution:

If $f(x)$ is even

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Since $f(x)$ is even.

$$I = 2 \times 2 \times 2 \int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\,dz\,dy\,dx$$

$$= 8 \int_a^b \int_a^b \left[x^2 z + y^2 z + \frac{z^3}{3}\right]_0^c dy\,dx$$

$$= 8 \int_0^a \int_0^b \left(cx^2 + cy^2 + \frac{c^3}{3}\right) dy\,dx$$

$$= 8 \int_0^a \left[cx^2 y + \frac{cy^3}{3} + \frac{c^3}{3} y\right]_0^b dx$$

$$= 8 \int_0^a \left(bcx^2 + \frac{b^3}{3} c + \frac{c^3}{3} b \right) dx$$

$$= 8 \left[bc \frac{x^3}{3} + \frac{b^3}{3} cx + \frac{c^3}{3} bx \right]_0^a$$

$$= 8 \left(\frac{a^3}{3} bc + \frac{ab^3}{3} c + \frac{abc^3}{3} \right)$$

$$\boxed{\boldsymbol{I = 8 \frac{abc}{3} (a^2 + b^2 + c^2)}}$$

Example 9: Evaluate $\displaystyle \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(1 + x + y + z)^3}$

Solution:

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(1 + x + y + z)^3}$$

$$\Rightarrow \int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{(n + 1)\, a}$$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 + x + y + z)^{-3} \, dz\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(z + 1 + x + y)^{-2}}{-2} \right]_0^{1-x-y} dy\, dx$$

$$= \frac{-1}{2} \int_0^1 \int_0^{1-x} [(1 - x - y + 1 + x + y)^{-2} - (1 + x + y)^{-2}] \, dy\, dx$$

$$= \frac{-1}{2} \int_0^1 \int_0^{1-x} [(2)^{-2} - (1+x+y)^{-2}]\, dy\, dx]\, dy\, dx$$

$$= \frac{-1}{2} \int_0^1 \int_0^{1-x} \left(\frac{1}{4} - (1+x+y)^{-2} \right) dy\, dx$$

$$= \frac{-1}{2} \int_0^1 \left[\frac{1}{4} y - \frac{(y+x+1)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left(\frac{1}{4}(1-x) + (1-x+x+1)^{-1} \right) - \left[\frac{1}{4}(0) + 1\,(1+x)^{-1} \right] dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4} - \frac{x}{4} + \frac{1}{2} - \frac{1}{1+x} \right] dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{3}{4} - \frac{x}{4} - \frac{1}{1+x} \right] dx$$

$$= -\frac{1}{2} \left[\frac{3}{4} x - \frac{x^2}{8} - \log(1+x) \right]_0^1$$

$$= -\frac{1}{2} \left[\left(\frac{3}{4} - \frac{1}{8} - \log 2 \right) - (0 - 0 - 0) \right]$$

$$= -\frac{1}{2} \left[\frac{5}{8} - \log 2 \right]$$

$$\boxed{\boldsymbol{I = \frac{1}{2} \log 2 - \frac{5}{16}}}$$

Example 10: Evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

Solution:

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)}$$

$$a^2 = a^2 - x^2 - y^2 \;\Rightarrow\; a = \sqrt{a^2-x^2-y^2}$$

$$x^2 = z^2 \;\Rightarrow\; x = z$$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sin^{-1}\left(\frac{\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}}\right) dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sin^{-1}(1)\,dy\,dx$$

$$= \frac{\pi}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx$$

$$= \frac{\pi}{2} \int_0^a [\,y\,]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$= \frac{\pi}{2} \left[\frac{a^2}{2} \sin^{-1}\left(\frac{a}{a}\right) \right]$$

$$= \frac{\pi}{2} \left[\frac{a^2}{2} \times \frac{\pi}{2} \right]$$

$$\boxed{I = \frac{a^2 \pi^2}{8}}$$

4.7 VOLUME INTEGRAL

Volume, $v = \iiint_v dz\, dy\, dx$

Where v is the volume of the given surface

Example 1: Find the volume of sphere $x^2 + y^2 + z^2 = a^2$

Solution:

$$\boxed{\textbf{Volume of sphere } = 8 \times \textbf{ volume in } 1^{\text{st}} \textbf{ octant}}$$

$$x^2 + y^2 + z^2 = a^2 \qquad \text{... (1)}$$

Put $y = 0, z = 0$ in (1)

$$x^2 = a^2$$

$\Rightarrow$ $\boldsymbol{x = a}$

Put $z = 0$ in (1)

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$\Rightarrow y = \sqrt{a^2 - x^2}$$

(1) $\Rightarrow$ $z^2 = a^2 - x^2 - y^2$

$$z = \sqrt{a^2 - x^2 - y^2}$$

Limits

x	0	a
y	0	$\sqrt{a^2-x^2}$
z	0	$\sqrt{a^2-x^2-y^2}$

$$v = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$$

$$v = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [\,z\,]_0^{\sqrt{a^2-x^2-y^2}}\, dy\, dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$$

$$\Rightarrow \text{Put } \sqrt{a^2-x^2} = t$$

$$a^2 - x^2 = t^2$$

$$= 8\int_0^a \int_0^t \sqrt{t^2-y^2}\, dy\, dx$$

$$\boxed{\begin{aligned} \sqrt{a^2-x^2} &= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \\ \sqrt{t^2-y^2} &= \frac{y}{2}\sqrt{t^2-y^2} + \frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right) \end{aligned}}$$

$$\sin^{-1}(0) = 0$$

$$\sin^{-1}(1) = \frac{\pi}{2}$$

$$v = 8\int_0^a \left[\frac{y}{2}\sqrt{t^2-y^2} + \frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t dx$$

$$= 8 \int_0^4 \left[\left(\frac{t}{2}\sqrt{t^2 - t^2} + \frac{t^2}{2}\sin^{-1}\left(\frac{t}{t}\right)\right) - \left(\frac{0}{2}\sqrt{t^2 - 0^2} + \frac{t^2}{2}\sin^{-1}\left(\frac{0}{t}\right)\right)\right] dx$$

$$= 8 \int_0^a \left[\frac{t^2}{2}\sin^{-1}(1)\right] dx = \frac{8}{2}\int_0^a t^2 \frac{\pi}{2}\, dx$$

$$= \frac{8}{2}\cdot\frac{\pi}{2}\int_0^a (a^2 - x^2)\, dx = 2\pi \int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\pi\left[a^3 - \frac{a^3}{3}\right]$$

$$= 2\pi\left[\frac{3a^3 - a^3}{3}\right] = 2\pi\cdot\left(\frac{2a^3}{3}\right) = \frac{4}{3}\pi a^3 \text{ cubic units}$$

$$\boxed{\therefore \textbf{ Volume of sphere } v = \frac{4}{3}\pi a^3 \textbf{ cubic units}}$$

NOTE

- Find the volume of the sphere

$$x^2 + y^2 + z^2 = 3^2 \quad \boxed{a = 3}$$

$\Rightarrow$ **Ans:** $v = \frac{4}{3}\pi (3)^3$ cubic units

- Find the volume of the sphere

$$x^2 + y^2 + z^2 = 16 \quad \boxed{a = 4}$$

$\Rightarrow$ **Ans:** $v = \frac{4}{3}\pi (4)^3$ cubic units

- $x^2 + y^2 + z^2 = r^2 \Rightarrow v = \frac{4}{3}\pi (r)^3$ cubic units.

Example 2: Find the volume of the Ellipsoid

$$\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.}$$

Solution:

Volume $v=\iiint dz\,dy\,dx$

$$\left.\begin{array}{l}\text{Volume of the}\\ \text{Ellipsoid, } v\end{array}\right\} = 8\times \text{ volume in the } 1^{\text{st}} \text{ octant}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots (1)$$

Put $y=0, z=0$ in (1)

$$\frac{x^2}{a^2}=1$$

$$x^2=a^2$$

$$\boxed{\boldsymbol{x=a}}$$

Put $z=0$ in (1)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left[1-\frac{x^2}{a^2}\right]$$

$$y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$z^2 = c^2\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]$$

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Limits

x	0	a
y	0	$b\sqrt{1-\frac{x^2}{a^2}}$
z	0	$c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$v = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$v = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right] dy\,dx$$

$$= 8c \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$$

$$\boxed{\begin{aligned} &\text{Put} \quad t = b\sqrt{1-\frac{x^2}{a^2}} \\ &t^2 = b^2\left(1-\frac{x^2}{a^2}\right) \\ &\frac{t^2}{b^2} = \left(1-\frac{x^2}{a^2}\right) \end{aligned}}$$

$$\Rightarrow v = 8c \int_0^a \int_0^t \sqrt{\frac{t^2}{b^2} - \frac{y^2}{b^2}}\, dy\, dx$$

$$= \frac{8c}{b} \int_0^a \int_0^t \sqrt{t^2 - y^2}\, dy\, dx$$

$$= \frac{8c}{b} \int_0^a \left[\frac{y}{2}\sqrt{t^2-y^2} + \frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t dx$$

$$= \frac{8c}{b} \int_0^a \left(\frac{t^2}{2}\sin^{-1}\left(\frac{t}{t}\right)\right) dx$$

$$= \frac{8c}{b} \int_0^a \frac{t^2}{2}\left(\frac{\pi}{2}\right) dx = \frac{8c\,\pi}{4b} \int_0^a t^2\, dx$$

$$= \frac{2c}{b} \int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{2c\,\pi\,b^2}{b} \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2c\,\pi\,b \left[x - \frac{x^3}{3a^2} \right]_0^a$$

$$= 2c\,\pi\,b \left[a - \frac{a^3}{3a^2} \right] = 2c\,\pi\,b \left[\frac{3a^3 - a^3}{3a^2} \right]$$

$$= 2c\,\pi\,b \left(\frac{2a^3}{3a^2} \right)$$

$$v = \frac{4c\,\pi\,ba}{3}$$

$$\boxed{\therefore\ v = \frac{4abc\,\pi}{3} \text{ cubic units}}$$

Example 3: Find the volume of the tetrahedran bounded by the coordinate planes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution:

$$\text{Volume } = \int\int\int dz\, dy\, dx$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

Put $y = 0, z = 0$ in (1)

$$\frac{x}{a} = 1$$

$$\boxed{x = a}$$

Put $z = 0$ in (1)

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{y}{b} = 1 - \frac{x}{a}$$

$$y = b\left(1 - \frac{x}{a}\right)$$

Limits $\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b}$

x	0	a
y	0	$b\left(1 - \frac{x}{a}\right)$
z	0	$c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

$$v = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} [\,z\,]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\, dx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx$$

$$\boxed{\text{Put} \qquad t = b\left(1 - \frac{x}{a}\right)} \Rightarrow \frac{t}{b} = 1 - \frac{x}{a}$$

$$v = c \int_0^a \int_0^t \left(\frac{t}{b} - \frac{y}{b}\right) dy\, dx$$

$$= \frac{c}{b} \int_0^a \left[ty - \frac{y^2}{2}\right]_0^t dx$$

$$= \frac{c}{b} \int_0^a \left(t^2 - \frac{t^2}{2} \right) dx = \frac{c}{b} \int_0^a \left(\frac{2t^2 - t^2}{2} \right) dx$$

$$= \frac{c}{2b} \int_0^a t^2 \, dx = \frac{c}{2b} \int_0^a \left[b \left(1 - \frac{x}{a} \right) \right]^2 dx$$

$$= \frac{cb^2}{2b} \int_0^a \left(1^2 + \frac{x^2}{a^2} - \frac{2x}{a} \right) dx$$

$$= \frac{cb}{2} \left[x + \frac{x^3}{3a^2} - \frac{2x^2}{2a} \right]_0^a$$

$$= \frac{cb}{2} \left[a + \frac{a^3}{3a^2} - \frac{2\,(a)^2}{2a} \right]$$

$$= \frac{cb}{2} \left[a + \frac{a}{3} - a \right]$$

$$I = \frac{abc}{6}$$

$$\boxed{\Rightarrow I = \frac{abc}{6} \text{ cubic units}}$$

4.8 CHANGE OF ORDER OF INTEGRATION

Change of order of integration

- Change of order of integration is done to make the evaluation of integral easier

Note 1

When all the **limits are constants**. We can change the order of integration as we like, the only point to be remembered is that the limits of x are to be retained for 'x' (and those of y used for y only)

Note 2

When the **limits are variable** the change in the order of integration changes the limits of integration. **To find** the **new limits** it is always **Draw a Rough sketch** of the region of integration.

Problems based on change of order of integration

Example 1: Change the order of integration in

$$\int_0^a \int_x^a (x^2 + y^2)\, dy\, dx$$ **and hence evaluate it**

Solution:

- The region of integration is bounded by

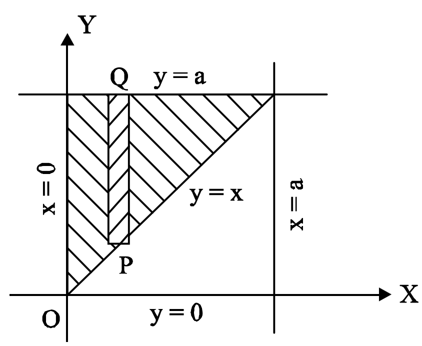
$$x = 0, x = a$$

$$y = x, y = a$$

The region of integration is a triangle

Hence $x = 0$ to a represents vertical path

$y = x$ to a represents vertical strip PQ sliding area.



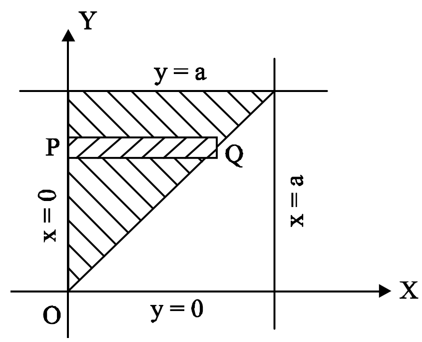
Rough Diagram

- To change the order of integration to change the **vertical path into horizontal path** and to change the **vertical strip to horizontal strip** RS.
- $x = 0$ to y represents Horizontal strip RS sliding area

Hence by changing the order we get

$$\int_0^a \int_0^y (x^2 + y^2)\, dx\, dy$$

$$= \int_0^a \left[\frac{x^3}{3} + y^2 x\right]_0^y dy$$



$$= \int_0^a \left[\frac{y^3}{3} + y^3\right] dy$$

$$= \left[\frac{y^4}{3 \times 4} + \frac{y^4}{4}\right]_0^a$$

$$= \frac{a^4}{12} + \frac{a^4}{4}$$

$$= \frac{a^4 + 3a^4}{12}$$

$$= \frac{4a^4}{12}$$

$$\boxed{I = \frac{a^4}{3}}$$

Example 2: Change the order of integration for

$$\int_0^1 \int_0^x f(x, y)\, dx\, dy$$

[not in correct form]

Solution:

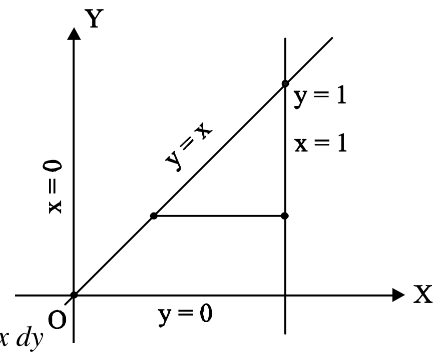
$$I = \int_0^1 \int_0^x f(x, y)\, dy\, dx$$

Given limits

x	0	1
y	0	x

Changed limits

x	y	1
y	0	1



$$\Rightarrow I = \int_0^1 \int_y^1 f(x, y)\, dx\, dy$$

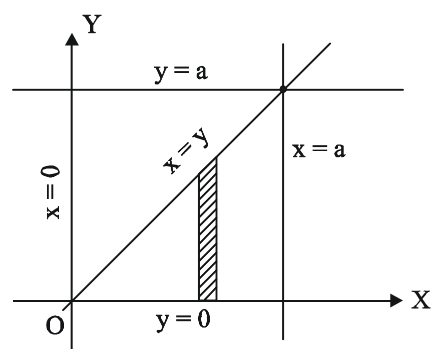
Example 3: $\displaystyle\int_0^a \int_y^a f(x, y)\, dx\, dy$

Solution:

$$I = \int_0^a \int_y^a f(x, y)\, dx\, dy \qquad \ldots (1)$$

Given limits

x	y	a
y	0	a



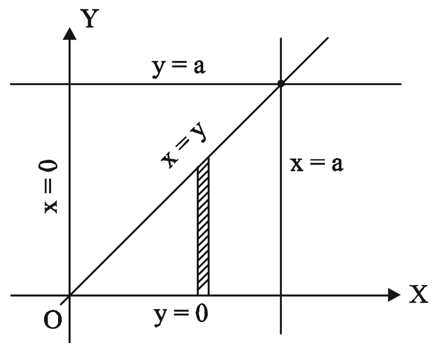
$$\Rightarrow I = \int_0^a \int_0^x f(x, y)\, dy\, dx$$

Example 4: Changed the order of integration and hence evaluate $\displaystyle\int_0^a \int_x^a (x^2 + y^2)\, dx\, dy$

Solution:

Given: $\displaystyle\int_0^a \int_x^a (x^2 + y^2)\, dx\, dy$

$$I = \int_0^a \int_x^a (x^2 + y^2)\, dy\, dx \qquad \ldots (1)$$



Given limits

x	0	a
y	x	a

Changed limits

x	y	a
y	0	a

$$\Rightarrow I = \int_0^a \int_y^a (x^2 + y^2)\, dx\, dy$$

$$= \int_0^a \left[\frac{x^3}{3} + y^2 x \right]_y^a dy$$

$$= \int_0^a \left[\left(\frac{a^3}{3} + y^2 a \right) - \left(\frac{y^3}{3} + y^3 \right) \right] dy$$

$$= \int_0^a \left[\frac{a^3}{3} + ay^2 - \frac{y^3}{3} - y^3 \right] dy$$

$$= \left[\frac{a^3}{3} y + \frac{ay^3}{3} - \frac{y^4}{3 \times 4} - \frac{y^4}{4} \right]_0^a$$

$$= \left[\frac{a^4}{3} + \frac{a^4}{3} - \frac{a^4}{12} - \frac{a^4}{4} \right]$$

$$= a^4 \left[\frac{1}{3} + \frac{1}{3} - \frac{1}{12} - \frac{1}{4} \right] = a^4 \left[\frac{1}{3} \right]$$

$$\boxed{\boldsymbol{I = \frac{a^4}{3}}}$$

Example 5: Changed the order of integration and hence evaluate $\displaystyle\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Solution:

$$I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$$

Given limits

x	0	∞
y	x	∞

Changed limits

x	0	y
y	0	∞

$$I = \int_0^\infty \int_0^y \frac{e^{-y}}{y} \cdot dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [\, x \,]_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (y)\, dy = \int_0^\infty e^{-y}\, dy$$

$$I = \left[\frac{e^{-y}}{-1} \right]_0^\infty$$

$$= -\,[e^{-\infty} - e^{-0}]$$

$$= -\,[0 - 1] = 1$$

$$\boxed{\Rightarrow \boldsymbol{I = 1}}$$

Example 6: Change the order of integration and hence evaluate $\displaystyle\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$

Solution:

$$I = \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx \qquad \text{... (1)}$$

Given limits

x	0	$4a$
y	$\frac{x^2}{4a}$	$2\sqrt{ax}$

Changed limit

x	$\frac{y^2}{4a}$	$2\sqrt{ay}$
y	0	$4a$

$$\Rightarrow I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a} y\left[\frac{x^2}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^{4a} y\left[(2\sqrt{ay})^2 - \left(\frac{y^2}{4a}\right)^2\right] dy$$

$$= \frac{1}{2}\int_0^{4a} y\left(4ay - \frac{y^4}{16a^2}\right) dy$$

$$= \frac{1}{2} \int_0^{4a} \left(4ay^2 - \frac{y^5}{16a^2} \right) dy$$

$$= \frac{1}{2} \left[\frac{4ay^3}{3} - \frac{y^6}{6 \times 16a^2} \right]_0^{4a}$$

$$= \frac{1}{2} \left[\frac{4a\,(4a)^3}{3} - \frac{(4a)^6}{96a^2} \right]$$

$$= \frac{1}{2} \left[\frac{256a^4}{3} - \frac{4096a^6}{96a^2} \right]$$

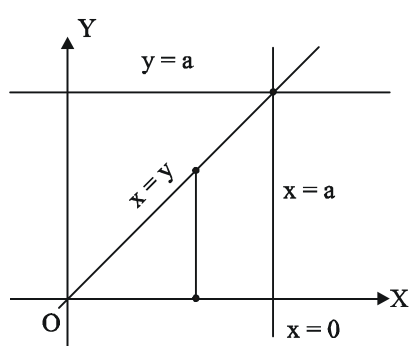
$$= \frac{1}{2} \left[\frac{256a^4}{3} - \frac{128a^4}{3} \right]$$

$$= \frac{1}{2} \left[\frac{128}{3} \right] a^4 = \frac{64}{3} a^4$$

$$\boxed{\boldsymbol{I = \frac{64}{3} a^4}}$$

Example 7: Changed the order of integration and hence evaluate $\displaystyle\int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$**.**

Solution:



$$I = \int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$$

Given limits

x	y	a
y	0	a

Changed limits

x	0	a
y	0	x

$$I = \int_0^a \int_0^x \frac{x}{x^2 + y^2}\, dy\, dx$$

$$= \int_0^a \int_0^x \frac{x}{x^2 + y^2}\, dy\, dx$$

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)}$$

$$= \int_0^a x\left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^a \left[\tan^{-1}\left(\frac{x}{x}\right) - \tan^{-1}\left(\frac{0}{x}\right)\right) dx$$

$$= \int_0^a \left(\frac{\pi}{4} - 0\right) dx$$

$$= \frac{\pi}{4} \int_0^4 dx$$

$$= \frac{\pi}{4} [\, x \,]_0^a$$

$$= \frac{\pi}{4} (a - 0)$$

$$\boxed{I = \frac{\pi a}{4}}$$

NOTE

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\Rightarrow \frac{1}{2}\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\log(x^2+a^2) + c$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right) + c$$

$$\log a + \log b = \log ab$$

$$\log a - \log b = \log \frac{a}{b}$$

Example 8: Change the order of integration and hence evaluate $\displaystyle\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$**.**

Solution:

$$I = \int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

Given limits

x	y	a
y	0	a

Changed limit

x	0	a
y	0	x

$$I = \int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$$

$$= \int_0^a x\left[\int_0^a \frac{1}{\sqrt{x^2+y^2}}\,dy\right]dx$$

$$= \int \frac{dx}{\sqrt{x^2+y^2}} = \log\,(x+\sqrt{x^2+a^2})$$

$$I = \int_0^a x\left[\log\,(y+\sqrt{y^2+x^2})\right]_0^x dx$$

$$= \int_0^a x\,[\log\,(x+\sqrt{x^2+x^2}) - \log\,(0+\sqrt{0^2+x^2})]\,dx$$

$$= \int_0^a x\,[(\log x+\sqrt{2x^2} - \log\,(\sqrt{x^2})]\,dx$$

$$= \int_0^a x\,[\log\,(x+\sqrt{2}x) - \log\,(x)]\,dx$$

$$= \int_0^a x\,[\log\,(1+\sqrt{2})\,x - \log x]\,dx$$

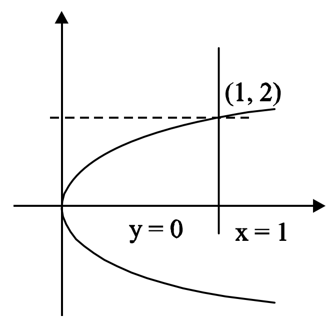
$$= \int_0^a x\log\left[\frac{(1+\sqrt{2})}{x}\,x\right]dx$$

$$= \int_0^a x\log\left(\frac{1+\sqrt{2}}{1}\right)dx = \log\,(1+\sqrt{2})\int_0^a x\,dx$$

$$= \log\,(1+\sqrt{2})\left[\frac{x^2}{2}\right]_0^a = \log\,(1+\sqrt{2})\left(\frac{a^2}{2}-0\right)$$

$$\Rightarrow\ I = \log\,(1+\sqrt{2})\,\frac{a^2}{2}$$

Example 9: Change the order of integration and hence evaluate $\displaystyle\int_0^1 \int_0^{2\sqrt{x}} x^2\, dy\, dx.$



Solution:

$$I = \int_0^1 \int_0^{2\sqrt{x}} x^2\, dy\, dx$$

Given limits

x	0	1
y	0	$2\sqrt{x}$

Changed limits

x	$\frac{y^2}{4}$	1
y	0	2

$$I = \int_0^2 \int_{y^2/4}^{1} x^2\, dx\, dy$$

$$= \int_0^2 \left[\frac{x^3}{3}\right]_{\frac{y^2}{4}}^{1} dy$$

$$= \frac{1}{3}\int_0^2 \left[x^3\right]_{\frac{y^2}{4}}^{1} dy = \frac{1}{3}\int_0^1 \left[1^3 - \left(\frac{y^2}{4}\right)^3\right] dy$$

$$= \frac{1}{3}\int_0^2 \left(1 - \frac{y^6}{64}\right) dy$$

$$= \frac{1}{3}\left[y - \frac{y^7}{448}\right]_0^2 = \frac{1}{3}\left[2 - \frac{2^7}{448}\right]$$

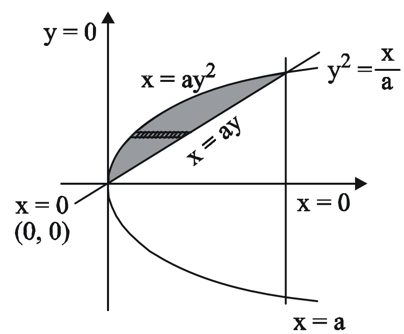
$$= \frac{1}{3}\left[2 - \frac{128}{448}\right] = \frac{1}{3}\left[2 - \frac{2}{7}\right]$$

$$= \frac{1}{3}\left[\frac{14-2}{7}\right] = \frac{1}{3}\left[\frac{12}{7}\right] = \frac{4}{7}$$

$$\boxed{\Rightarrow I = \frac{4}{7}}$$

Example 10: Change the order of integration and hence evaluate $\displaystyle\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dy\, dx$

Solution:



$$I = \int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dy\, dx$$

Given limits

x	0	a
y	$\frac{x}{a}$	$\sqrt{\frac{x}{a}}$

Hints:

$x = 0$	$x = a$
$y = \frac{x}{a}$	$y = \sqrt{\frac{x}{a}}$
$\boldsymbol{ay = x}$	$y^2 = \frac{x}{a}$

$$\boxed{ay^2 = x}$$

Changed limits

x	ay^2	ay
y	0	1

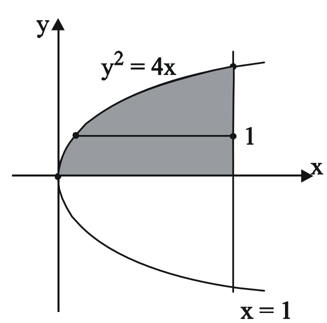
$$I = \int_0^1 \int_{ay^2}^{ay} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_{ay^2}^{ay} dy$$

$$= \int_0^1 \left[\left(\frac{(ay)^3}{3} + y^2 (ay) \right) - \left(\frac{(ay^2)^3}{3} + y^2 (ay^2) \right) \right] dy$$

$$= \int_0^1 \left[\frac{a^3 y^3}{3} + ay^3 - \frac{a^3 y^6}{3} - ay^4 \right] dy$$

$$= \left[\frac{a^3 y^4}{12} + \frac{ay^4}{4} - \frac{a^3 y^7}{21} - \frac{ay^5}{5} \right]_0^1$$



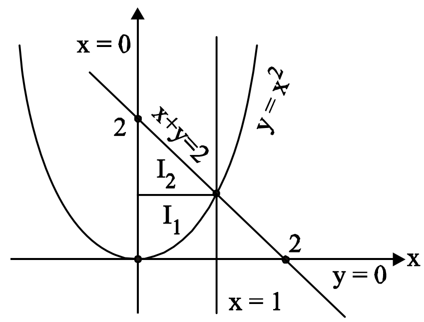
$$= \left[\frac{a^3 y^4}{12} + \frac{ay^4}{4} - \frac{a^3 y^7}{21} - \frac{ay^5}{5} \right]_0^1$$

$$I = \left[\frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} \right]$$

$$= \left[\frac{7a^3 - 4a^3}{84} + \frac{5a - 4a}{20} \right]$$

$$\boxed{I = \left[\frac{3a^3}{84} + \frac{a}{20} \right]}$$

Example 11: Change the order of integration and hence evaluate $\displaystyle\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$



Solution:

$$I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$$

Given limit

x	0	1
y	x^2	$2-x$

Changed limit

$\Rightarrow I_1$

x	0	$\sqrt{y}$
y	0	1

$\Rightarrow I_2$

x	0	$2-y$
y	1	2

$$\boxed{I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx = I_1 + I_2}$$

$$\Rightarrow I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy = \frac{1}{2}\int_0^1 \left[x^2 y\right]_0^{\sqrt{y}} dy$$

$$= \frac{1}{2}\int_0^1 [(\sqrt{y})^2 y]\, dy = \frac{1}{2}\int_0^1 y^2\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3} - 0\right] = \frac{1}{6}$$

$$\boxed{\Rightarrow \boldsymbol{I_1 = \frac{1}{6}}}$$

$$I_2 = \int_0^1 \int_0^{2-y} xy\, dy\, dx$$

$$\Rightarrow I_2 = \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

$$= \int_1^2 \left[\frac{x^2}{2}y\right]_0^{2-y} dy = \int_1^2 \frac{(2-y)^2}{2}\, y\, dy$$

$$= \frac{1}{2}\int_1^2 (4 + y^2 - 4y)\, y\, dy$$

$$= \frac{1}{2}\int_1^2 (4y + y^3 \;\; 4y^2)\, dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left(\frac{4\,(2)^2}{2} + \frac{(2)^4}{4} - \frac{4\,(2)^3}{3}\right) - \left(\frac{4\,(1)^2}{2} + \frac{(1)^4}{4} - \frac{4\,(1)^3}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{16}{2} + \frac{16}{4} - \frac{32}{3} - \frac{4}{2} - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{2}\left[12 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{2}\left(\frac{5}{12}\right) = \frac{5}{24}$$

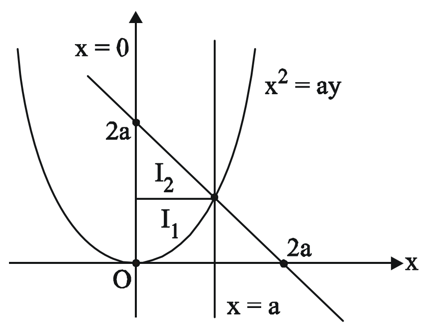
$$\boxed{I_1 = \frac{5}{24}}$$

$$\Rightarrow I = I_1 + I_2$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$\boxed{I = \frac{3}{8}}$$

Example 12: Change the order of integration and hence evaluate $\displaystyle\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx$



Solution:

Given: $I = \displaystyle\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx$ ... (1)

$x = 0$	$x = a$
$y = \frac{x^2}{a}$	$y = 2a - x$
$\Rightarrow x^2 = ay$	$\Rightarrow x + y = 2a$

Given limits

x	0	a
y	$\frac{x^2}{a}$	$2a - x$

Change limits

$I_1 \Rightarrow$

x	0	$\sqrt{ay}$
y	0	a

$I_2 \Rightarrow$

x	0	$2a-y$
y	a	$2a$

$I = I_2 + I_2 \Rightarrow$ (1) becomes

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy\, dx\, dy$$

$$= \int_0^a \left[\frac{x^2}{2} y\right]_0^{\sqrt{ay}} dy$$

$$= \int_0^a \left[\frac{(\sqrt{ay})^2}{2} y\right] dy = \int_0^a \left[\frac{a\,y^2}{2}\right] dy$$

$$= \frac{1}{2} \int_0^a ay^2\, dy = \frac{a}{2} \int_0^a y^2\, dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{a}{2}\left[\frac{a^3}{3}\right] = \frac{a^4}{6}$$

$$\boxed{\boldsymbol{I_1 = \frac{a^4}{6}}}$$

$$I_2 = \int_a^{2a} \int_0^{2a-y} xy\, dx\, dy$$

$$= \int_{a}^{2a} \left[\frac{x^2}{2} y \right]_0^{2a-y} dy$$

$$= \frac{1}{2} \int_{a}^{2a} [(2a-y)^2 y]\, dy = \frac{1}{2} \int_{a}^{2a} (4a^2 + y^2 - 4ay)\, y\, dy$$

$$= \frac{1}{2} \int_{a}^{2a} (4a^2 y + y^3 - 4ay^2)\, dy$$

$$= \frac{1}{2} \left[\frac{4a^2 y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3} \right]_a^{2a}$$

$$= \frac{1}{2} \left[\left(\frac{4a^2 (2a)^2}{2} + \frac{(2a)^4}{4} - \frac{4a\,(2a)^3}{3} \right) - \left(\frac{4a^2\, a^2}{2} + \frac{a^4}{4} - \frac{4a^4}{3} \right) \right]$$

$$= \frac{1}{2} \left[\frac{16a^4}{2} + \frac{16a^4}{4} - \frac{24a^4}{3} - \frac{4a^4}{2} - \frac{a^4}{4} + \frac{4a^4}{3} \right]$$

$$= \frac{a^4}{2} \left[\frac{16}{2} + \frac{16}{4} - \frac{24}{3} - \frac{4}{2} - \frac{1}{4} + \frac{4}{3} \right]$$

$$= \frac{a^4}{2} \left(8 + 4 - \frac{24}{3} - 2 - \frac{1}{4} + \frac{4}{3} \right)$$

$$= \frac{a^4}{2} \left[10 - \frac{24}{3} - \frac{1}{4} + \frac{4}{3} \right]$$

$$\boxed{\boldsymbol{I_2 = \frac{5a^4}{24}}}$$

$$\Rightarrow I = I_1 + I_2$$

$$= \frac{a^4}{b} + \frac{5a^4}{24}$$

$$\boxed{\boldsymbol{I = \frac{3}{8} a^4}}$$

TWO MARKS QUESTIONS AND ANSWERS

1. Evaluate $\displaystyle\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta.$

Solution:

$$\int_0^{\pi}\left(\int_0^{\sin\theta} r\,dr\right)d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\times 2\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\times\frac{\pi}{2}$$

$$\boxed{\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta = \frac{\pi}{4}}$$

2. Evaluate $\displaystyle\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$

Solution:

$$I = \int_0^{\pi}\left(\int_0^{a} r\,dr\right)d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} d\theta$$

$$= \frac{a^2}{2} (\theta)_0^{\pi}$$

$$= \frac{a^2}{2} (\pi)$$

$$\boxed{\therefore \int_0^{\pi} \int_0^{a} rdr\, d\theta = \frac{\pi a^2}{2}}$$

3. Evaluate $\int_0^{\pi/2} \int_0^{\sin\theta} rdr\, d\theta$.

Solution:

$$I = \int_0^{\pi/2} \int_0^{\sin\theta} rdr\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_0^{\sin} \theta\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2} \times \frac{\pi}{2}\right)$$

$$\boxed{I = \frac{\pi}{8}}$$

4. Evaluate: $\displaystyle\int_0^2 \int_0^\pi \boldsymbol{r \sin^2 \theta \, d\theta \, dr}$.

Solution:

$$\int_0^\pi \int_0^2 \sin^2\theta \, r dr \, d\theta = \int_0^\pi \sin^2\theta \left(\frac{r^2}{2}\right)_0^2 d\theta$$

$$= \int_0^\pi \sin^2\theta \left(\frac{4}{2}\right) d\theta$$

$$= 2 \times 2 \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= 4 \times \frac{1}{2} \times \frac{\pi}{2} = \pi$$

$$\boxed{\therefore \int_0^\pi \int_0^2 \sin^2\theta \, r d\theta \, dr = \pi}$$

5. Evaluate $\displaystyle\int_1^b \int_1^a \frac{dx\,dy}{xy}$

Solution:

$$I = \int_1^b \int_1^a \frac{dx\,dy}{xy} = \int_1^b \frac{dy}{y} \times \int_1^a \frac{dx}{x}$$

$$= (\log y)_1^b \times (\log x)_1^a$$

$$= (\log b - \log 1)(\log a - \log 1) \quad (\because \log 1 = 0)$$

$$I = (\log b)(\log a)$$

6. Evaluate: $\displaystyle\int_1^3 \int_1^2 \frac{1}{xy}\, dx\, dy$

Solution:

$$I = \int_1^3 \int_1^2 \frac{dx\, dy}{xy} = \int_1^2 \frac{dx}{x} \times \int_1^3 \frac{dy}{y}$$

$(\because \log 1 = 0)$

$$= (\log x)_1^2 \times (\log y)_1^3$$

$$= (\log 2 - \log 1) \times (\log 3 - \log 1)$$

$$\boxed{I = \log 2 \log 3}$$

7. Express $\displaystyle\int_0^\infty \int_0^\infty f(x, y)\, dx\, dy$ **in polar co-ordinates.**

Solution:

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2 + y^2 &= r^2 \\ dx\, dy &= r\,dr\, d\theta \end{aligned}}$$

$\theta \to 0$ to $\pi/2$

$r \to 0$ to ∞

$$I = \int_0^{\pi/2} \int_0^\infty f(r, \theta)\, r\,dr\, d\theta$$

8. Evaluate: $\displaystyle\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dy\, dx$

Solution:

$$I = \int_0^1 \left(\int_0^{x^2} (x^2 + y^2)\, dy \right) dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)_0^{x^2} dx$$

$$= \int_0^1 \left(x^4 + \frac{x^6}{3} \right) dx$$

$$= \left(\frac{x^5}{5} + \frac{x^7}{21} \right)_0^1$$

$$= \frac{1}{5} + \frac{1}{21}$$

$$\boxed{I = \frac{26}{105}}$$

9. Evaluate: $\displaystyle\int_0^1 \int_0^y \int_0^{x+y} dx\, dy\, dz$

Solution:

$$I = \int_0^1 \int_0^y \int_0^{x+y} dx\, dy\, dz = \int_0^1 \int_0^y \int_0^{x+y} dz\, dx\, dy$$

$$= \int_0^1 \int_0^y \left(z \right)_0^{x+y} dx\, dy = \int_0^1 \left(\int_0^y (x+y)\, dx \right) dy$$

$$= \int_0^1 \left(\frac{x^2}{2} + yx \right)_0^y dy$$

$$= \int_0^1 \left(\frac{y^2}{2} + y^2\right) dy$$

$$= \left(\frac{y^3}{6} + \frac{y^3}{3}\right)_0^1$$

$$= \frac{1}{6} + \frac{1}{3}$$

$$= \frac{3}{6}$$

$$\boxed{\boldsymbol{I = \frac{1}{2}}}$$

10. Evaluate: $\displaystyle\int_0^a \int_0^b \int_0^c dx\, dy\, dz.$

Solution:

$$I = \int_0^a \int_0^b \int_0^c dx\, dy\, dz$$

$$= \int_0^c dx \times \int_0^b dy \times \int_0^a dz$$

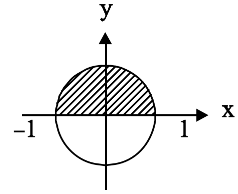
$$= (x)_0^c \times (y)_0^b \times (z)_0^a$$

$$= (c - 0)\,(b - 0)\,(a - 0)$$

$$\boxed{\boldsymbol{I = abc}}$$

11. Evaluate $\displaystyle\iint_R dx\, dy$ **where** R **is the shaded region in the fig.**

Solution:



Here circle is given and radius $= 2$

Area of a circle $= \pi r^2 = \pi (2)^2 = 4\pi$

Area is half of a circle $= \dfrac{4\pi}{2} = 2\pi$

12. Express the region $x \geq 0$; $y \geq 0$; $z \geq 0$; $x^2 + y^2 + z^2 \leq 1$ by Triple integration.

Solution:

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$$

Put $z = 0$ in (1)	$x^2 + y^2 + z^2 = 1 \quad \ldots (1)$	Put $y = 0$
$x^2 + y^2 = 1$	$x^2 = 1 - x^2 - y^2$	$z = 0$ in (1)
$y^2 = 1 - x^2$		$x^2 = 1$
$\boxed{y = \sqrt{1-x^2}}$	$\boxed{z = \sqrt{1-x^2-y^2}}$	$\boxed{x = 1}$

13. Evaluate $\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} d\theta\, dr$

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} d\theta\, dr = \int_{-\pi/2}^{\pi/2} [\theta]_0^{2\cos\theta}\, dr = \int_{-\pi/2}^{\pi/2} 2\cos\theta\, dr$$

$$= 2[\sin\theta]_{-\pi/2}^{\pi/2} = 2\left[\sin\frac{\pi}{2} - \sin\left(\frac{-\pi}{2}\right)\right] = 2\left[1 + \sin\frac{\pi}{2}\right]$$

$$= 2[1+1]$$

$$\boxed{I = 4}$$

14. Evaluate: $\int_0^a \int_0^b (x+y)\, dx\, dy$

Solution:

$$I = \int_0^a \int_0^b (x+y)\, dx\, dy$$

$$= \int_0^a \left(\frac{x^2}{2} + yx \right)_0^b dy$$

$$= \int_0^a \left(\frac{b^2}{2} + by \right) dy$$

$$= \left(\frac{b^2}{2} y + \frac{by^2}{2} \right)_0^a$$

$$I = \frac{b^2 a}{2} + \frac{ba^2}{2} = \frac{ab}{2}(a+b)$$

15. Integrate $\int_1^3 \int_3^4 \int_1^4 xyz\, dz\, dy\, dx$.

Solution:

$$I = \int_1^3 \int_3^4 \int_1^4 xyz\, dz\, dy\, dx$$

$$= \int_1^3 x\, dx \times \int_3^4 y dy \times \int_1^4 z dz$$

$$= \left(\frac{x^2}{2} \right)_1^3 \times \left(\frac{y^2}{2} \right)_3^4 \times \left(\frac{z^2}{2} \right)_1^4$$

$$= \left(\frac{9}{2} - \frac{1}{2} \right) \times \left(\frac{16}{2} - \frac{9}{2} \right) \times \left(\frac{16}{2} - \frac{1}{2} \right)$$

$$= \frac{8}{2} \times \frac{7}{2} \times \frac{15}{2}$$

$$= 7 \times 15$$

$$\boxed{I = 105}$$

16. Evaluate: $\int_{1}^{2}\int_{0}^{x^2} xdy\,dx$

Solution:

$$I = \int_{1}^{2}\int_{0}^{x^2} xdy\,dx$$

$$= \int_{1}^{2} (xy)_{0}^{x^2}\,dx = \int_{1}^{2} x(x^2)\,dx$$

$$= \int_{1}^{2} x^3\,dx$$

$$= \left(\frac{x^4}{4}\right)_{1}^{2}$$

$$= \frac{16}{4} - \frac{1}{4}$$

$$\boxed{I = \frac{15}{4}}$$

17. Find the value of $\int_{0}^{\infty}\int_{0}^{y} \frac{e^{-y}}{y}\,dx\,dy$

Solution:

$$I = \int_{0}^{\infty}\int_{0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y} x \right)_0^y dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \times y dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$\boldsymbol{I = 1}$$

18. Evaluate: $\displaystyle\int_0^a \int_0^b \int_0^c e^{x+y+z}\, dz\, dy\, dx.$

Solution:

$$I = \int_0^a \int_0^b \int_0^c e^{x+y+z}\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b \int_0^c e^x e^y e^z\, dz\, dy\, dx$$

$$= \int_0^c e^x\, dx \times \int_0^b e^y\, dy \times \int_0^a e^z\, dz$$

$$= \left(e^x \right)_0^c \times \left(e^y \right)_0^b \times \left(e^z \right)_0^a$$

$$= (e^c - e^0)\,(e^b - e^0)\,(e^a - e^0)$$

$$I = (e^c - 1)\,(e^b - 1)\,(e^a - 1)$$

19. Evaluate: $\displaystyle\int_0^1 \int_0^2 \int_0^3 \boldsymbol{e^x\, e^y\, e^z\, dz\, dy\, dx}$.

Solution:

$$\int_0^1 \int_0^2 \int_0^3 e^x\, e^y\, e^z\, dz\, dy\, dx = \int_0^1 e^x\, dx \times \int_0^2 e^y\, dy \times \int_0^3 e^z\, dz$$

$$= \left(e^x \right)_0^1 \times \left(e^y \right)_0^2 \times \left(e^z \right)_0^3$$

$$= (e^1 - e^0)\,(e^2 - e^0)\,(e^3 - e^0)$$

$$I = (e^1 - 1)\,(e^2 - 1)\,(e^3 - 1)$$

20. Evaluate: $\displaystyle\int_0^a \int_0^{\sqrt{ay}} \boldsymbol{xy\, dx\, dy}$

Solution:

$$\int_0^a \left(\int_0^{\sqrt{ay}} xy\, dx \right) dy = \int_0^a \left(\frac{x^2}{2} y \right)_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2} \int_0^a (ay)\, y dy$$

$$= \frac{a}{2} \int_0^a \frac{y^2}{2}\, dy$$

$$= \frac{a}{2} \left(\frac{y^3}{3} \right)_0^a$$

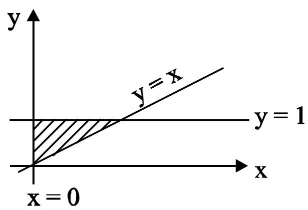
$$= \frac{a}{2} \left(\frac{a^3}{3} \right)$$

$$\boxed{\boldsymbol{I = \frac{a^4}{6}}}$$

21. Find the area bounded by the lines $x = 0$; $y = 1$ and $y = x$ using double integration.

Solution:

$$I = \int_0^1 \int_x^1 dy\, dx$$



$$= \int_0^1 (y)_x^1\, dx$$

$$= \int_0^1 (1 - x)\, dx$$

$$= \left[x - \frac{x^2}{2} \right]_0^1 = 1 - \frac{1}{2}$$

$$\boxed{I = \frac{1}{2}}$$

22. Change the Order of Integration $\int_0^1 \int_y^1 dx\, dy$.

Solution:

$$\int_0^1 \int_y^1 dx\, dy = \int_0^1 \int_0^x dy\, dx$$

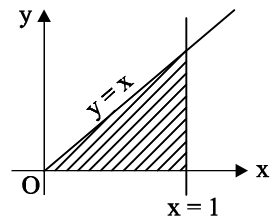
23. Change the order of Integration $\int_0^1 \int_0^x f(x, y)\, dy\, dx$

Solution:

$$I = \int_0^1 \int_0^x f(x, y)\, dy\, dx = \int_0^1 \int_y^1 f(x, y)\, dx\, dy$$

24. Sketch the region of $\displaystyle\int_0^1 \int_0^x f(x, y)\, dy\, dx$

Solution:



$$x \Rightarrow 0 \text{ to } 1$$
$$y \Rightarrow 0 \text{ to } x$$

25. Reduction formula:

Solution:

$$I = \int_0^{\pi/2} \cos^n x\, dx \text{ (or)}$$

$$\int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \text{If } n \text{ is even} \\ \dfrac{n-1}{n-0} \times \dfrac{n-3}{n-2} \times \dots \dfrac{1}{2} \cdot \dfrac{\pi}{2} \end{cases}$$

If n is odd

$$\frac{n-1}{n-0} \times \frac{n-3}{n-2} \times \dots \frac{2}{3} \times 1$$

EXERCISE

Double integration in cartesian co-ordinates

1. Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

2. Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$

3. Evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4-a^2x^2}}\,dy\,dx$

4. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\,(x+y)\,dx\,dy$

5. Evaluate $\int_0^5 \int_0^2 (x^2+y^2)\,dx\,dy$

Double integration in polar coordinates

1. Evaluate $\int_0^{\pi/2} \int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

2. Evaluate $\int_0^{\pi/4} \int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

3. Evaluate $\int_0^{2\pi} \int_{a\sin\theta}^{a} r\,dr\,d\theta$

4. Evaluate $\displaystyle\int_{b/2}^{b} \int_{0}^{\pi/2} r d\,\theta\, dr$

Change of order of integration

1. Evaluate $\displaystyle\int_{0}^{a} \int_{\frac{x}{a}}^{\frac{\sqrt{x}}{0}} (x^2 + y^2)\, dx\, dy$

2. Evaluate $\displaystyle\int_{0}^{\infty} \int_{0}^{x} xe^{-x^2/y}\, dy\, dx$

3. Evaluate $\displaystyle\int_{0}^{a} \int_{y}^{a} \frac{x^2}{\sqrt{x^2 + y^2}}\, dx\, dy$

4. Evaluate $\displaystyle\int_{1}^{4} \int_{2/y}^{2\sqrt{y}} dx\, dy$

5. Evaluate $\displaystyle\int_{0}^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\, dx\, dy$

6. Evaluate $\displaystyle\int_{0}^{a} \int_{y}^{a} \frac{x}{x^2 + y^2}\, dx\, dy$

7. Evaluate $\displaystyle\int_{0}^{2a} \int_{0}^{\sqrt{2ax - x^2}} \frac{x}{x^2 + y^2}\, dx\, dy$

8. Evaluate $\displaystyle\int_{0}^{2a} \int_{0}^{\sqrt{2x - x^2}} dy\, dx$

Area enclosed by plane curves [Cartesian co-ordinates]

1. Evaluate: $\int\int xy\,dx\,dy$ taken over the positive quadrant bounded by the (a) circle $x^2 + y^2 = a^2$ (b) $\frac{x}{a} + \frac{y}{b} = 1$

2. Evaluate $\int\int x^2 y\,dy\,dx$ over the region for which $x, y \geq 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$

3. Evaluate: $\int\int (x + y)^2\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

4. Find, by double integration, the area enclosed by the curves $y = \frac{3x}{x^2 + 2}$ and $4y = x^2.$

5. Evaluate: $\int\int_s (xy - y^2)\,dx\,dy,$ where s is the triangle with vertices (0, 0), (0, 1) and (1, 1).

Area enclosed by plane curves [Polar co-ordinates]

1. Evaluate: $\int\int r \sin\theta\,dr\,d\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the initial line.

2. Find by double integration the area bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta.$

3. Find the area which is inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta)$

4. Find the area common to $r = a\sqrt{2}$ and $r = 2a\cos\theta.$

Triple Integrals

1. Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$

2. Evaluate $\iiint xyz\, dx\, dy\, dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$

3. Evaluate: $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$

4. Evaluate: $\displaystyle\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

5. Evaluate: $\displaystyle\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$

Volume of Solids

1. Evaluate: $\displaystyle\iiint_v xyz\, dx\, dy\, dz$, where V is the region of space inside the tetrahedron bounded by the planes $x = 0, y = 0,$ $z = 0$ and $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$

2. Evaluate: $\displaystyle\iiint_v (x + y + z)\, dx\, dy\, dz,$ where V is the region of space inside the cylinder $x^2 + y^2 = a^2$

3. $\iiint\limits_{S} xyz \, dx \, dy \, dz$ where

   $S = [(x, y, z) : x^2 + y^2 + z^2 \leq 1, x \geq 0, z \geq 0]$

4. Calculate the volume of the solid bounded by the surface $x = 0$, $y = 0$, $x + y + z = 1$, and $z = 0$.

5. Evaluate: $\iiint xyz \, dx \, dy \, dz$ over the first octant of $x^2 + y^2 + z^2 = a^2$

UNIT - 5

ORDINARY DIFFERENTIAL EQUATIONS

5.1 INTRODUCTION AND APPLICATIONS

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities. In fact many physical laws are expressed mathematically in the form of differential equations.

Definition

A differential equation is an equation involving one dependent variable and its derivatives with respect to one (or) more independent variables, eg: 1

$$\frac{d^2 y}{dx^2} + 6\frac{dy}{dx} + y = x^2$$

- It is the principal language of science.
- It plays an important role in Science, Engineering and Social sciences.
- It is applied in the field of disease environment, Prey-Predator Model etc.

Definition

An equation involving **one dependent variable** and its derivatives with respect to **one or more independent variables** is called differential equation.

If $y = f(x)$ is a given function then its derivative is $\frac{dy}{dx}$.

Rate of change of 'y' with respect to 'x'.

Here y – dependent variable

x – Independent variable

Differential equations is an equation in which **differential Co-efficients** occur.

Differential equations are of two types, ordinary differential equations and partial differential equations.

Examples

$$\frac{dy}{dx} = x + 5, \quad y' + 7y = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \; \frac{d^2 y}{dx^2} + 7\frac{dy}{dx} = 0$$

5.1.1 Order of the differential equations

The order of a differential equation is the order of the highest order derivatives occurring in it.

5.1.2 Degree of the differential equation

The degree of the differential equation is the degree of the highest order derivatives which occurs in it, after the differential equation has been made free from radicals and fractions.

5.1.3 Find the order and degree of the following differential equations

(i) $\frac{d^3 y}{dx^3} + \left(\frac{d^2 y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^5 + y = 7$

Order: 3

degree: 1

(ii) $y = 4\frac{dy}{dx} + 3x\frac{dx}{dy}$

$$y = 4\frac{dy}{dx} + 3x\frac{1}{\left(\frac{dy}{dx}\right)}$$

Multiply $\frac{dy}{dx}$

$$\frac{dy}{dx}y = 4\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + 3x$$

$$\frac{dy}{dx}y = 4\left(\frac{dy}{dx}\right)^2 + 3x$$

Highest order: 1

degree: 2

(iii) $$\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/4}$$

Raising to the power 4 on both sides, we get

$$\left(\frac{d^2y}{dx^2}\right)^4 = \left(\left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/4}\right)^4$$

Order $= 2$

degree $= 4$

(iv) $(1 + y')^2 = y'^2$

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$1 + y'^2 + 2y' = y'^2$$

$$1 + 2y' = y'^2 - y'^2$$

$$1 + 2y' = 0$$

$$2\frac{dy}{dx} + 1 = 0$$

Order $= 1$

degree $= 1$

$$\boxed{y' = \frac{dy}{dx}}$$

Practice problem

1. $\frac{dy}{dx} + y = x^2$

2. $y'' + 3y'^2 + y^3 = 0$

3. $y' + (y'')^2 = x\,(x + y'')^2$

4. $\left(\frac{dy}{dx}\right)^2 + x = \frac{dx}{dy} + x^2$

5.2 Second order linear differential equations with constant coefficients

Solution of the differential equation is

$$y = \text{C.F} + \text{P.I}$$

C.F – Complementary function

P.I – Particular integral

Complementary function rules

1. Roots are different and real.

$$\text{C.F} = Ae^{m_1 x} + Be^{m_2 x}$$

2. Roots are equal and real

$$\text{C.F} = (A + Bx)\,e^{mx}$$

3. Roots are imaginary

$$\text{C.F} = e^{\alpha x}\,(A \cos \beta x + B \sin \beta x)$$

Notations

$$\frac{d^2 y}{dx^2} = D^2 y = y''$$

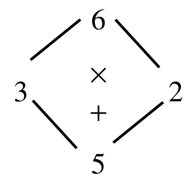
$$\frac{dy}{dx} = Dy = y'$$

For C.F

To find the auxiliary equations

$$\boxed{\textbf{Put } D = m}$$

Solving quadratic equations

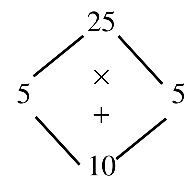


1. $x^2 + 5x + 6 = 0$

$(x+3)\,(x+2) = 0$

$$\boxed{x = -3,\ x = -2}$$

Roots are different



2. $x^2 + 10x + 25 = 0$

$(x+5)\,(x+5) = 0$

$$\boxed{x = -5,\ x = -5}$$

Roots are Same

3. $x^2 + x + 1 = 0$

$ax^2 + bx + c = 0$

$$a = 1, b = 1, c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$x = \frac{-1 \pm i\sqrt{3}}{2}$$

$$x = \frac{-1 - i\sqrt{3}}{2}, \ \frac{-1 + i\sqrt{3}}{2}$$

Imaginary roots.

Rules for Particular Integral

Type I e^{ax}	Replace D by a
Type II $\sin ax$ $\cos ax$	Replace D^2 by $-(a^2)$
Type III x, x^2, x^3	Convert the denominator to $(1+x)^{-1}$ or $(1-x)^{-1}$
Type IV $e^{ax}\cos ax$ $e^{ax}\sin ax$	Replace D by $D+a$ (or) D by $D-a$ and proceed by type II
Type V $e^{ax}x^2$ $e^{ax}x^3$	Replace D by $D+a$ (or) D by $D-a$ and proceed by type III

Example 1: Solve $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 12y = 0$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \text{... (1)}$$

C.F

$D^2 y + 7Dy + 12y = 0$

$(D^2 + 7D + 12)\, y = 0 \qquad \text{... (2)}$

The Auxiliary Equation is

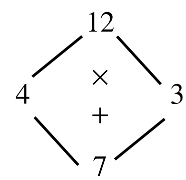
Put $D = m$, in (2)

$$m^2 + 7m + 12 = 0$$

$$(m+4)(m+3) = 0$$

$$m + 4 = 0 \qquad m + 3 = 0$$

$$m_1 = -4 \qquad m_2 = -3$$



Roots are real and different,

$$\text{C.F} = Ae^{m_1 x} + Be^{m_2 x}$$

$$\text{C.F} = Ae^{-4x} + Be^{-3x}$$

$$y = Ae^{-4x} + Be^{-2x} + 0$$

$$\boxed{\textbf{P.I} = \mathbf{0}}$$

Practice problem

1. $(D^2 - 5D + 16)\, y = 0$
2. $(D^2 - 2D + 1)\, y = 0$
3. $(D^2 + 10D + 9)\, y = 0$

Example 2: Solve $(D^2 + 6D + 9)\, y = 0$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \text{... (1)}$$

C.F:

The Auxiliary Equation is

Put $D = m$

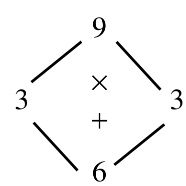
$$(D^2 + 6D + 9)\, y = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m + 3)\,(m + 3) = 0$$

$$m + 3 = 0 \qquad m + 3 = 0$$

$$m = -3 \qquad m = -3$$



Roots are real and equal

$$\text{C.F} = (Ax + B)\, e^{mx}$$

$$\text{C.F} = (Ax + B)\, e^{-3x}$$

$$\boxed{\mathbf{P.I = 0}}$$

$$\boxed{\boldsymbol{y = (Ax + B)\, e^{-3x} + 0}}$$

The general form of the linear differential equation of second order is

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Where P and Q are constant R is a function of x or constant.

The general form of Linear differential equation n^{th} order is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} \dots + a_n\, y = f(x) \qquad \dots (1)$$

When $f(x) = 0$ eqn (1) is called homogeneous Linear differential equation.

When $f(x) \neq 0$ eqn (1) is called non-homogeneous Linear differential equations.

The general solution is

***y* = Complementary function (CF) + Particular Integral (P.I)**

Rules of C.F Roots	C.F
1. Roots are reals different m_1, m_2 $(m_1 \neq m_2)$	$Ae^{m_1 x} + Be^{m_2 x}$
Ex: $m = 3, 2$	$Ae^{3x} + Be^{2x}$
Ex: $m = 3, 2, 1$	$Ae^{3x} + Be^{2x} + Ce^{x}$
2. Roots are real & equal $m_1 = m_2 = m$	$(Ax + B)\, e^{mx}$ (or) $(A + Bx)\, e^{mx}$
Ex: $m = 3, 3$	$(Ax + B)\, e^{3x}$
Ex: $m = 2, 2, 2$	$(Ax^2 + Bx + C)\, e^{2x}$
3. Roots are Imaginary	$e^{\alpha x}\, [A \cos \beta x + B \sin \beta x]$
Ex: $4 \pm i3$ $\alpha = 4, \beta = 3$	$e^{4x}\, [A \cos 3x + B \sin 3x]$

5.2.1 Type I: RHS $= 0$

Example 1: Solve $(D^2 - 5D + 6)\, y = 0$.

Solution:

$$(D^2 - 5D + 6)\, y = 0$$

The Auxiliary equation is

Put $D = m$

$$m^2 - 5m + 6 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1, b = -5, c = 6$$

$$\frac{+5 \pm \sqrt{25 - 4\,(1)\,(6)}}{2\,(1)} = \frac{5 \pm \sqrt{1}}{2}$$

$$\frac{5 \pm 1}{2} = \frac{5+1}{2}, \frac{5-1}{2}$$

$$= \frac{6}{2}, \frac{4}{2} = 3, 2$$

RHS $= 0$ (no Particular Solution) $(m = 2, 3)$

$$y = \text{C.F} = Ae^{2x} + Be^{3x}$$

Example 2: Solve $(D^2 + 6D + 9)\, y = 0$

Solution:

$$(D^2 + 6D + 9)\, y = 0$$

The Auxiliary equation is put $D = m$

$$m^2 + 6m + 9 = 0$$

$$a = 1, b = 6, c = 9$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 - 4\,(1)\,(9)}}{2\,(1)}$$

$$= \frac{-6 \pm 0}{2} = \frac{-6}{2} = -3$$

so $m = -3, -3$ (equal), P.I $= 0$

$y = \text{C.F} + 0 = (Ax + B)\, e^{-3x}$

Aliter:

$$m^2 + 6m + 9$$

$$m^2 + 3m + 3m + 9$$

$$m\,(m+3) + 3\,(m+3)$$

$$(m+3)\,(m+3) = 0$$

$$m = -3, -3$$

$$y = (Ax + B)\, e^{-3x}$$

Example 3: Solve $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0$ (or) $y'' - 6y' + 13y = 0$

Solution:

$$\boxed{\begin{array}{l} \dfrac{d^2}{dx^2} = D^2 \\ \dfrac{d}{dx} = D \end{array}}$$

Given $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + By = 0$

(ie) $D^2 y - 6Dy + 13y = 0$

$$(D^2 - 6D + 13)\, y = 0$$

The Auxiliary equation is put $D = m$

$$\boxed{\begin{array}{l} a = 1 \\ b = -6 \\ c = 13 \end{array}}$$

$$m^2 - 6m + 13 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 52}}{2\,(1)}$$

$$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm i4}{2}$$

$$= \frac{2\,(3 \pm i2)}{2} = 3 \pm i2$$

$$m = 3 \pm i2$$

$$\alpha = 3,\ \beta = 2$$

$$y = \text{C.F} = e^{3x}\,[A\cos 2x + B\sin 2x]$$

Example 4: **Solve** $\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ **(Or)** $y'' + 5y' + 6y = 0$ **(Or)** $D^2 y + 5Dy + 6y = 0$

Solution:

$$D^2 y + 5Dy + 6y = 0$$

$$(D^2 + 5D + 6)\, y = 0$$

$$y = \text{C.F} + \text{P.I} \qquad \dots (1)$$

C.F:

$$(D^2 + 5D + 6)\, y = 0$$

The Auxiliary equation is

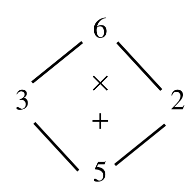
Put $\quad D = m$

$$m^2 + 5m + 6 = 0$$

$$(m + 3)\,(m + 2) = 0$$

$$m + 3 = 0, \quad m + 2 = 0$$

$$m_1 = -3 \qquad m_2 = -2$$



The roots are real and distinct

$$\text{C.F} = Ae^{m_1 x} + Be^{m_2 x}$$

$$\text{C.F} = Ae^{-3x} + Be^{-2x}$$

$$\text{P.I} = 0$$

$$y = Ae^{-3x} + Be^{-2x}$$

Example 5: **Solve** $(D^2 + 10D + 25)\, y = 0$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \text{... (1)}$$

C.F:

$(D^2 + 10D + 25)\, y = 0$

The Auxiliary equation is

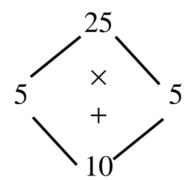
Put $\qquad D = m$

$m^2 + 10m + 25 = 0$

$(m + 5)\,(m + 5) = 0$

$m + 5 = 0 \qquad m + 5 = 0$

$m = -5 \qquad m = -5$



The roots are real and equal

$\therefore$ C.F $= (Ax + B)\, e^{mx}$

C.F $= (Ax + B)\, e^{-5x}$

P.I $= 0$

$y = (Ax + B)\, e^{-5x}$

Example 6: **Solve** $(D^2 y + Dy + y) = 0$

Solution:

$$(D^2 + D + 1)\, y = 0$$

$$y = \text{C.F} + \text{P.I} \qquad \text{... (1)}$$

C.F:

$$(D^2 + D + 1)\, y = 0$$

The Auxiliary equation is

Put $D = m$

$$m^2 + m + 1 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 1, b = 1, c = 1$

$$= \frac{-1 \pm \sqrt{1^2 - 4\,(1)\,(1)}}{2\,(1)}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

$$m = \frac{-1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$$\alpha = \frac{-1}{2} \quad \beta = \frac{\sqrt{3}}{2}$$

Roots are complex

$$\text{C.F} = e^{\alpha x}\,(A \cos \beta\, x + B \sin \beta\, x)$$

$$= e^{-1/2x}\left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\right)$$

Practice problems

1. Solve $y'' + 2y' + y = 0$
2. Solve $\dfrac{d^2 y}{dx^2} + 3\,\dfrac{dy}{dx} + 2y = 0$
3. Solve $D^2 y + 8\,\dfrac{dy}{dx} + 16y = 0$
4. Solve $y'' - 2y' + 2y = 0$

Example 7: Solve $(D^2+1)\,y=0$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \dots (1)$$

C.F:

$$(D^2+1)\,y=0$$

The Auxiliary equation is,

Put $\quad D = m$

$$m^2+1=0$$

$$m^2=-1$$

$$m=\pm\sqrt{-1}$$

$$m=\pm i\sqrt{1}$$

$$m=\pm 1i$$

$$m=0\pm 1i$$

$$y = A\cos x + B\sin x$$

$$\boxed{\begin{array}{c} m=0\pm 1i \\ m=\alpha\pm i\beta \\ \alpha=0,\ \beta=1 \end{array}}$$

Roots are complex

$$\text{C.F} = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$\boxed{\textbf{P.I} = \mathbf{0}}$$

$$\boxed{y = e^{0x}(A\cos 1x + B\sin 1x)}$$

Example 8: Solve $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$ (or)

$$D^3y - 6D^2y + 11Dy - 6y = 0 \text{ (or)}$$

$$(D^3 - 6D^2 + 11D - 6)\,y = 0$$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \dots (1)$$

C.F: The Auxiliary equation is put $D = m$

$$m^3 - 6m^2 + 11m - 6 = 0 \quad \text{... (2)}$$

Solve equation (2) we get

$$\boxed{\begin{matrix} \boldsymbol{m_1 = 1} \\ \boldsymbol{m_2 = 2} \\ \boldsymbol{m_3 = 3} \end{matrix}}$$

Roots are real and different

$\text{C.F} = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x}$

$\text{C.F} = Ae^{1x} + Be^{2x} + Ce^{3x}$

$$\boxed{\textbf{P.I} = \mathbf{0}}$$

$y = Ae^x + Be^{2x} + Ce^{3x}$

Example 9: Solve $(D^3 - 3D^2 + 3D - 1)\, y = 0$

Solution:

$$y = \text{C.F} + \text{P.I} \quad \text{... (1)}$$

C.F: The Auxiliary equation is, put $D = m$

$$m^3 - 3m^2 + 3m - 1 = 0 \quad \text{... (2)}$$

Solve equation (2) we get

$$\boxed{\begin{matrix} \boldsymbol{m_1 = 1} \\ \boldsymbol{m_2 = 1} \\ \boldsymbol{m_3 = 1} \end{matrix}}$$

Roots are real and equal

$\text{C.F} = e^{mx}(A + Bx + Cx^2)$

$\because \text{C.F} = e^{1x}(A + Bx + Cx)^2 \quad \text{P.I} = 0$

$$\boxed{\boldsymbol{y = e^x (A + Bx + Cx^2)}}$$

5.2.2 Type - II [RHS $= e^{ax}$]

General form:

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + y = e^{ax}$$

e^{ax} **; Replace D by 'a'**

Example 1: Solve $(D^2 - 4D + 13)\, y = e^{2x}$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \ldots (1)$$

C.F:

$$(D^2 - 4D + 13)\, y = 0$$

The Auxiliary equation is, put $D = m$

$$m^2 - 4m + 13 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 1, b = -4, c = 13$

$$m = \frac{4 \pm \sqrt{16 - 4\,(1)\,(13)}}{2\,(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm i\sqrt{36}}{2}$$

$$= \frac{4 \pm i6}{2}$$

$$= \frac{2\,(2 \pm i3)}{2}$$

$\alpha = 2, \beta = 3$

$$\text{C.F} = e^{2x}\,(A\cos 3x + B\sin 3x)$$

$$\text{P.I} = \frac{e^{2x}}{D^2 - 4D + 13}$$

Replace D by 'a', $a = 2$

$$= \frac{e^{2x}}{2^2 - 4\,(2) + 13}$$

$$= \frac{e^{2x}}{4 - 8 + 13}$$

$$\text{P.I} = \frac{e^{2x}}{9}$$

$$y = e^{2x}\,(A\cos 3x + B\sin 3x) + \frac{e^{2x}}{9}$$

Example 2: Solve $(y'' + 7y' + 12y) = 14e^{-3x}$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \text{... (1)}$$

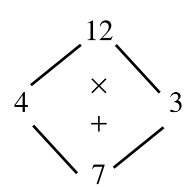
$$D^2 y + 7Dy + 12y = 14e^{-3x}$$

$$(D^2 + 7D + 12)\,y = 14e^{-3x}$$

$$\boxed{\begin{array}{l} y'' = D^2 y \\ y' = Dy \end{array}}$$

C.F:

$$(D^2 + 7D + 12)\,y = 0$$



The Auxiliary equation is put $D = m$

$$m^2 + 7m + 12 = 0$$

$$m + 3 = 0 \qquad m + 4 = 0$$

$$m_1 = -3 \qquad m_2 = -4$$

The roots are different,

$$\text{C.F} = Ae^{m_1 x} + Be^{m_2 x}$$

$$\text{C.F} = Ae^{-3x} + Be^{-4x}$$

$$\text{P.I} = \frac{14e^{-3x}}{D^2 + 7D + 12} \qquad \text{... (1)}$$

$$= \frac{14e^{-3x}}{(-3)^2 + 7(-3) + 12} \qquad e^{-3x} \Rightarrow e^{ax}, \; a = -3$$

$$= \frac{14e^{-3x}}{9 - 21 + 12}$$

$$= \frac{14e^{-3x}}{0}$$

$$= \frac{x\,(14e^{-3x})}{2D + 7}$$

Note:
If denominator = 0 go to equation (1) Differentiate the denominator and multiply with 'x' in the numerator

Replace D by a, $a = -3$

$$= \frac{x\,(14e^{-3x})}{2(-3) + 7}$$

$$= \frac{x\,(14e^{-3x})}{-6 + 7} = 14xe^{-3x}$$

$$\text{P.I} = x\,(14e^{-3x}) \qquad \text{... (**)}$$

$$y = \text{C.F} + \text{P.I}$$

$$y = Ae^{-3x} + Be^{-4x} + 14x\,e^{-3x}$$

Example 3: Find the Particular Integral of $(D^2 - 1)\, y = (e^x + 1)^2$

Solution:

$$(D^2 - 1)\, y = (e^x)^2 + 1^2 + 2e^x \ (1)$$

$$(D^2 - 1)\, y = e^{2x} + 1 + 2e^x$$

$$\text{P.I} = \text{P.I}_1 + \text{P.I}_2 + \text{P.I}_3 \qquad \text{... (1)}$$

$$\text{P.I}_1 = \frac{e^{2x}}{D^2 - 1}$$

$\boldsymbol{D = a = 2}$

$$= \frac{e^{2x}}{2^2 - 1}$$

$$= \frac{e^{2x}}{4 - 1}$$

$$\text{P.I}_1 = \frac{e^{2x}}{3}$$

$$\text{P.I}_2 = \frac{1e^{0x}}{D^2 - 1}$$

$\boldsymbol{D = a = 0}$

$$= \frac{e^{0x}}{0^2 - 1}$$

$$\text{P.I}_2 = \frac{e^{0x}}{-1}$$

$$\text{P.I}_2 = \frac{1}{-1}$$

$$\text{P.I}_2 = -1$$

$$\text{P.I}_3 = \frac{2e^x}{D^2 - 1}$$

$\boldsymbol{D = a = 1}$

$$= \frac{2e^x}{1^2 - 1}$$

$$= \frac{2e^x}{0}$$

$$= \frac{2xe^x}{2D}$$

$$= \frac{2xe^x}{2}$$

$$\text{P.I}_3 = xe^x$$

$$\because \text{P.I} = \frac{e^{2x}}{3} + xe^x - 1$$

Example 4: Find the Particular Integral of $\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 5y = \cosh x.$

Solution: Note

$$\boxed{\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} \\ \sinh x &= \frac{e^x - e^{-x}}{2} \end{aligned}} \longrightarrow \text{Hyperbolic functions}$$

Given: $D^2 y + 4Dy + 5y = \cosh x$

$$(D^2 + 4D + 5)\, y = \frac{e^x + e^{-x}}{2}$$

$$(D^2 + 4D + 5)\, y = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

$$\text{P.I} = \text{P.I}_1 + \text{P.I}_2 \qquad \text{... (1)}$$

$$\text{P.I}_1 = \frac{\frac{e^x}{2}}{D^2 + 4D + 5}$$

$$= \frac{\frac{1}{2} e^x}{1^2 + 4\,(1) + 5} = \frac{\frac{1}{2} e^x}{10}$$

$$\text{P.I}_1 = \frac{e^x}{20}$$

$$\text{P.I}_2 = \frac{\frac{e^{-x}}{2}}{D^2 + 4D + 5}$$

$$\text{P.I}_2 = \frac{\frac{e^{-x}}{2}}{D^2 + 4D + 5}$$

$$= \frac{\frac{1}{2} e^{-x}}{(-1)^2 + 4\,(-1) + 5}$$

$$= \frac{\frac{1}{2} \cdot e^{-x}}{1 - 4 + 5}$$

$$= \frac{\frac{1}{2} e^{-x}}{2}$$

$$\text{P.I}_2 = \frac{e^{-x}}{4}$$

$$\text{P.I} = \frac{e^x}{20} + \frac{e^{-x}}{4}$$

Practice problems

1. Solve $\dfrac{d^2 y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sinh x$
2. Solve $(D^2 - 4)\, y = e^{2x} + e^{-4x}$
3. Solve $(D^2 - 4)\, y = 10$
4. Solve $(D^2 - 4D + 4)\, y = e^{-4x}$
5. Solve $y'' + 7y' - 8y = e^{2x+3}$

5.2.3 Type - III RHS $\sin ax$ (or) $\cos ax$

$\text{P.I} = \dfrac{1}{f(D)} \sin ax$ or $\dfrac{1}{f(D)} \cos ax$
Replace D^2 by $-(a^2)$

Example 1: Solve $(D^2 + 10D + 25)\, y = \sin 2x$

Solution:

$$y = \text{C.F} + \text{P.I} \qquad \dots (1)$$

C.F:

Consider $(D^2 + 10D + 25)\, y = 0$

The Auxiliary equation is

Put $D = m$

$$m^2 + 10m + 25 = 0$$

$$m = -5, -5 \qquad \text{(Roots same)}$$

$$\text{C.F} = e^{-5x}(Ax + B)$$

$$\text{P.I} = \frac{\sin 2x}{D^2 + 10D + 25} \qquad [\text{Replace } D^2 = -a^2 = -(2)^2 = -4]$$

$$= \frac{\sin 2x}{-4 + 10D + 25}$$

$$= \frac{\sin 2x}{10D + 21}$$

$$= \frac{\sin 2x}{10D + 21} \times \frac{10D - 21}{10D - 21}$$

$$= \frac{\sin 2x\,(10D - 21)}{(10D)^2 - (21)^2}$$

$$= \frac{10D\,(\sin 2x) - 21 \sin 2x}{100D^2 - 441}$$

$$= \frac{10\,(2 \cos 2x) - 21 \sin 2x}{100\,(-4) - 441}$$

$$= \frac{20 \cos 2x - 21 \sin 2x}{-400 - 441}$$

$$\text{P.I} = \frac{20 \cos 2x - 21 \sin 2x}{-841}$$

$$y = e^{-5x}\,(Ax + B) + \frac{20 \cos 2x - 21 \sin 2x}{-841}$$

Example 2: Solve $(D^2 + 2D + 2)\,y = \cos 2x$

Solution:

$$y = \text{C.F} + \text{P.I} \quad \ldots (1)$$

C.F: $(D^2 + 2D + 2)\,y = 0$

The Auxiliary equation is

Put $D = m$

$$m^2 + 2m + 2 = 0$$

$$\boxed{a = 1, b = 2, c = 2}$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4\,(1)\,(2)}}{2\,(1)}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm i\,\sqrt{4}}{2}$$

$$= \frac{-2 \pm i2}{2}$$

$$= \frac{2\,(-1 \pm i)}{2}$$

$$m = -1 \pm i$$

$$\alpha = -1,\ \beta = 1$$

$$\text{C.F} = e^{-x}\,(A\cos x + B\sin x)$$

$$\text{C.F} = e^{-x}\,(A\cos x + B\sin x)$$

$$\text{P.I} = \frac{\cos 2x}{D^2 + 2D + 2}$$

$$= \frac{\cos 2x}{-4 + 2D + 2} \qquad [\text{Replace } D^2 = -4]$$

$$= \frac{\cos 2x}{2D - 2}$$

$$= \frac{\cos 2x}{2D - 2} \times \frac{2D + 2}{2D + 2}$$

$$= \frac{2D\,(\cos 2x) + 2\cos 2x}{(2D)^2 - (2)^2}$$

$$= \frac{2\,(-\,2\sin 2x) + 2\cos 2x}{4D^2 - 4}$$

$$= \frac{-\,4\sin 2x + 2\cos 2x}{4\,(-\,4) - 4}$$

$$\text{P.I} = \frac{-\,4\sin 2x + 2\cos 2x}{-\,16 - 4}$$

$$= \frac{-\,4\sin 2x + 2\cos 2x}{-\,20}$$

$$y = \text{C.F} + \text{P.I}$$

$$y = e^{-x}\,(A\cos x + B\sin x) + \frac{4\sin 2x + 2\cos 2x}{20}$$

Example 3: Find the P.I of $(D^2 + 4)\,y = \cos 2x$

Solution:

$$\text{P.I} = \frac{\cos 2x}{D^2 + 4} \qquad [\text{Replace } D^2 = -\,(a^2) = -\,(2)^2 = -\,4]$$

$$= \frac{\cos 2x}{-\,4 + 4}$$

$$= \frac{\cos 2x}{0}$$

$$= \frac{x\cos 2x}{2D}$$

$$= \frac{x}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2}\left[\frac{\sin 2x}{2}\right]$$

$$\text{P.I} = \frac{x\sin 2x}{4}$$

Example 4: Find the P.I. of $(D^2+1)\,y = \sin x$

Solution:

$$\text{P.I} = \frac{\sin x}{D^2+1}$$

$$= \frac{\sin x}{-1+1}$$

$$= \frac{\sin x}{0}$$

$$= \frac{x \sin x}{2D}$$

$$= \frac{x}{2}\int \sin x \, dx$$

$$= \frac{x}{2}[-\cos x]$$

$$\text{P.I} = -\frac{x\cos x}{2}$$

Example 5: Find the P.I. of $(D^2-3D+2)\,y = 2\cos(2x+3)$

Solution:

$$\text{P.I} = \frac{2\cos(2x+3)}{D^2-3D+2}$$

$$= \frac{2\cos(2x+3)}{-4-3D+2}$$

$$= \frac{2\cos(2x+3)}{-3D-2} = \frac{2\cos(2x+3)}{-3D-2} \times \frac{-3D+2}{-3D+2}$$

$$= \frac{-6D[\cos(2x+3)] + 2[2\cos(2x+3)]}{(-3D)^2-(2)^2}$$

$$= \frac{-6[-2\sin(2x+3)] + 4\cos(2x+3)}{9D^2-4}$$

$$= \frac{12 \sin (2x+3) + 4 \cos (2x+3)}{9(-4)-4}$$

$$= \frac{12 \sin (2x+3) + 4 \cos (2x+3)}{-40}$$

$$= \frac{4 [3 \sin (2x+3) + \cos (2x+3)}{-40}$$

$$\text{P.I} = \frac{3 \sin (2x+3) + \cos (2x+3)}{-10}$$

Practice problems

1. Solve $(D^2 - 2D + 5)\, y = \cos 2x$
2. Solve $(D^2 + 4D + 3)\, y = \sin x$
3. Solve $(D^2 + 5D + 4)\, y = \sin^2 x$

Formula

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{\cos 2x}{2}$$

Example 6: Solve $(D^2 + 1)\, y = \cos^2 x$

Solution:

$$y = \text{C.F} + \text{P.I}$$

We know that $\cos^2 x = \frac{1+\cos 2x}{2}$

$$(D^2+1)\, y = \frac{1}{2} e^{0x} + \frac{\cos 2x}{2}$$

$$y = C.F + \text{P.I}_1 + \text{P.I}_2 \qquad \ldots (1)$$

C.F:

The Auxiliary equation is $m^2+1=0$

$$m^2=-1$$

$$m=\pm 1i$$

$$\alpha=0,\ \beta=1$$

$$\text{C.F}=e^{\alpha x}\,[A\cos\beta x+\beta\sin\beta x]$$

$$\text{C.F}=e^{0x}\,[A\cos x+B\sin x]$$

$$\text{P.I}_1=\frac{\frac{1}{2}e^{0x}}{D^2+1}$$

$$=\frac{\frac{1}{2}e^{0x}}{0^2+1}=\frac{1}{2}e^{0x}=\frac{1}{2}$$

$$\text{P.I}_2=\frac{\frac{1}{2}\cos 2x}{D^2+1}$$

$$=\frac{\frac{1}{2}\cos 2x}{-4+1}$$

$$=\frac{\frac{1}{2}\cos 2x}{-3}$$

$$\text{P.I}_2=\frac{\cos 2x}{-6}$$

$$\therefore\ y=A\cos x+B\sin x+\frac{1}{2}-\frac{\cos 2x}{6}$$

Example 7: **Solve** $(D^2 + 16)\, y = e^{-3x} + \cos 4x$

Solution:

$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$

C.F:

The Auxiliary equation is,

Put $D = m$

$$m^2 + 16 = 0$$

$$m^2 = -16$$

$$m = 0 \pm 4i$$

$\therefore$ C.F $= e^{0x}\,(A \cos 4x + B \sin 4x)$

$$\text{P.I}_1 = \frac{e^{-3x}}{D^2 + 16}$$

$$= \frac{e^{-3x}}{(-3)^2 + 16}$$

$$= \frac{e^{-3x}}{9 + 16}$$

$$\boxed{\mathbf{P.I_1} = \frac{e^{-3x}}{25}}$$

$$\text{P.I}_2 = \frac{\cos 4x}{D^2 + 16}$$

$$= \frac{\cos 4x}{-16 + 16}$$

$$= \frac{\cos 4x}{0}$$

$$= \frac{x \cos 4x}{2D + 0}$$

$$= \frac{x}{2} \int \cos 4x \, dx$$

$$= \frac{x}{2} \frac{\sin 4x}{4}$$

$$\text{P.I}_2 = \frac{x \sin 4x}{8}$$

$$\boxed{\boldsymbol{y = A \cos 4x + B \sin 4x + \frac{e^{-3x}}{25} + \frac{x \sin 4x}{8}}}$$

5.2.4 Type - IV [RHS $= x^n$]

The following formulae are important

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

Hint:

$\boldsymbol{\frac{1}{1-x} = (1-x)^{-1}}$	$\boldsymbol{\frac{1}{1+x} = (1+x)^{-1}}$

Example 1: **Solve** $\boldsymbol{\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2}$

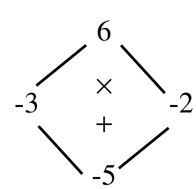
Solution:

Given: $(D^2 - 5D + 6)\, y = x^2$

$$y = \text{C.F} + \text{P.I} \qquad \ldots (1)$$

C.F:

$$(D^2 - 5D + 6)\, y = 0$$



The Auxiliary equation, $D = m$

$$m^2 - 5m + 6 = 0$$

$$(m-3)(m-2) = 0$$

$$m-3=0 \qquad m-2=0$$

$$m_1=3 \qquad m_2=2$$

$\therefore$ Roots are different

$$\text{C.F} = Ae^{3x} + Be^{2x}$$

$$\text{P.I} = \frac{1}{D^2-5D+6}(x^2)$$

$$= \frac{1}{6+D^2-5D}(x^2)$$

$$= \frac{1}{6\left[1+\left(\frac{D^2-5D}{6}\right)\right]}(x^2)$$

$$= \frac{1}{6}\left[1+\left(\frac{D^2-5D}{6}\right)\right]^{-1}(x^2)$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2-5D}{6}\right)+\left(\frac{D^2-5D}{6}\right)^2 - \ldots\right](x^2)$$

$$= \frac{1}{6}\left[x^2-\left(\frac{D^2-5D}{6}\right)x^2+\ldots\right]$$

$$= \frac{1}{6}\left[x^2-\frac{D^2}{6}(x^2)+\frac{5D}{6}(x^2)\right]$$

$$\text{P.I} = \frac{1}{6}\left[x^2-\frac{2}{6}+\frac{10x}{6}\right]$$

$$\boxed{\boldsymbol{y = Ae^{3x} + Be^{2x} + \frac{x^2}{6} - \frac{2}{36} + \frac{10x}{36}}}$$

Example 2: Solve $(D^2 + 1)\, y = x$

Solution:

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = 0 \pm 1i$$

$$\text{C.F} = e^{0x}\,(A \cos x + B \sin x)$$

$$\text{P.I} = \frac{1}{D^2 + 1}\,(x)$$

$$= \frac{1}{1 + D^2}\,(x)$$

$$= (1 + D^2)^{-1}\,(x)$$

$$= [1 - D^2 + (D^2)^2 + \ldots]\,(x)$$

$$= [x - D^2\,(x)]$$

$$= x - 0$$

$$\text{P.I} = x$$

$$\boxed{\boldsymbol{y = A \cos x + B \sin x + x}}$$

Example 3: Find the P.I of $(D^2 + 5D + 4)\, y = x^2 + 7x + 9$

Solution:

$$\text{P.I} = \frac{1}{D^2 + 5D + 4}\,(x^2 + 7x + 9)$$

$$= \frac{1}{4\left[1 + \left(\dfrac{5D + D^2}{4}\right)\right]}\,(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{5D + D^2}{4}\right)\right]^{-1}[x^2 + 7x + 9]$$

$$= \frac{1}{4}\left[1 - \left(\frac{5D + D^2}{4} \right) + \left(\frac{5D + D^2}{4} \right)^2 - \ldots \right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[x^2 + 7x + 9 - \frac{5D}{4}(x^2 + 7x + 9) - \frac{D^2}{4}[x^2 + 7x + 9) \right]$$

$$= \frac{1}{4}\left[x^2 + 7x + 9 - \frac{5}{4}(2x + 7) \frac{-1}{4}(2) \right]$$

$$\text{P.I} = \frac{1}{4}\left[x^2 + 7x + 9 - \frac{10x}{4} - \frac{35}{4} - \frac{2}{4} \right]$$

Example 4: Find the P.I. of $(D^2 + D + 10)\, y = x^2 + 3$

Solution:

$$\text{P.I} = \frac{1}{D^2 + D + 10}(x^2 + 3)$$

$$= \frac{1}{10\left[1 + \left(\frac{D + D^2}{10} \right) \right]}(x^2 + 3)$$

$$= \frac{1}{10}\left[1 + \left(\frac{D + D^2}{10} \right) \right]^{-1}(x^2 + 3)$$

$$= \frac{1}{10}\left[1 - \left(\frac{D + D^2}{10} \right) + \left(\frac{D + D^2}{10} \right)^2 - \ldots \right](x^2 + 3)$$

$$= \frac{1}{10}\left[(x^2 + 3) - \frac{D}{10}(x^2 + 3) - \frac{D^2}{10}(x^2 + 3) \right]$$

$$\boxed{\mathbf{P.I} = \left[\frac{x^2}{10} + \frac{3}{10} - \frac{2x}{100} - \frac{2}{100} \right]}$$

Practice problem

1. Solve $(D^2 + 3D + 2)\, y = x^2$
2. Solve $(D^2 + D + 1)\, y = x^3 + x^2$
3. Solve $(D^2 - 7D + 12)\, y = x^2 + 2x$

5.2.5 Type - V $e^{ax} \sin ax$ or $e^{ax} \cos ax$

Example 1: **Solve $(D^2 + D + 1)\, y = e^{-x} \sin 2x$**

Solution:

$$\text{C.F} = e^{-1/2x}\left[A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\right]$$

$$\text{P.I} = \frac{1}{D^2 + D + 1} e^{-x} \sin 2x$$

Replace D by $D - 1$

$$= \frac{e^{-x}}{(D-1)^2 + (D-1) + 1}$$

$$= \frac{e^{-x}}{D^2 + 1 - 2D + D - 1 + 1} \sin 2x$$

$$= \frac{e^{-x}}{D^2 - D + 1} \sin 2x$$

Replace D^2 by $-(a^2)$

$$= \frac{e^{-x}}{-4 - D + 1}$$

$$= \frac{e^{-x}}{-D - 3} \sin 2x$$

$$= e^{-x}\left[\frac{\sin 2x}{-D-3} \times \frac{-D+3}{-D+3}\right]$$

$$= e^{-x}\left[\frac{-D(\sin 2x) + 3 \sin 2x}{(-D)^2 - (3)^2}\right]$$

$$= e^{-x}\left[\frac{-2 \cos 2x + 3 \sin 2x}{D^2 - 9}\right]$$

$$= e^{-x}\left[\frac{-2 \cos 2x + 3 \sin 2x}{-4 - 9}\right]$$

$$= e^{-x}\left[\frac{-2 \cos 2x + 3 \sin 2x}{-13}\right]$$

$y = \text{C.F} + \text{P.I}$

$$\boxed{y = e^{-1/2\,x}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] - \frac{e^{-x}}{13}\,[\,3\sin 2x - 2\cos 2x\,]}$$

Example 2: Find particular integral of

$$(D^2 - 2D + 1)\,y = e^x \cos x$$

Solution:

$$\text{P.I} = \frac{1}{D^2 - 2D + 1}\,e^x \cos x$$

$$= \frac{e^x \cos x}{(D+1)^2 - 2\,(D+1) + 1} \qquad \text{Replace } D \text{ by } D+1$$

$$= \frac{e^x}{D^2 + 2D + 1 - 2D - 2 + 1}\cos x$$

$$= \frac{e^x}{D^2}\cos x$$

$$= \frac{e^x}{D}\left[\int (\cos x\,dx)\right]$$

$$= e^x\left[\int \sin x dx\right]$$

$$= e^x\,(-\cos x)$$

$$\boxed{\textbf{P.I} = -\,e^x \cos x}$$

Example 3: Find the particular integral of

$$(D^2 + 4D + 4)\,y = e^{-2x}\sin 2x$$

Solution:

$$\text{P.I} = \frac{1}{D^2 + 4D + 4}\,e^{-2x}\sin 2x \qquad \text{Replace } D \text{ by } D-2$$

$$= \frac{e^{-2x}}{(D-2)^2 + 4\,(D-2) + 4} \sin 2x$$

$$= \frac{e^{-2x}}{(D-2)^2 + 4\,(D-2) + 4} \sin 2x$$

$$= \frac{e^{-2x}}{D^2 + 4 - 4D + 4D - 8 + 4} \sin 2x$$

$$= \frac{e^{-2x}}{D^2} \sin 2x$$

Replace D^2 by $-(a^2)$

$$\boxed{\mathbf{P.I} = \frac{e^{-2x} \sin 2x}{-4}}$$

Practice problems

1. Solve $(D^2 - 2D + 2)\, y = e^x \sin x$
2. Solve $(D^2 + 4D + 3)\, y = e^x \cos 2x$
3. Solve $(D^2 + 6D + 9)\, y = e^{3x} \cos 3x$

5.2.6 Type - VI RHS $= e^{ax}\, x^n$

Example 1: Find the particular integral of

$$(D^2 + 2D + 1)\, y = e^{-x}\, x^2$$

Solution:

$$\text{P.I} = \frac{1}{D^2 + 2D + 1}\, e^{-x} \cdot x^2$$

Replace D by $D-1$

$$= \frac{e^{-x}}{(D-1)^2 + 2\,(D-1) + 1}\, x^2$$

$$= \frac{e^{-x}}{D^2 - 2D + 1 + 2D - 2 + 1}\, x^2$$

$$= \frac{e^{-x}}{D^2}(x^2)$$

$$= e^{-x} \int \left(\int x^2 \, dx \right)$$

$$= e^{-x} \left[\int \frac{x^3}{3} \, dx \right]$$

$$= e^{-x} \left[\frac{x^4}{12} \right]$$

$$\boxed{\mathbf{P.I} = \frac{e^{-x} \cdot x^4}{12}}$$

Example 2: Find the particular integral of $(D^2 + 4D + 3)\, y = xe^{3x}$.

Solution:

$$y = \text{C.F} + \text{P.I}$$

$$\text{C.F} = Ae^{-x} + Be^{-3x}$$

$$\text{P.I} = \frac{1}{D^2 + 4D + 3} xe^{3x}$$

$$= \frac{e^{3x}}{(D+3)^2 + 4\,(D+3) + 3} x \qquad \text{Replace } D \text{ by } D+3$$

$$= \frac{e^{3x}}{D^2 + 9 + 6D + 4D + 12 + 3} x$$

$$= \frac{e^{3x}}{D^2 + 10D + 24} x$$

$$= \frac{e^{3x}}{24 + D^2 + 10D} x$$

$$= \frac{e^{3x}}{24\left[1+\left(\frac{D^2+10D}{24}\right)\right]} x$$

$$= \frac{e^{3x}}{24}\left[1+\left(\frac{D^2+10D}{24}\right)\right]^{-1} \cdot x$$

$$= \frac{e^{3x}}{24}\left[1-\left(\frac{D^2+10D}{24}\right)+\left(\frac{D^2+10D}{24}\right)^2 - \dots\right] x$$

$$= \frac{e^{3x}}{24}\left[x-\frac{D^2}{24}(x)-\frac{10D}{24}(x)\right]$$

$$= \frac{e^{3x}}{24}\left[x-0-\frac{10}{24}\right]$$

$$= \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

$$\boxed{\mathbf{P.I} = \frac{xe^{3x}}{24} - \frac{5e^{3x}}{288}}$$

Example 3: Solve $(D^2 - 2D + 2)\, y = e^x x^2$.

Solution:

$$y = \text{C.F} + \text{P.I}$$

$$\text{C.F} = e^x [A \cos x + B \sin x]$$

$$\text{P.I} = \frac{1}{D^2-2D+2} e^x x^2$$

Replace D by $D+1$

$$= \frac{e^x}{(D+1)^2 - 2(D+1) + 2} x^2$$

$$= \frac{e^x}{D^2+1+2D-2D-2+2} x^2$$

$$= \frac{e^x}{D^2+1} x^2$$

$$= e^x [1 + D^2]^{-1} x^2 = e^x [1 - D^2 + D^4] x^2$$

$$= e^x [x^2 - D^2 (x^2)]$$

$$\text{P.I} = e^x [x^2 - 2]$$

$$Y = \text{C.F} + \text{P.I}$$

$$\boxed{\boldsymbol{y = e^x [A \cos x + B \sin x] + e^x (x^2 - 2)}}$$

Practice problem

1. Solve $(D^2 - 2D + 1)\, y = e^x (3x^2 - 2)$
2. Solve $(D^2 - 2D + 1)\, y = x^2 e^{3x}$
3. Solve $(D^2 + 6D + 9)\, y = e^{-2x} x^3$

Example 4: Find P.I. of $(D^2 + 4D + 4)\, y = e^{-2x} x^2$

Solution:

$$\text{P.I} = \frac{1}{D^2 + 4D + 4} e^{-2x} \cdot x^2$$

Replace D by $D - 2$

$$= \frac{e^{-2x}}{(D-2)^2 + 4(D-2) + 4} x^2$$

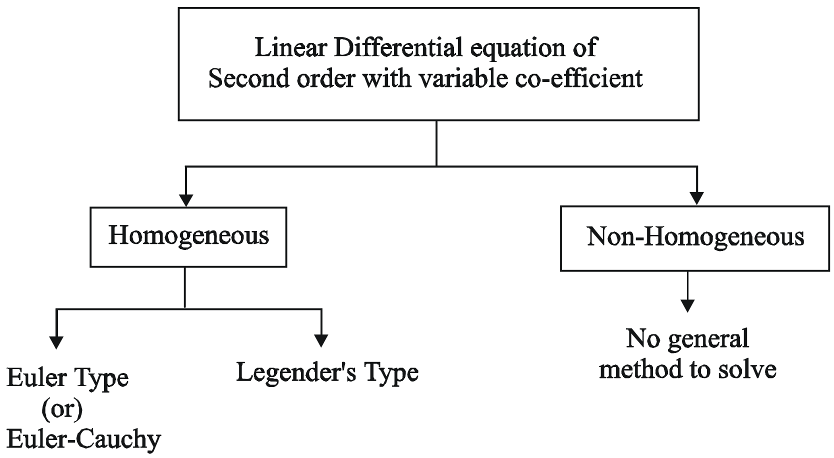
$$= \frac{e^{-2x}}{D^2 + 4 - 4D + 4D - 8 + 4}$$

$$= \frac{e^{-2x}}{D^2} x^2$$

$$= e^{-2x} \int\int x^2\, dx$$

$$\boxed{\textbf{P.I} = e^{-2x} \frac{x^4}{12}}$$

5.3 LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH VARIABLE CO-EFFICENT



5.3.1 Homogeneous equations of Euler type [Cauchy's type]

Formula:

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D'(D'-1) \\ x^3 D^3 &= D'(D'-1)(D'-2) \\ x^4 D^4 &= D'(D'-1)(D'-2)(D'-3) \end{aligned}}$$

Example 1: **Solve** $x^2 \dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} + y = 0$

Solution:

$$x^2 D^2 y - xDy + y = 0$$

$$(x^2 D^2 - xD + 1)\, y = 0 \qquad \ldots (*)$$

Put $\quad x = e^z$

$$z = \log x$$

$$x^2 D^2 = D'(D'-1)$$

$$xD = D'$$

$$[D'(D'-1) - D' + 1]\, y = 0$$

$$[D'^2 - D' - D' + 1]\, y = 0$$

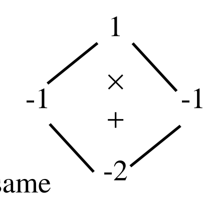
$$\boxed{\mathbf{[D'^2 - 2D' + 1]\, y = 0}}$$

The auxiliary equation is

Put $\quad D' = m$

$$m^2 - 2m + 1 = 0$$

$$(m-1)(m-1) = 0$$



$$m - 1 = 0 \qquad m - 1 = 0$$

$$\boxed{m = 1 \qquad m = 1}$$ Roots same

$$y = (Az + B)\, e^{1z}$$

$$y = (A \log x + B)\, x$$

$$\boxed{\therefore\ \mathbf{y = x\,(A \log x + B)}}$$

Example 2: Solve $x^2 y'' + 2xy' + 2y = 0$

Solution:

$$x^2 D^2 y + 2xDy + 2y = 0$$

$$(x^2 D^2 + 2xD + 2)\, y = 0 \qquad \text{... (*)}$$

$$\boxed{\begin{gathered}\boldsymbol{x = e^z}\\ \boldsymbol{z = \log x}\\ \boldsymbol{x^2 D^2 = D'(D'-1)}\\ \boldsymbol{xD = D'}\end{gathered}}$$

$$[D'(D'-1) + 2D' + 2]\, y = 0$$

$$[D'^2 - D' + 2D' + 2]\, y = 0$$

$$[D'^2 + D' + 2]\, y = 0$$

The auxiliary equation is put $D' = m$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, b = 1, c = 2$$

$$m = \frac{-1 \pm \sqrt{1 - 4\,(1)\,(2)}}{2.1}$$

$$m = \frac{-1 \pm \sqrt{1-8}}{2}$$

$$= \frac{-1 \pm \sqrt{-7}}{2}$$

$$\boxed{\begin{gathered}\boldsymbol{m = \frac{-1}{2} \pm i\,\frac{\sqrt{7}}{2}}\\ \boldsymbol{\alpha \pm i\,\beta}\end{gathered}}$$

Hint

$$e^{2z} = (e^z)^2$$

$$e^{-2z} = (e^z)^2$$

$$\boxed{\begin{gathered}\boldsymbol{\sqrt{x} = x^{1/2}}\\ \boldsymbol{\frac{1}{\sqrt{x}} = x^{-1/2}}\end{gathered}}$$

$$\alpha = \frac{-1}{2} \quad \beta = \frac{\sqrt{7}}{2}$$

$$y = e^{-\alpha z}\,[A \cos \beta\, z + B \sin \beta\, z]$$

$$y = e^{-1/2\, z}\left[A \cos \frac{\sqrt{7}}{2}\, z + B \sin \frac{\sqrt{7}}{2}\, z\right]$$

$$= (e^z)^{-1/2}\left[A\cos\frac{\sqrt{7}}{2}\log x + B\sin\frac{\sqrt{7}}{2}\log x\right]$$

$$= x^{-1/2}\left[A\cos\frac{\sqrt{7}}{2}\log x + B\sin\frac{\sqrt{7}}{2}\log x\right]$$

$$= \frac{1}{\sqrt{x}}\left[A\cos\frac{\sqrt{7}}{2}\log x + B\sin\frac{\sqrt{7}}{2}\log x\right]$$

Example 3: Solve $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$

Solution:

Given: $xD^2y + Dy = 0$ ... (1)

Multiply by x

$$x^2D^2y + xDy = 0$$

$$\boxed{\begin{array}{ll} \text{Put} & \boldsymbol{x = e^z} \\ & \boldsymbol{z = \log x} \\ & \boldsymbol{x^2D^2 = D'(D'-1)} \\ & \boldsymbol{xD = D'} \end{array}}$$

$$[x^2D^2 + xD]\,y = 0$$

$$[D'(D'-1) + D']\,y = 0$$

$$[D'^2 - D' + D']\,y = 0$$

$$[D'^2]\,y = 0,\ m = 0, 0$$

$$y = [Az + B]\,e^{0z}$$

$$= [A\log x + B]\,(x)^0$$

$$\boxed{\boldsymbol{y = A\log x + B}}$$

Example 4: **Solve** $\boldsymbol{x^2 \dfrac{d^2 y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = x^2 + \dfrac{1}{x^2}}$

Solution:

$$x^2 D^2 y + 4xDy + 2y = x^2 + \frac{1}{x^2}$$

$$[x^2 D^2 + 4xD + 2]\, y = x^2 + \frac{1}{x^2} \qquad \text{... (1)}$$

Put $\boxed{\begin{aligned} &\boldsymbol{x = e^z} \\ &\boldsymbol{z = \log z} \\ &\boldsymbol{x^2 D^2 = D'(D' - 1)} \\ &\boldsymbol{xD = D'} \end{aligned}}$ Put in (1)

$$[D'(D' - 1) + 4D' + 2]\, y = (e^z)^2 + \frac{1}{(e^z)^2}$$

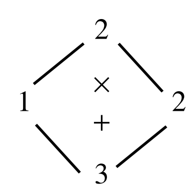
$$[D'^2 - D' + 4D' + 2]\, y = e^{2z} + \frac{1}{e^{2z}}$$

$$[D'^2 + 3D' + 2]\, y = e^{2z} + e^{-2z}$$

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

C.F:

Put $D' = m$



$$m^2 + 3m + 2 = 0$$

$$m + 1 = 0,\ m + 2 = 0$$

$$m_1 = -1,\ m_2 = -2$$

$$\text{C.F} = Ae^{m_1 z} + Be^{m_2 z}$$

$$\text{C.F} = Ae^{-1z} + Be^{-2z}$$

$$\text{C.F} = Ae^{-1z} + Be^{-2z}$$

$$= A\,[e^z]^{-1} + B\,[e^z]^{-2}$$

$$= A\,[x]^{-1} + B\,[x]^{-2}$$

$$\boxed{\mathbf{C.F} = \boldsymbol{A}\left[\frac{\mathbf{1}}{\boldsymbol{x}}\right] + \boldsymbol{B}\left[\frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{2}}}\right]}$$

$$\text{P.I}_1 = \frac{1}{D'^2 + 3D' + 2}\, e^{2z}$$

$a = 2$

D' by a

$$= \frac{e^{2z}}{2^2 + 3\,(2) + 2}$$

$$\boxed{\mathbf{P.I_1} = \frac{\boldsymbol{e}^{\mathbf{2z}}}{\mathbf{12}}}$$

$$\text{P.I}_1 = \frac{(e^z)^2}{12} = \frac{x^2}{12}$$

$$\text{P.I}_2 = \frac{1}{D'^2 + 3D' + 2}\, e^{-2z}$$

$$= \frac{e^{-2z}}{(-2)^2 + 3\,(-2) + 2}$$

$$= \frac{e^{-2z}}{4 - 6 + 2}$$

$$= \frac{e^{-2z}}{6 - 6} = \frac{e^{-2z}}{0}$$

$$= \frac{ze^{-2z}}{2D' + 3}$$

$$= \frac{ze^{-2z}}{2\,(-2) + 3}$$

$$= \frac{ze^{-2z}}{-4 + 3}$$

$$\text{P.I}_2 = \frac{ze^{-2z}}{-1}$$

$$= \frac{z\,(e^z)^{-2}}{-1}$$

$$= \frac{z\,[x]^{-2}}{-1}$$

$$= -z\,\frac{1}{x^2}$$

$$\text{P.I}_2 = \frac{-\log x}{x^2}$$

$$\boxed{\boldsymbol{y = \frac{A}{x} + \frac{B}{x^2} + \frac{x^2}{12} - \frac{\log x}{x^2}}}$$

Example 5: Solve $(x^2 D^2 + 3xD + 4)\,y = \cos(\log x)$

Solution:

Put $\boldsymbol{x = e^z}$

$\boldsymbol{z = \log x}$

$\boldsymbol{x^2 D^2 = D'\,(D' + 1)}$

$\boldsymbol{xD = D'}$

$$[D'\,(D' - 1) + 3D' + 4]\,y = \cos(z)$$

$$[D'^2 - D' + 3D' + 4]\,y = \cos z$$

$$[D'^2 + 2D' + 4]\,y = \cos z$$

$$y = \text{C.F} + \text{P.I}$$

C.F:

The Auxiliary is Put $D' = m$

$$m^2 + 2m + 4 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, b = 2, c = 4$$

$$= \frac{-2 \pm \sqrt{4 - 4\,(1)\,(4)}}{2\,(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm i\sqrt{12}}{2}$$

$$= \frac{-2 \pm i2\sqrt{3}}{2}$$

$$= \frac{2\,[-1 \pm i\sqrt{3}]}{2}$$

$$m = -1 \pm i\sqrt{3}$$

$$= \alpha \pm i\,\beta$$

$$\alpha = -1, \beta = \sqrt{3}$$

$$\begin{array}{r|l} 2 & 12 \\ \hline 2 & 6 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

$$\text{C.F} = e^{-1z}\,[A \cos \sqrt{3}z + B \sin \sqrt{3}z]$$

$$= [e^z]^{-1}\,[A \cos \sqrt{3}z + B \sin \sqrt{3}z]$$

$$= x^{-1}\,[A \cos \sqrt{3} \log x + B \sin \sqrt{3}z]$$

$$\text{C.F} = \frac{1}{x}\,[A \cos \sqrt{3} \log x + B \sin \sqrt{3} \log x]$$

$$\text{P.I} = \frac{1}{D'^2 + 2D' + 4} \cos z$$ Replace D'^2 by $(-a^2)$

$$= \frac{\cos z}{-1 + 2D' + 4}$$

$$= \frac{\cos z}{2D' + 3} \times \frac{2D' - 3}{2D' - 3}$$

$$= \frac{2D' (\cos z) - 3 \cos z}{(2D')^2 - (3)^2}$$

$$= \frac{2 (- \sin z) - 3 \cos z}{4D'^2 - 9}$$

$$= \frac{- 2 \sin z - 3 \cos z}{4 (- 1) - 9}$$

$$= \frac{- 2 \sin z - 3 \cos z}{- 4 - 9}$$

$$= \frac{- 2 \sin z - 3 \cos z}{- 13}$$

$$\text{P.I} = \frac{- [2 \sin z + 3 \cos z]}{- 13}$$

$$\text{P.I} = \frac{1}{13} [2 \sin \log x + 3 \cos \log x]$$

$y = \text{C.F.} + \text{P.I.}$

$$\boxed{\boldsymbol{y = \frac{1}{x} [A \cos \sqrt{3} \log x + B \sin \sqrt{3} \log x] + \frac{1}{13} [2 \sin \log x + 3 \cos \log x]}}$$

Example 6: Solve $x^2 D^2 y - xDy + y = \log x$

Solution:

Given: $x^2 D^2 y - xDy + y = \log x$

$$(x^2 D^2 - xD + 1) y = \log x \qquad \dots (1)$$

$$\boxed{\begin{aligned} \text{Put}\quad & \boldsymbol{x = e^z} \\ & \boldsymbol{z = \log x} \\ & \boldsymbol{x^2 D^2 = D' (D' - 1)} \\ & \boldsymbol{xD = D'} \end{aligned}} \quad \text{in (1)}$$

$$[D' (D' - 1) - D' + 1]\, y = z$$

$$[D'^2 - D' - D']\, y = z$$

$$[D'^2 - 2D' + 1]\, y = z$$

$$y = \text{C.F} + \text{P.I} \qquad \text{... (1)}$$

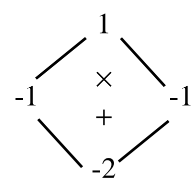
C.F: $[D'^2 - 2D' + 1]\, y = 0$

Put $\quad D' = m$

$$m^2 - 2m + 1 = 0$$

$$(m - 1)\,(m - 1) = 0$$

$$m = 1,\ 1$$



$$\text{C.F} = (Az + B)\, e^{1z}$$

$$\text{C.F} = (A \log x + B)\, x$$

$$\text{P.I} = \frac{1}{D'^2 - 2D' + 1}\, z$$

$$= \frac{1}{1\left[1 - \left(\dfrac{2D' - D'^2}{1}\right)\right]}\, z$$

$$= \frac{1}{1 - (2D' - D'^2)}\, z$$

$$= [1 - (2D' - D'^2)]^{-1}\, z$$

$$= [1 + (2D' - D'^2) + (2D' - D'^2)^2 + \ldots]\, z$$

$$= [z + (2D' - D'^2)\ z]$$

$$= [z + 2D'\ (z) - D'^2\ (z)]$$

$$= z + 2\ (1)$$

$$\text{P.I} = z + 2$$

$$\boxed{\textbf{P.I} = \log x + 2}$$

$$y = \text{C.F.} + \text{P.I.}$$

$$\boxed{y = (A \log x + B) + \log x + 2}$$

Example 7: $\boldsymbol{(x^2\,D^2 - xD + 4)\ y = x^2 \sin(\log x)}$

Solution:

Given: $(x^2\,D^2 - xD + 4)\ y = x^2 \sin(\log x)$ ... (1)

Put	$\boldsymbol{x = e^z}$
	$\boldsymbol{z = \log x}$
	$\boldsymbol{x^2 D^2 = D'(D' - 1)}$
	$\boldsymbol{xD = D'}$ Put in (1)

$$[D'(D' - 1) - D' + 4]\ y = (e^z)^2 \sin(z)$$

$$[D'^2 - D' - D' + 4]\ y = e^{2z} \sin z$$

$$[D'^2 - 2D' + 4]\ y = e^{2z} \sin z$$

$$y = \text{C.F} + \text{P.I}$$

C.F: $[D'^2 - 2D' + 4]\ y = 0$

Put $D' = m$

$$m^2 - 2m + 4 = 0$$

$$a = 1,\ b = -2,\ c = 4$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{2 \pm \sqrt{4 - 4\,(1)\,(4)}}{2.1}$$

$$= \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$\begin{array}{r|l} 2 & 12 \\ \hline 2 & 6 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm i2\sqrt{3}}{2}$$

$$= \frac{2\,(1 \pm i\sqrt{3})}{2}$$

$$\alpha = 1,\ \beta = \sqrt{3}$$

$$\text{C.F} = e^{1z}\,[A\cos\sqrt{3}z + B\sin\sqrt{3}z]$$

$$\text{C.F} = x\,[A\cos\sqrt{3}\log x + B\sin\sqrt{3}\log x]$$

$$\text{P.I} = \frac{1}{D'^2 - 2D' + 4}\,e^{2z}\sin z$$

$$= \frac{e^{2z}}{(D'+2)^2 - 2\,(D'+2) + 4}\sin z$$

Replace D' by $D' + 2$

$$= \frac{e^{2z}}{D'^2 + 4 + 4D' - 2D' - 4 + 4}\sin z$$

$$= \frac{e^{2z}}{D'^2 + 2D' + 4}\sin z$$

$$= \frac{e^{2z}}{-1 + 2D' + 4}\sin z$$

$$= \frac{e^{2z}}{2D' + 3}\sin z$$

$$= \frac{e^{2z}}{2D'+3}\left[\sin z \times \frac{2D'-3}{2D'-3}\right]$$

$$= e^{2z}\left[\frac{2D'(\sin z) - 3\sin z}{(2D')^2 - (3)^2}\right]$$

$$= e^{2z}\left[\frac{2\cos z - 3\sin z}{4D'^2 - 9}\right]$$

$$= e^{2z}\left[\frac{2\cos z - 3\sin z}{4(-1) - 9}\right] \qquad [\text{Replace } D'^2 = -1]$$

$$= e^{2z}\left[\frac{2\cos z - 3\sin z}{-4-9}\right]$$

$$= e^{2z}\left[\frac{2\cos z - 3\sin z}{-13}\right]$$

$$= (e^z)^2\left[\frac{2\cos z - 3\sin z}{-13}\right]$$

$$\text{P.I} = (x)^2\,\frac{[2\cos\log x - 3\sin\log x]}{-3}$$

$$y = \text{C.F} + \text{P.I}$$

$$\boxed{\begin{aligned} y = x\,[A\cos\sqrt{3}\log x + B\sin\sqrt{3}\log x] \\ -\frac{x^2}{3}[2\cos(\log x) - 3\sin\log x] \end{aligned}}$$

Example 8: $(x^2D^2 - xD + 2)\,y = x^2\log x$

Solution:

$$\boxed{\begin{aligned} \text{Put}\quad & x = e^z \\ & z = \log x \\ & x^2D^2 = D'(D'-1) \\ & xD = D' \end{aligned}} \quad \text{Put in (1)}$$

$$[D'(D'-1) - D' + 2]\,y = (e^z)^2\,z$$

$$[D'^2 - 2D' + 2]\, y = e^{2z}\, z$$

$$[D'^2 - 2D' + 2]\, y = e^{2z}\, z$$

$$y = \text{C.F} + \text{P.I}$$

C.F: Put $D' = m$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, b = -2, c = 2$$

$$= \frac{2 \pm \sqrt{4 - 4\,(1)\,(2)}}{2\,(1)} = \frac{2 \pm i2}{2}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm i\sqrt{4}}{2}$$

$$= \frac{2\,[1 \pm i]}{2}$$

$$\alpha = 1, \beta = 1$$

$$\text{C.F} = e^{1z}\,[A \cos 1z + B \sin 1z]$$

$$= e^{z}\,[A \cos z + B \sin z]$$

$$\boxed{\textbf{C.F} = \boldsymbol{x}\,[\boldsymbol{A} \cos (\log \boldsymbol{x}) + \boldsymbol{B} \sin (\log \boldsymbol{x})]}$$

$$\text{P.I} = \frac{1}{D'^2 - 2D' + 2}\, e^{2z} \cdot z$$

$$= \frac{e^{2z}}{(D' + 2)^2 - 2\,(D' + 2) + 2} \qquad \text{Replace } D' \text{ by } D' + 2$$

$$= \frac{e^{2z}}{D'^2 + 4 + 4D' - 2D' - 4 + 2}\, z$$

$$= \frac{e^{2z}}{D'^2 + 2D' + 2} z$$

$$= \frac{e^{2z}}{2 + 2D' + D'^2} z$$

$$= \frac{e^{2z}}{2\left[1 + \left(\frac{2D' + D'^2}{2}\right)\right]} \cdot z$$

$$= \frac{e^{2z}}{2}\left[1 + \left(\frac{2D' + D'^2}{2}\right)\right]^{-1} z$$

$$= \frac{e^{2z}}{2}\left[1 - \left(\frac{2D' + D'^2}{2}\right) + \left(\frac{2D' + D'^2}{2}\right)^2 + \ldots\right] z$$

$$= \frac{e^{2z}}{2}\left[z - \frac{2D'}{2}(z) - \frac{D'^2}{2}(z)\right]$$

$$= \frac{e^{2z}}{2}[z - 1]$$

$$= \frac{(e^z)^2}{2}[\log x - 1] = \frac{(x)^2}{2}[\log x - 1]$$

$$y = \text{C.F} + \text{P.I}$$

$$\boxed{\boldsymbol{y = x\,[A \cos(\log x) + B \sin(\log x)] + \frac{x^2}{2}(\log x - 1)}}$$

Example 9: **Solve** $\boldsymbol{\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12 \log x}{x^2}}$ ... (1)

Solution:

Multiply by (x^2)

$$x^2 \frac{d^2 y}{dx^2} + \frac{x^2}{x}\frac{dy}{dx} = \frac{x^2\, 12 \log x}{x^2}$$

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} = 12 \log x \qquad \text{... (2)}$$

$$\boxed{\begin{aligned} &\boldsymbol{x = e^z}\\ &\boldsymbol{z = \log x}\\ &\boldsymbol{x^2 D^2 = D' (D' - 1)}\\ &\boldsymbol{xD = D'} \end{aligned}}$$

$$[x^2 D^2 + xD]\, y = 12 \log x$$

$$[D' (D' - 1) + D']\, y = 12z$$

$$[D'^2 - D' + D]\, y = 12z$$

$$[D'^2]\, y = 12z$$

$$y = \text{C.F} + \text{P.I}$$

C.F: $[D'^2]\, y = 0$

Put $D' = m$

$$m^2 = 0$$

$$m = 0, m = 0$$

Roots are equal

$$\text{C.F} = [Az + B]\, e^{0z}$$

$$= [A \log x + B]\, [e^z]^0$$

$$= [A \log x + B]\, (x)^0$$

$$\text{C.F} = A \log x + B$$

$$\text{P.I} = \frac{1}{D'^2}\, 12z$$

$$= \frac{12z}{D'^2}$$

$$= \frac{12}{D'} \int z dz$$

$$= \frac{12}{D'}\left[\frac{z^2}{2}\right]$$

$$= \frac{12}{2}\int z^2\, dz$$

$$= 6\int z^2\, dz$$

$$= 6\,\frac{z^3}{3}$$

$$\text{P.I} = 2z^3$$

$$\text{P.I} = 2\,(\log x)^3$$

$$y = \text{C.F} + \text{P.I}$$

$$\boxed{\boldsymbol{y = A \log x + B + 2\,(\log x)^3}}$$

Example 10: Reduce the equation $(x^2 D^2 + xD + 1)\, y = 0$ into the ordinary differential equation.

Solution:

Given: $(x^2 D^2 + xD + 1)\, y = 0$

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D'(D'-1) \end{aligned}}$$

$$[D'(D'-1) + D' + 1]\, y = 0$$

$$(D'^2 - D' + D' + 1]\, y = 0$$

$$(D'^2 + 1)\, y = 0$$

Example 11: Reduce the equation $x^2 y'' + 3xy' + 3y = x$ into ordinary differential equation.

Solution:

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D'(D'-1) \end{aligned}}$$

$$x^2 y'' + 3xy' + 3y = x$$

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 3y = x$$

$$(x^2 D^2 + 3xD + 3)\, y = x$$

$$D'(D'-1) + 3D' + 3)\, y = e^z$$

$$(D'^2 - D' + 3D' + 3)\, y = e^z$$

$$\boxed{(D'^2 + 2D' + 3)\, y = e^z}$$

Example 12: Transform: $(x^2 D^2 - 3xD)\, y = \dfrac{\sin(\log x)}{x}$ into a ordinary differential equations.

Solution:

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D'(D'-1) \end{aligned}}$$

$$(x^2 D^2 - 3 x D) y = \frac{\sin(\log x)}{x}$$

$$[D' (D' - 1) - 3D') z = \frac{\sin(z)}{e^z}$$

$$(D'^2 - D' - 3D') z = \frac{\sin(z)}{e^z}$$

$$\boxed{(D'^2 - 4D') z = e^{-z} \sin z} \qquad D' \to \frac{d}{dz}$$

Example 13: Solve: $(x^2 D^2 + 4xD + 2) y = 6x$.

Solution:

Given: $(x^2 D^2 + 4xD + 2) y = 6x$

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D' (D' - 1) \end{aligned}}$$

$$[D' (D' - 1) + 4D' + 2] z = 6e^z$$

$$[D'^2 - D' + 4D' + 2] z = 6e^z$$

$$[D'^2 + 3D' + 2] z = 6e^z$$

The solution is $y = \text{C.F} + \text{P.I}$

To find: C.F

The Auxiliary equation is

$$\boxed{D' = m}$$

$$m^2 + 3m + 2 = 0$$

$$m = -1, -2 \qquad \text{(real and unequal)}$$

$$\text{C.F} = Ae^{-z} + Be^{-2z}$$

$$= A(e^z)^{-1} + B(e^z)^{-2}$$

$$= Ax^{-1} + Bx^{-2}$$

$$\text{C.F.} = \frac{A}{x} + \frac{B}{x^2}$$

$$\text{P.I} = \frac{6e^z}{D'^2 + 3D' + 2} \qquad \boldsymbol{D' \to 1}$$

$$= \frac{6e^z}{(1)^2 + 3\,(1) + 2}$$

$$= \frac{6e^z}{1 + 3 + 2}$$

$$= \frac{6e^z}{6}$$

$$\text{P.I} = e^z$$

$$\text{P.I} = x$$

The solution is $y = \text{C.F} + \text{P.I}$

$$\boxed{\boldsymbol{y = A/x + B/x^2 + x}}$$

Example 14: Solve: $(x^2 D^2 - 3xD - 5)\, y = \sin(\log x)$

Solution:

Given: $(x^2 D^2 - 3xD - 5)\, y = \sin(\log x)$

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D'(D' - 1) \end{aligned}}$$

$$(D'(D'-1)-3D'-5)\,y=\sin z$$

$$(D'^2-4D'-5)\,y=\sin z$$

$$z=\text{C.F}+\text{P.I}$$

To find: C.F

The Auxiliary equation $\boldsymbol{D'=m}$

$$m^2-4m-5=0$$

$$m=5,\ -1 \qquad \text{(real and unequal)}$$

$$\text{C.F}=Ae^{5z}+Be^{-z}$$

$$\text{P.I}=\frac{\sin z}{D'^2-4D'-5}$$

$$=\frac{\sin z}{(-1)-4D'-5}$$

$$=\frac{\sin z}{-4D'-6}$$

$$=\frac{\sin z}{-4D'-6}\times\frac{-4D'+6}{-4D'+6}$$

$$=\frac{\sin z\,(-4D'+6)}{16D'^2-36}$$

$$=\frac{-4D'(\sin z)+6\sin z}{16(-1)-36}$$

$$=\frac{-4(\cos z)+4\sin z}{-16-36}$$

$$\text{P.I}=\frac{-4\cos z+4\sin z}{-52}$$

The solution is

$$y=\text{C.F}+\text{P.I}$$

$$=Ae^{5z}+Be^{-z}+\left(\frac{-4\cos\ \log x+4\sin\log x}{-52}\right)$$

$$\boxed{\boldsymbol{y=Ax^5+\frac{B}{x}+\frac{4\cos\log x}{52}-\frac{4\sin\log x}{52}}}$$

Example 15: **Solve $(x^2 D^2 - 12xD - 4)\, y = x^2 + 2 \log x$.**

Solution:

Given: $(x^2 D^2 - 12xD - 4y = x^2 + 2 \log x$

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D'(D'-1) \end{aligned}}$$

$$(D'(D'-1) - 12D' - 4)\, z = (e^z)^2 + 2z$$

$$(D'^2 - 13D' - 4)\, z = e^{2z} + 2z$$

$$y = \text{C.F.} + \text{P.I.}_1 + \text{P.I.}_2$$

To find: C.F

The Auxiliary equation $D' = m$

$$m^2 - 13m - 4 = 0$$

$$m = \frac{13 \pm \sqrt{(-13)^2 - 4(1)(-4)}}{2}$$

$$= \frac{13 \pm \sqrt{169 + 16}}{2}$$

$$= \frac{13 \pm \sqrt{185}}{2}$$

$$m = \frac{13 + \sqrt{185}}{2}, \frac{13 - \sqrt{185}}{2} \quad \text{(real and unequal)}$$

$$\text{C.F} = Ae^{\left(\frac{13+\sqrt{185}}{2}\right)z} + Be^{\left(\frac{13-\sqrt{185}}{2}\right)z}.$$

$$\text{P.I}_1 = \frac{e^{2z}}{D'^2 - 13D' - 4}$$

$$\boxed{D' \to 2}$$

$$= \frac{e^{2z}}{4 - 26 - 4}$$

$$= \frac{e^{2z}}{-26}$$

$$\text{P.I} = \frac{-e^{2z}}{26}$$

$$\text{P.I}_2 = \frac{2z}{D'^2 - 13D'^2 - 4}$$

$$= \frac{2z}{-4 - 13D' + D'^2}$$

$$= \frac{2z}{-4\left[1 + \frac{-13D' + D'^2}{-4}\right]}$$

$$= \frac{2z}{-4\left[1 + \left(\frac{13D'}{4} - \frac{D'^2}{4}\right)\right]}$$

$$= \frac{2z}{-4\left[1 + \left(\frac{13D'}{4} - \frac{D'^2}{4}\right)\right]}$$

$$= \frac{1}{-4}\left[\left(1 + \left(\frac{13D'}{4} - \frac{D'^2}{4}\right)^{-1}\right)\right] 2z$$

$$= \frac{1}{-4}\left[\left(1 - \left(\frac{13D'}{4} - \frac{D'^2}{4}\right) + \left(\frac{13D'}{4} - \frac{D'^2}{4}\right)^2 \ldots\right)\right] 2z$$

Omitting higher order

$$= \frac{-1}{4}\left[2z - \frac{13D'}{4}(2z) + \frac{D'^2}{4}(2z)\right]$$

$$= \frac{-1}{4}\left[2z - \frac{13}{4}(2) + 0\right]$$

$$= \frac{-2z}{4} + \frac{13}{8}$$

$$\text{P.I} = \frac{13}{8} - \frac{z}{2}$$

$$z = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$$= Ae^{\left(\frac{13+\sqrt{185}}{2}\right)z} + Be^{\left(\frac{13-\sqrt{135}}{2}\right)z} - \frac{e^{2z}}{26} + \frac{13}{8} - \frac{z}{2}$$

$$y = Ae^{\left(\frac{13+\sqrt{185}}{2}\right)\log x} + Be^{\left(\frac{13-\sqrt{135}}{2}\right)\log x} - \frac{x^2}{26} + \frac{13}{8} - \frac{\log x}{2}$$

$$y = A\,e^{\left(\frac{13+\sqrt{185}}{2}\right)} + Be^{\left(\frac{13-\sqrt{135}}{2}\right)} - \frac{x^2}{26} + \frac{13}{8} - \frac{\log x}{2}$$

Example 16: Solve: $[x^2 D^2 - xD + 1]\, y = \left(\frac{\log x}{x}\right)^2$

Solution:

Given: $(x^2 D^2 - xD + 1)\, y = \left(\frac{\log x}{x}\right)^2$

$$x = e^z$$

$$\log x = z$$

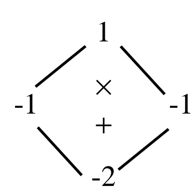
$$xD = D'$$

$$x^2 D^2 = D'(D'-1)$$

$$(D'(D'-1) - D' + 1)\, z = \left(\frac{z}{e^z}\right)^2$$

$$(D'^2 - D' - D' + 1)\, z = \frac{z^2}{e^{2z}}$$

$$(D'^2 - 2D' + 1)\, z = z^2 \cdot e^{-2z}$$



The solution is $z = \text{C.F} + \text{P.I}$

To find: C.F

The Auxiliary equation $\boldsymbol{D' = m}$

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1 \qquad \text{(real and equal)}$$

$$\text{C.F} = (Az + B)\, e^{1z}$$

$$\text{P.I} = \frac{z^2 \cdot e^{-2z}}{D'^2 - 2D' + 1}$$

$$\text{P.I} = \frac{z^2\ e^{-2z}}{(D' - 1)^2}$$

Replace D' by $D' - 2$

$$= e^{-2z} \frac{z^2}{[(D' - 2) - 1]^2}$$

$$= e^{-2z} \frac{z^2}{(D' - 3)^2}$$

$$= \frac{e^{-2z}}{9} \cdot \frac{z^2}{\left(1 - \frac{D'}{3}\right)^2}$$

$$= \frac{e^{-2z}}{9} \cdot \left[1 - \frac{D'}{3}\right]^{-2} z^2$$

$$= \frac{e^{-2z}}{9} \cdot \left[1 + 2\frac{D'}{3} + 3\left(\frac{D'}{3}\right)^2 + \dots\right] z^2$$

Omitting higher powers

$$= \frac{e^{-2z}}{9}\left[z^2 + \frac{4z}{3} + \frac{2}{3}\right]$$

$$= \frac{(e^z)^{-2}}{9} [3z^2 + 4z + 2]$$

$$= \frac{x^{-2}}{27} [3 (\log x)^2 + 4 \log x + 2]$$

$$y = \text{C.F.} + \text{P.I.}$$

$$\boxed{\boldsymbol{Y = (A \log x + B)x + \frac{1}{27x^2} [3 (\log x)^2 + 4 (\log x) + 2]}}$$

Example 17: **Solve** $(x^2 D^2 - 2xD - 4)\, y = \log x + \pi$

Solution:

Given: $(x^2 D^2 - 2xD - 4)\, y = \log x + \pi$

$$\boxed{\begin{aligned} x &= e^z \\ \log x &= z \\ xD &= D' \\ x^2 D^2 &= D'(D'-1) \end{aligned}}$$

$$(D'(D'-1) - 2D' - 4)\, z = z + \pi$$

$$(D'^2 - D' - 2D' - 4)\, z = z + \pi$$

$$(D'^2 - 3D' - 4)\, z = z + \pi\, e^{0z}$$

Auxiliary equation $(D' = m)$

$$m^2 - 3m - 4 = 0$$

$$(m+1)(m-4) = 0$$

$$m = -1, 4$$

$$\text{C.F} = Ae^{-z} + Be^{4z}$$

$$\text{C.F} = Ax^{-1} + Bx^4$$

$$\text{P.I}_1 = \frac{1}{D'^2 - 3D' - 4}$$

$$\text{P.I}_1 = \frac{1}{-4\left[1 - \left(D'^2 - 3D'\right)4\right]}\, z$$

$$= -\frac{1}{4}\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]^{-1} z$$

$$= -\frac{1}{4}\left[1 + \left(\frac{D'^2 - 3D'}{4}\right)\right] z$$

$$= -\frac{1}{4}\left[z + \left(\frac{-3}{4}\right)\right] = \frac{-z}{4} + \frac{3}{16}$$

$$= -\frac{1}{16}(4z - 3)$$

$$\text{P.I}_1 = \frac{1}{-16}(4 \log x - 3)$$

$$\text{P.I}_2 = \frac{1}{(D'^2 - 3D' - 4)} \pi e^{0z}$$

$$= \frac{\pi e^{0z}}{(0 - 0 - 4)} \quad (D' = 0)$$

$$\text{P.I}_2 = \frac{-\pi e^{0z}}{4} = \frac{-\pi}{4}$$

The solution is

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$$\boxed{\boldsymbol{y = Ax^{-1} + Bx^4 - \frac{1}{16}(4 \log x - 3) - \frac{\pi}{4}}}$$

5.4 LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

General equation of the form

$$(ax + b)^n \frac{d^n y}{dx^n} + K_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + K_n y = Q$$

where $K's$ are constants Q is a function of x is called Legendre's Linear differential equations.

Substitution to make Legender into differential equation with constant coefficients.

$$\boxed{\begin{gathered} \boldsymbol{(ax + b) = e^z} \\ \boldsymbol{z = \log (ax + b)} \\ \boldsymbol{(ax + b)^2 D^2 = a^2 D' (D' - 1)} \\ \boldsymbol{(ax + b) D = aD'} \end{gathered}}$$

Note

Legender's linear differential equations can be reduced to Euler-Cauchy type by putting $\boldsymbol{ax+b=t}$.

Example 1: Convert $\boldsymbol{(3x+2)^2 \frac{d^2 y}{dx^2} + 3\,(3x+2)\frac{dy}{dx} - 36y = 0}$ into a linear ODE.

Solution:

Given: $(3x+2)^2 D^2 y + 3\,(3x+2)\,Dy - 36y = 0$

$$[(3x+2)^2 D^2 + 3\,(3x+2)\,D - 36]\,y = 0$$

Put
$$\boxed{\begin{aligned} &\boldsymbol{3x+2 = e^z} \\ &\boldsymbol{z = \log(3x+2)} \\ &\boldsymbol{(3x+2)^2 D^2 = 3^2 D'(D'-1)} \\ &\qquad\qquad\quad \boldsymbol{= 9\,(D'^2 - D')} \\ &\quad \boldsymbol{(3x+2)\,D = 3D'} \end{aligned}}$$

$$[9D'^2 - 9D' + 3\,(3D') - 36]\,y = 0$$

$$[9D'^2 - 9D' + 9D' - 36]\,y = 0$$

$$[9D'^2 - 36]\,y = 0$$

Example 2: Transform into linear differential equation with constant co-efficients $\boldsymbol{(2x+3)^2 D^2 y - (2x+3)\,Dy - 12y = 6x}$

Solution:

$$(2x+3)^2 D^2 - (2x+3)\,D - 12]\,y = 6x$$

Put
$$\boxed{\begin{aligned} &\boldsymbol{2x+3 = e^z} \\ &\boldsymbol{z = \log(2x+3)} \\ &\boldsymbol{(2x+3)^2 D^2 = 2^2 D'(D'-1)} \\ &\qquad\qquad\quad \boldsymbol{= 4\,[D'^2 - D']} \\ &\quad \boldsymbol{(2x+3)\,D = 2D'} \end{aligned}}$$

$$2x + 3 = e^z$$

$$2x = e^z - 3$$

$$x = \frac{e^z - 3}{2}$$

$$[4\,[D'^2 - D'] - 2D' - 12]\; y = 6\left[\frac{e^z - 3}{2}\right]$$

$$[4D'^2 - 4D' - 2D' - 12]\; y = 3\,[e^z - 3]$$

$$[4D'^2 - 6D' - 12]\; y = 3e^z - 9$$

Example 3: Solve $\boldsymbol{(x+2)^2 \dfrac{d^2 y}{dx^2} - (x+2)\dfrac{dy}{dx} + y = 3x + 4}$

Solution:

$$[(x+2)^2 D^2 - (x+2)\, D + 1]\; y = 3x + 4$$

Put

$$\boxed{\begin{aligned} \boldsymbol{x + 2} &= \boldsymbol{e^z} \\ \boldsymbol{z} &= \boldsymbol{\log (x+2)} \\ \boldsymbol{(x+2)^2 D^2} &= \boldsymbol{1^2 D' (D' - 1)} \\ &= \boldsymbol{D'^2 - D'} \\ \boldsymbol{(x+2)\, D} &= \boldsymbol{1D'} \end{aligned}}$$

$$x + 2 = e^z$$

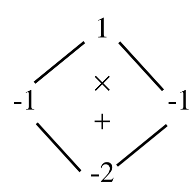
$$x = e^z - 2$$

$$[D'^2 - D' - D' + 1]\; y = 3\,(e^z - 2) + 4$$

$$[D'^2 - 2D' + 1]\; y = 3e^z - 6 + 4$$

$$[D'^2 - 2D' + 1]\; y = 3e^z - 2e^{0z}$$

$$y = \text{C.F} + \text{P.I}_1 - \text{P.I}_2$$



C.F: $[D'^2 - 2D' + 1]\; y = 0$

The Auxiliary equation is

$$D' = m$$

$$m^2 - 2m + 1 = 0$$

$$(m - 1)\,(m - 1) = 0$$

$$m = 1, 1$$

Roots are real and equal

$$\text{C.F} = [Az + B]\, e^{z}$$

$$\text{C.F} = [A \log (x+2) + B]\,(x+2)$$

$$\text{P.I}_1 = \frac{1}{D'^2 - 2D' + 1}\, 3e^{z}$$

$$= \frac{3e^{z}}{1^2 - 2\,(1) + 1}$$

Replace D' by 1

$$\text{P.I}_1 = \frac{3e^{z}}{1 - 2 + 1}$$

$$= \frac{3e^{z}}{2-2} = \frac{3e^{z}}{0}$$

$$= \frac{z3e^{z}}{2D' - 2}$$

$$= \frac{3ze^{z}}{2-2}$$

$$= \frac{3ze^{2}}{0}$$

$$\text{P.I}_1 = \frac{z^2\, 3e^{z}}{2}$$

$$= \frac{3}{2}\, z^2\, e^{z}$$

$$= \frac{3}{2}\, [(\log (x+2)]^2\, (x+2)]$$

$$\text{P.I}_2 = \frac{1}{D'^2 - 2D' + 1}\, 2e^{0z}$$

$$= \frac{2e^{0z}}{0^2 - 2\,(0) + 1}$$

$$= \frac{2e^{0z}}{1}$$

$$\text{P.I}_2 = 2$$

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

$$= [A \log (x+2) + B]\,(x+2) + \frac{3}{2}\,[(\log (x+2)]^2\,(x+2) + 2$$

Example 4: Solve
$[(x+1)^2 D^2 + (x+1) D + 1]\, y = 4 \cos [\log (x+1]$

Solution:

Given: $[(x+1)^2 D^2 + (x+1) D + 1]\, y = 4 \cos [\log (x+1)]$

Put
$$\boxed{\begin{aligned} (x+1) &= e^z \\ z &= \log (x+1) \\ (x+1)^2 D^2 &= 1^2 D' (D'-1) \\ (x+1) D &= 1D' \end{aligned}}$$

$$x + 1 = e^z$$

$$x = e^z - 1$$

$$4 \cos [\log (x+1)] = 4 \cos [z]$$

$$[D' (D'-1) + D' + 1]\, y = 4 \cos z$$

$$[D'^2 - D' + D' + 1]\, y = 4 \cos z$$

$$[D'^2 + 1]\, y = 4 \cos z \qquad \text{(Type II)}$$

$$y = \text{C.F} + \text{P.I}$$

The Auxiliary equation is

$$D' = m$$

$$[m^2 + 1] = 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$= \sqrt{-1}$$

$$m = \pm 1i$$

$$m = 0 \pm 1i$$

$$\alpha = 0, \beta = 1$$

$$\text{C.F} = e^{0z}\,[A \cos \beta z + B \sin \beta z]$$

$$\text{C.F} = e^{0z}\,[A \cos z + B \sin z]$$

$$\text{P.I} = \frac{1}{D'^2 + 1}\, 4 \cos z$$

$$= \frac{4}{D'^2 + 1} \cos z$$

$$= \frac{4 \cos z}{-1 + 1} \qquad \text{Replace } D'^2 \text{ by } (-1)$$

$$= \frac{4 \cos z}{0}$$

Replace D^2 by $-(a^2)$

$$= \frac{z 4 \cos z}{2D'}$$

$$= \frac{4z}{2} \frac{\cos z}{D'}$$

$$= \frac{4z}{2} \int \cos z dz$$

$$\text{P.I} = 2z \sin z$$

$$\text{P.I} = 2 \log [x + 1] \sin [\log (x + 1)]$$

$$\boxed{\begin{aligned} \boldsymbol{y} &= \boldsymbol{A} \cos (\log (\boldsymbol{x} + 1)) + \boldsymbol{B} \sin (\log (\boldsymbol{x} + 1))] \\ &\quad + 2 \log (\boldsymbol{x} + 1) \sin (\log (\boldsymbol{x} + 1))] \end{aligned}}$$

Example 5: Solve $(2x+7)^2 D^2 y - 6(2x+7)Dy + 8y = 8x$

Solution:

Put

$$\boxed{\begin{aligned} 2x+7 &= e^z \\ z &= \log(2x+7) \\ (2x+7)^2 D^2 &= 2^2 D'(D'-1) \\ (2x+7)D &= 2D' \end{aligned}}$$

$$(2x+7)^2 D^2 y - 6(2x+7)Dy + 8y = 8x$$

$$[4D'(D'-1) - 6(2D') + 8]\, y = 8\left[\frac{e^z - 7}{2}\right]$$

$$[4D'^2 - 4D' - 12D' + 8]\, y = 4[e^z - 7]$$

$$[4D'^2 - 16D' + 8]\, y = 4e^z - 28e^{0z}$$

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2 \quad \text{... (1)}$$

C.F:

The Auxiliary equation put $D' = m$

$$4m^2 - 16m + 8 = 0$$

$$a = 4;\ b = -16;\ c = 8$$

$$m = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(4)(8)}}{2(4)}$$

$$= \frac{16 \pm \sqrt{256 - 128}}{8}$$

$$= \frac{16 \pm \sqrt{128}}{8}$$

$$= \frac{16 \pm \sqrt{64 \times 2}}{8}$$

$$= \frac{16 \pm 8\sqrt{2}}{8}$$

$$= \frac{8\,(2 \pm \sqrt{2})}{8}$$

$$m = 2 \pm \sqrt{2}$$

$$\text{C.F} = A\,e^{(2+\sqrt{2})\,z} + Be^{(2-\sqrt{2})\,z}$$

$$\text{C.F} = A\,(e^{z})^{(2+\sqrt{2})} + B\,(e^{z})^{(2-\sqrt{2})}$$

$$\text{C.F} = A\,(2x+7)^{(2+\sqrt{2})} + B\,(2x+7)^{(2-\sqrt{2})}$$

$$\text{P.I}_1 = \frac{4e^{z}}{4D'^2 - 16D' + 8}$$

$$= \frac{4e^{z}}{4\,(1) - 16\,(1) + 8}$$

$$= \frac{4e^{z}}{12 - 16}$$

$$= -\frac{4e^{z}}{4}$$

$$\text{P.I} = -\,(2x+7)$$

$$\text{P.I}_2 = \frac{-\,28e^{0z}}{4D'^2 - 16D' + 8}$$

$$= \frac{-\,28\,(1)}{8}$$

$$\text{P.I}_2 = \frac{-\,7}{2}$$

$$\because (1) \Rightarrow \boxed{\boldsymbol{A\,(2x+7)^{(2+\sqrt{2})} + B\,(2x+7)^{(2-\sqrt{2})} - [2x+7] - \frac{7}{2}}}$$

Example 6: Solve
$$\mathbf{(3x+2)^2 D^2 y + 3\,(3x+2)\,Dy - 36y = 3x^2 + 4x + 1}$$

Solution:

$$\boxed{\begin{aligned} &e^z = 3x+2 \\ &z = \log(3x+2) \\ &(3x+2)\,D = 3D' \\ &(3x+2)^2 D^2 = 3^2 D'\,(D'-1) \end{aligned}} \qquad \Rightarrow x = \frac{e^z-2}{3}$$

$$(3x+2)^2 D^2 y + 3\,(3x+2)\,Dy - 36y = 3x^2 + 4x + 1$$

$$[9D'\,(D'-1) + 3\,(3)\,D' - 36]\,y = 3x^2 + 4x + 1$$

$$[9D'^2 - 9D' + 9D' - 36]\,y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

$$[9D'^2 - 36]\,y = 3\left(\frac{(e^z-2)^2}{9}\right) + \frac{4e^z-8}{3} + 1$$

$$= \frac{(e^z-2)^2}{3} + \frac{4e^z-8}{3} + 1$$

$$= \frac{(e^z)^2 + 4 - 4e^z + 4e^z - 8}{3} + 1$$

$$= \frac{e^{2z}-4}{3} + 1$$

$$\boxed{[9D'^2 - 36]\,y = \frac{e^{2z}}{3} - \frac{1}{3}}$$

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2 \qquad \text{... (1)}$$

$$3x + 2 = e^z$$

$$3x = e^z - 2$$

$$x = \frac{e^z-2}{3}$$

C.F: The Auxiliary equation put $D' = m$

$$9m^2 - 36 = 0$$

$$9m^2 = 36$$

$$m^2 = \frac{36}{9}$$

$$m^2 = 4$$

$$m = \pm 2$$

Roots are real and unequal

$$\text{C.F} = Ae^{2z} + Be^{-2z}$$

$$= A\,(e^z)^2 + B\,(e^z)^{-2}$$

$$= A\,(3x+2)^2 + B\,(3x+2)^{-2}$$

$$\text{P.I}_1 = \frac{\frac{e^{2z}}{3}}{9D'^2 - 36}$$

Replace D'^2 by 4

$$= \frac{\frac{e^{2z}}{3}}{0}$$

$$= \frac{\frac{ze^{2z}}{3}}{18D'}$$

$$= \frac{\frac{ze^{2z}}{3}}{18\,(2)}$$

$$= \frac{ze^{2z}}{108}$$

$$\text{P.I}_1 = \frac{\log\,(3x+2)\,(3x+2)^2}{108}$$

$$\text{P.I}_2 = \frac{-\frac{e^{0z}}{3}}{9D'^2 - 36}$$

$$= \frac{\frac{-1}{3}}{0 - 36}$$

$$= \frac{1}{36 \times 3}$$

$$\text{P.I}_2 = \frac{1}{108}$$

$$\boxed{\mathbf{(1) \Rightarrow} \ \boldsymbol{A\ (3x+2)^2 + B\ (3x+2)^{-2}} + \frac{\mathbf{log\ (3x+2)\ (3x+2)^2}}{\mathbf{108}} + \frac{\mathbf{1}}{\mathbf{108}}}$$

5.5 SYSTEM OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT CO-EFFICIENTS

5.5.5 Simultaneous linear equations

Linear differential equations in which there are two or more dependent variables and a single independent variable. Such equations are known as simultaneous linear equations.

Here, we shall deal with systems of linear equations with constant coefficients only. Such a system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the dependent variables is obtained in a similar manner.

Consider the simultaneous equation in two dependent variables x and y and one independent variable t.

$$f_1\ (D)\ x + g_1\ (D)\ y = h_1\ (t) \qquad \text{... (1)}$$

$$f_2\ (D)\ x + g_2\ (D)\ y = h_2\ (t) \qquad \text{... (2)}$$

where f_1, f_2, g_1 and g_2 are polynomials in the operator D, $\left[D \equiv \frac{d}{dt} \right]$

The number of independent arbitrary constants appearing in the general solution of the system of differential equation (1) & (2) is equal to the degree of D in the coefficient determinant

$$\Delta = \begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix} \text{ provided } \Delta \neq 0$$

Note: If $\Delta = 0$, then the system is dependent.

5.5.2 Problems of simultaneous linear equations

Example 1: Solve $\frac{dx}{dt} - y = 0 = 0, \frac{dy}{dt} + x = 0$

Solution:

Given: $Dx - y = 0$

$$Dy + x = 0$$

$$Dx - y = 0 \quad \text{... (1)}$$

$$x + Dy = 0 \quad \text{... (2)}$$

$$\Delta = \begin{vmatrix} x & y \\ D & -1 \\ 1 & D \end{vmatrix}$$

$$\Delta = D^2 - (-1)$$

$$\Delta = D^2 + 1, \ \Delta \neq 0$$

$$\Delta_y = \begin{vmatrix} x & R \\ D & 0 \\ 1 & 0 \end{vmatrix}$$

$$\Delta_y = 0 - 0$$

$$\Delta_y = 0$$

$$(D^2 + 1)\, y = 0$$

$$y = \text{C.F} + \text{P.I}$$

C.F: $(D^2 + 1)\, y = 0$

Put $D = m$

The Auxiliary equation is

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$\sqrt{m^2} = \sqrt{-1}$$

$$m = \pm 1i$$

$$m = 0 \pm 1i$$

$$\alpha = 0, \beta = 1$$

$$\text{C.F} = e^{\alpha t}\,[A \cos \beta t + B \sin \beta t]$$

$$= e^{0t}\,[A \cos 1t + B \sin 1t]$$

$$\text{C.F} = A \cos t + B \sin t$$

$$\text{P.I} = 0$$

$$\boxed{\boldsymbol{y = A \cos t + B \sin t}}$$

$$Dy = A\,(-\sin t) + B \cos t$$

$(2) \Rightarrow \quad x + Dy = 0$

$$x = -Dy$$

$$x = -[-A \sin t + B \cos t]$$

$$\boxed{\boldsymbol{x = A \sin t - B \cos t}}$$

Example 2: **Solve** $\boldsymbol{\frac{dx}{dt} + y = e^t, x - \frac{dy}{dt} = t}$

Solution:

Given: $Dx + y = e^t$ ... (1)

$x - Dy = t$ ... (2)

$$\Delta = \begin{vmatrix} x & y \\ D & 1 \\ 1 & -D \end{vmatrix}$$

$$= -D^2 - 1$$

$$\Delta = -(D^2 + 1), \Delta \neq 0$$

$$\Delta_y = \begin{vmatrix} x & R \\ D & e^t \\ 1 & t \end{vmatrix}$$

$$= D\,(t) - e^t$$

$$\Delta y = 1 - e^t$$

$$-(D^2 + 1)\,y = 1 - e^t$$

$\times$ by (-1) above

$$(D^2 + 1) = e^t - 1e^{0t}$$

$$y = \text{C.F} + \text{P.I}_1 + \text{P.I}_2$$

C.F: $(D^2 + 1)\,y = 0$

Put $D = m$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = 0 \pm i$$

$$\alpha = 0, \beta = 1$$

$$\text{C.F} = e^{0t}\,[A \cos t + B \sin t]$$

C.F $= A\cos t + B\sin t$

$$\text{P.I}_1 = \frac{1}{D^2+1} e^t \quad \text{Replace } D \text{ by } 1$$

$$= \frac{e^t}{1^2+1}$$

$$\text{P.I}_1 = \frac{e^t}{2}$$

$$\text{P.I}_2 = \frac{1}{D^2+1} e^{0t}$$

$$= \frac{e^{0t}}{0+1} \quad \text{Replace } D \text{ by } 0$$

$$(*) \Rightarrow \text{P.I}_2 = 1$$

$$y = A\cos t + B\sin t + \frac{e^t}{2} - 1$$

$$\boxed{\boldsymbol{y = A\cos t + B\sin t + \frac{1}{2}e^t - 1}}$$

$$Dy = A(-\sin t) + B\cos t + \frac{1}{2}e^t - 0$$

$$(2) \Rightarrow \quad x - Dy = t$$

$$x = t + Dy$$

$$x = t + (-A\sin t) + B\cos t + \frac{e^t}{2}$$

$$\boxed{\boldsymbol{x = -A\sin t + B\cos t + \frac{e^t}{2} + t}}$$

Example 3: Solve $\boldsymbol{\frac{dx}{dt} - y = t\,;\ \frac{dy}{dt} + x = t^2}$.

Solution:

Given: $Dx - y = t$ ... (1)

$x + Dy = t^2$ ... (2)

$$\Delta = \begin{vmatrix} x & y \\ D & -1 \\ 1 & D \end{vmatrix}$$

$$\Delta = D^2 + 1$$

$$\Delta y = \begin{vmatrix} x & R \\ D & t \\ 1 & t^2 \end{vmatrix}$$

$$\Delta y = D\,(t^2) - t$$

$$= 2t - t$$

$$\Delta y = t$$

$$(D^2 + 1)\,y = t$$

$$y = \text{C.F} + \text{P.I}$$

C.F: $(D^2 + 1)\,y = 0$

$$\text{C.F} = A\cos t + B\sin t$$

$$\text{P.I} = \frac{1}{D^2 + 1}\,t$$

$$= \frac{1}{1 + D^2}\,t$$

$$= (1 + D^2)^{-1}\,t$$

$$\boxed{\mathbf{(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots}}$$

$$= [1 - D^2 + (D^2)^2]\,t$$

$$= t - D^2\,(t)$$

$$= t - 0$$

$$\text{P.I} = t$$

$$\boxed{\mathbf{y = A\cos t + B\sin t + t}}$$

$$Dy = A\,(-\sin t) + B\cos t + 1$$

$$Dy = -A\sin t + B\cos t + 1$$

(2) $\Rightarrow$

$$x = t^2 - Dy$$

$$x = t^2 - [-A \sin t + B \cos t + 1]$$

$$\boxed{\mathbf{x = t^2 + A \sin t - B \cos t - 1}}$$

$$x = A \sin t - B \cos t + t^2 - 1$$

Hence the desired solutions are

$$x = A \sin t - B \cos t + t^2 - 1 \qquad \text{... (I)}$$

$$y = A \cos t + B \sin t + t \qquad \text{... (II)}$$

Given:

$$\boxed{\mathbf{x\ (0) = 2}}$$

$$x\ (t) = A \sin t - B \cos t + t^2 - 1$$

Put $t = 0$

$$x\ (0) = A \sin 0 - B \cos 0 + 0^2 - 1$$

$$2 = A\ (0) - B\ (1) + 0 - 1$$

$$2 = 0 - B - 1$$

$$-B - 1 = 2$$

$$-B = 2 + 1$$

$$-B = 3$$

$$\boxed{\mathbf{B = -3}}$$

$$y\ (t) = A \cos t + B \sin t + t$$

Put $t = 0$

$$y\ (0) = A \cos 0 + B \sin 0 + 0$$

$$2 = A\ (1) + B\ (0) + 0$$

$$\boxed{A = 2}$$

$\text{I} \Rightarrow \quad x(t) = 2 \sin t + 3 \cos t + t^2 - 1$

$\text{I} \Rightarrow \quad y(t) = 2 \cos t - 3 \sin t + t$

Example 4: Solve $\frac{dx}{dt} + 2y = -\sin t, \frac{dy}{dt} - 2x = \cos t$ given $x = 1$ and $y = 0$ at $t = 0$.

Solution:

Given: $Dx + 2y = -\sin t$ ... (1)

$-2x + Dy = \cos t$ ... (2)

$$\Delta = \begin{vmatrix} x & y \\ D & 2 \\ -2 & D \end{vmatrix}$$

$$= D^2 - (-4)$$

$$\Delta = D^2 + 4$$

$$\Delta \neq 0$$

$$\Delta = D^2 + 4$$

$$\Delta y = \begin{vmatrix} x & R \\ D & -\sin t \\ -2 & \cos t \end{vmatrix}$$

$$= D(\cos t) - (2 \sin t)$$

$$\Delta y = -1 \sin t - 2 \sin t$$

$$\Delta y = -3 \sin t$$

$$(D^2 + 4)\, y = -3 \sin t$$

$$y = \text{C.F} + \text{P.I}$$

C.F: $(D^2 + 4)\, y = 0$

Put $D = m$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = 0 \pm 2i$$

$$\alpha = 0, \beta = 2$$

$$\text{C.F} = e^{\alpha}t\,[A \cos \beta t + B \sin \beta t]$$

$$= e^{0t}\,[A \cos 2t + B \sin 2t]$$

$$\text{C.F} = A \cos 2t + B \sin 2t$$

$$\text{P.I} = \frac{-3}{D^2 + 4} \sin t$$

$$= \frac{-3}{-1 + 4} \sin t$$

$$= \frac{-3}{3} \sin t$$

$$\text{P.I} = -\sin t \qquad a = 1,\ -a^2 = -1$$

$$\boxed{\boldsymbol{y = A \cos 2t + B \sin 2t - \sin t}}$$

$$Dy = A(-2 \sin 2t) + B(2 \cos 2t) - \cos t$$

$$Dy = -2A \sin 2t + 2B \cos 2t - \cos t$$

$$(2) \Rightarrow \qquad -2x + Dy = \cos t$$

$$-2x = \cos t - Dy$$

$\times$ by (-1) above

$$2x = Dy - \cos t$$

$$2x = (-2A \sin 2t + 2B \cos 2t - \cos t - \cos t)$$

$$2x = -2A \sin 2t + 2B \cos 2t - 2 \cos t$$

$$x = \frac{2\,[-A \sin 2t + B \cos 2t - \cos t]}{2}$$

$$\boxed{\mathbf{x\ (t) = -A \sin 2t + B \cos 2t - \cos t}}$$

Conditions, $x = 1$, $y = 0$ at $t = 0$

The desired solutions are,

$x\ (t) = -A \sin 2t + B \cos 2t - \cos t$ ... (I)

$y\ (t) = A \cos 2t + B \sin 2t - \sin t$ ... (II)

(I) $\Rightarrow$ Put$t = 0$, in I

$x\ (0) = -A \sin 2\ (0) + B \cos 2\ (0) - \cos 0$

$1 = -A\ (0) + B\ (1) - 1$

$1 = 0 + B - 1$

$B - 1 = 1$

$B = 1 + 1 \quad \Rightarrow B = 2$

(II) $\Rightarrow$ Put $t = 0$, in II

$y\ (0) = A \cos 2\ (0) + B \sin 2\ (0) - \sin 0$

$0 = A\ (1) + B\ (0) - 0$

$$\boxed{\mathbf{0 = A}}$$

I $\Rightarrow$ $x\ (t) = 2 \cos 2t - \cos t$

II $\Rightarrow$ $y\ (t) = 2 \sin 2t - \sin t$

Try these

1. $\dfrac{dx}{dt} + 2y = \sin 2t, \dfrac{dy}{dt} - 2x = \cos 2t$

2. $\dfrac{dx}{dt} + 2y = 5e^t, \dfrac{dy}{dt} - 2x = 5e^t$

Example 5: Solve the simultaneous equations
$$\frac{dx}{dt} + 2x - 3 = 5t, \frac{dy}{dt} - 3x + 2y = 0$$

Solution:

Given: $Dx + 2x - 3y = 5t$

$$(D+2)\,x - 3y = 5t \quad \ldots (1)$$

$$Dy - 3x + 2y = 0$$

$$-3x + Dy + 2y = 0$$

$$-3x + (D+2)\,y = 0 \quad \ldots (2)$$

$$\Delta = \begin{vmatrix} x & y \\ D+2 & -3 \\ -3 & D+2 \end{vmatrix}$$

$$= (D+2)^2 - (9)$$

$$= D^2 + 4 + 4D - 9$$

$$\Delta = D^2 + 4D - 5$$

$$\Delta \neq 0$$

$$\Delta\, y = \begin{vmatrix} x & R \\ D+2 & 5t \\ -3 & 0 \end{vmatrix}$$

$$= (D+2)\,(0) - 15t$$

$$\Delta\, y = 0 - 15t$$

$$\boxed{[D^2 + 4D - 5]\, y = -15t}$$

Example 6: $\dfrac{dx}{dt} - 7x + y = 0, \dfrac{dy}{dt} - 2x - 5y = 0$

Solution:

Given: $Dx - 7x + y = 0$

$$(D-7)\,x+y=0 \quad \text{... (1)}$$

$$Dy-2x-5y=0$$

$$-2x+(D-5)\,y=0 \quad \text{... (2)}$$

$$\Delta = \begin{vmatrix} x & y \\ (D-7) & 1 \\ -2 & (D-5) \end{vmatrix}$$

$$=(D-7)\,(D-5)+2$$

$$=D^2-5D+35+2$$

$$\Delta = D^2-12D+37$$

$$\Delta \neq 0$$

$$\Delta_y = \begin{vmatrix} x & R \\ D-7 & 0 \\ -2 & 0 \end{vmatrix}$$

$$=(D-7)\,0-(-2\,(0))$$

$$\Delta\, y = 0$$

$$\Delta\, y = 0$$

$$[D^2-12D+37]\,y=0$$

Example 7: $\frac{dx}{dt}+\frac{dy}{dt}+3x=\sin t\ ;\ \frac{dx}{dt}+y-x=\cos t$

Solution:

Given: $Dx+Dy+3x=\sin t$

$$(D+3)\,x+Dy=\sin t \quad \text{... (1)}$$

$$Dx+y-x=\cos t$$

$$(D-1)\,x+y=\cos t \quad \text{... (2)}$$

$$\Delta = \begin{vmatrix} x & y \\ (D+3) & D \\ (D-1) & 1 \end{vmatrix}$$

$$= (D+3) - D\,(D-1)$$

$$= D + 3 - D^2 + D$$

$$= -D^2 + 2D + 3$$

$$\Delta = -D^2 + 2D + 3,\ \Delta \neq 0$$

$$\Delta\, y = \begin{vmatrix} x & R \\ D+3 & \sin t \\ D-1 & \cos t \end{vmatrix}$$

$$= [(D+3)\,(\cos t)] - [(D-1)\,(\sin t)]$$

$$= -\sin t + 3\cos t - [\cos t - \sin t]$$

$$= -\sin t + 3\cos t - \cos t + \sin t$$

$$\Delta\, y = 2\cos t$$

$$(-D^2 + 2D + 3)\, y = 2\cos t$$

$\times$ by (-1)

$$(D^2 - 2D - 3)\, y = -2\cos t$$

5.6 METHOD OF VARIATION OF PARAMETERS

This method is very useful in finding general solution of the second order linear differential equations

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2\, y = R$$

where R is a function of x

The general solution is given by

$$y = \text{C.F} + \text{P.I}$$

$$\text{C.F} = Af_1 + Bf_2$$

$$\text{P.I} = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2\, R}{w}\, dx$$

$$Q = \int \frac{f_1\, R}{w}\, dx$$

Here w is called the **wronskian** and $\boldsymbol{w}$ is defined by

$$w = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix}$$

$$\boxed{\boldsymbol{w = f_1 f'_2 - f_2 f'_1}}$$

5.6.1 Problems on method of variation of parameter

Example 1: Find the wronskian for the problem $(D^2 + 1)\, y = x$.

Solution:

Given: $(D^2 + 1)\, y = x$

The auxiliary equation is put $D = m$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm 1i$$

$$\alpha = 0, \beta = 1$$

$$\text{C.F} = e^{\alpha x}\, [A \cos \beta x + B \sin \beta x]$$

$$= e^{0x}\, [A \cos 1x + B \sin 1x]$$

$$\boxed{\begin{array}{c}\mathbf{C.F} = \boldsymbol{A}\ \mathbf{cos}\ \boldsymbol{x} + \boldsymbol{B}\ \mathbf{sin}\ \boldsymbol{x} \\ \mathbf{C.F} = \boldsymbol{A f_1} + \boldsymbol{B f_2}\end{array}}$$

$$f_1 = \cos x \qquad f_2 = \sin x$$

$$f'_1 = -\sin x \qquad f'_2 = \cos x$$

Wronskian,

$$w = \begin{vmatrix} f_1 & f'_1 \\ f_2 & f'_2 \end{vmatrix}$$

$$w = \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{\boldsymbol{w = 1}}$$

$$\boxed{\mathbf{WKT}\ \ \mathbf{cos^2}\,\boldsymbol{\theta} + \mathbf{sin^2}\,\boldsymbol{\theta} = \mathbf{1}}$$

Example 2: Find the wronskian of $(D^2 - 4D + 4)\, y = e^{2x}$

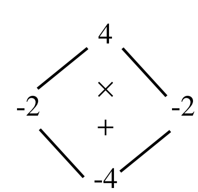
Solution:

Given: $(D^2 - 4D + 4)\, y = e^{2x}$

C.F: The auxiliary equation is

Put $D = m$

$$m^2 - 4m + 4 = 0$$



$$(m-2)\,(m-2) = 0$$

$$m_1 = 2,\ m_2 = 2$$

Roots are real and equal

$$\therefore \text{C.F} = (Ax + B)\, e^{mx}$$

$$= (Ax + B)\, e^{2x}$$

$$\boxed{\begin{array}{c}\mathbf{C.F} = \boldsymbol{Axe^{2x} + Be^{2x}} \\ \mathbf{C.F} = \boldsymbol{Af_1 + Bf_2}\end{array}}$$

$$f_1 = xe^{2x} \quad f_2 = e^{2x}$$

$$f_1 = x\, e^{2x}$$

$$f'_1 = e^{2x}(1) + x\,(2e^{2x})$$

$$f'_1 = e^{2x} + 2xe^{2x}$$

$$f_2 = e^{2x} \quad f'_2 = 2e^{2x}$$

Wronskian

$$w = \begin{vmatrix} f_1 & f'_1 \\ f_2 & f'_2 \end{vmatrix}$$

$$= \begin{vmatrix} xe^{2x} & e^{2x} + 2xe^{2x} \\ e^{2x} & 2e^{2x} \end{vmatrix}$$

$$= 2x\, e^{2x}\, e^{2x} - e^{2x}\,[e^{2x} + 2xe^{2x}]$$

$$= 2x\,(e^{2x})^2 - (e^{2x})^2 - 2x\,(e^{2x})^2$$

$$= -\,(e^{2x})^2$$

$$\boxed{\boldsymbol{w = -\,e^{4x}}}$$

Example 3: Find the wronskian of $y'' + 3y' + 2y = x^2$.

Solution:

Given: $(D^2 + 3D + 2)\, y = x^2$

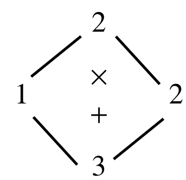
C.F: The auxiliary equation is

$$D = m$$

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m_1 = -1, m_2 = -2$$



Roots are real and distinct

$$\text{C.F} = Ae^{m_1 x} + Be^{m_2 x}$$

$$\boxed{\begin{array}{c} \textbf{C.F} = \boldsymbol{Ae^{-1x} + Be^{-2x}} \\ \textbf{C.F} = \boldsymbol{Af_1 + Bf_2} \end{array}}$$

$$f_1 = e^{-x} \qquad f_2 = e^{-2x}$$

$$f'_1 = -e^{-x} \qquad f'_2 = -2e^{-2x}$$

Wronskian

$$w = \begin{vmatrix} f_1 & f'_1 \\ f_2 & f'_2 \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & -e^{-x} \\ e^{-2x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-1x}e^{-2x} - [(-e^{-1x}(e^{-2x})]$$

$$= -2e^{-3x} + 1e^{-3x}$$

$$= -1e^{-3x}$$

$$\boldsymbol{w = -e^{-3x}}$$

Practice problem

I. Find the wronksian of the following

1. $(D^2 + 4)\, y = \text{cosec } 4x$
2. $(D^2 + 6^2)\, y = \cot 6x$
3. $(D^2 + 3^2)\, y = \sec 3x$

Formulae

$$\int x dx = \frac{x^2}{2} + c$$

$$\int x^2 \, dx = \frac{x^3}{3} + c$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos ax \, dx = \frac{\sin ax}{a} + c$$

$$\int \sin bx \, dx = \frac{-\cos bx}{b} + c$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\int \tan x \, dx = \log(\sec x)$$

$$\int \tan ax \, dx = \frac{\log(\sec ax)}{a}$$

$$\int \sec x \, dx = \log[\sec x + \tan x]$$

$$\int \sec ax \, dx = \frac{\log[\sec ax + \tan ax]}{a}$$

$$\int \operatorname{cosec} ax \, dx = \log[\operatorname{cosec} x - \cot x]$$

$$\int \cot x \, dx = \log(\sin x)$$

$$\int \cot ax \, dx = \frac{\log(\sin ax)}{a}$$

$$\sec ax = \frac{1}{\cos ax} \qquad \tan ax = \frac{\sin ax}{\cos ax}$$

$$\operatorname{cosec} ax = \frac{1}{\sin ax} \qquad \cot ax = \frac{\cos ax}{\sin ax}$$

Example 4: Solve $(D^2 + a^2)\, y = \operatorname{cosec} ax$ by the method of variation of parameters.

Solution:

Given: $(D^2 + a^2)\, y = \operatorname{cosec} ax$

$$y = \text{C.F} + \text{P.I} \qquad \dots (1)$$

C.F: The auxiliary equation is

Put $D = m$

$$m^2 + a^2 = 0$$

$$m^2 = -a^2$$

$$\sqrt{m^2} = \pm\sqrt{-a^2}$$

$$m = \pm ia$$

$$\boxed{\begin{array}{c} \boldsymbol{m = 0 \pm ia} \\ \boldsymbol{m = \alpha \pm i\beta} \end{array}}$$

$$\alpha = 0,\ \beta = a$$

$$\text{C.F} = e^{\alpha x}\,[A\cos\beta x + B\sin\beta x]$$

$$= e^{0x}\,[A\cos ax + B\sin ax]$$

$$\text{C.F} = A\cos ax + B\sin ax \qquad \dots (2)$$

$$\text{C.F} = Af_1 + Bf_2$$

$$f_1 = \cos ax \qquad f_2 = \sin ax$$

$$f'_1 = -a\sin ax \qquad f'_2 = a\cos ax$$

Wronskian

$$w = \begin{vmatrix} f_1 & f'_1 \\ f_2 & f'_2 \end{vmatrix}$$

$$= \begin{vmatrix} \cos ax & -a\sin ax \\ \sin ax & a\cos ax \end{vmatrix}$$

$$= a\cos^2(ax) + a\sin^2(ax)$$

$$= a[\cos^2(ax) + \sin^2(ax)]$$

$$= a[1]$$

$$\boxed{\boldsymbol{w = a}}$$

P.I $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 R}{w}$$

$w = a, f_2 = \sin ax \;\; R = \operatorname{cosec} ax$

$$= -\int \frac{\sin ax \times \operatorname{cosec} ax}{a}\, dx$$

$$= -\frac{1}{a}\int \sin ax \frac{1}{\sin ax}\, dx \quad \left[\because \operatorname{cosec} ax = \frac{1}{\sin ax}\right]$$

$$= \frac{-1}{a}\int dx$$

$$\boxed{\boldsymbol{P = \frac{-1}{a}x + c}}$$

$$Q = \int \frac{f_1 R}{w}$$

$w = a,\; f_1 = \cos ax,\; R = \operatorname{cosec} ax$

$$= \int \frac{\cos ax \times \operatorname{cosec} ax}{a}\, dx$$

$$= \frac{1}{a} \int \cos ax \, \frac{1}{\sin ax} \, dx$$

$$= \frac{1}{a} \int \frac{\cos ax}{\sin ax} \, dx \qquad \left(\frac{\cos ax}{\sin ax} = \cot ax \right)$$

$$= \frac{1}{a} \int \cot ax \, dx$$

$$Q = \frac{1}{a} \, \frac{\log (\sin ax)}{a}$$

$$\text{P.I} = \frac{-x}{a} \cos ax + \frac{\log (\sin ax)}{a^2} \sin ax \qquad \dots (4)$$

$$y = \text{C.F} + \text{P.I}$$

$$\boxed{\boldsymbol{y = A \cos ax + B \sin ax - \frac{x}{a} \cos ax + \frac{\log (\sin ax) \sin ax}{a^2}}}$$

Try solve: $(D^2 + 2^2) \, y = \text{cosec } 2x$

Hint: In the above Replace 'a' by '2'

Example 5: Solve $(D^2 + a^2) \, y = \tan ax$ by the method of variation of parameters.

Solution:

Given: $(D^2 + a^2) \, y = \tan ax$

$$y = \text{C.F} + \text{P.I} \qquad \dots (1)$$

C.F: The auxiliary equation is

Put $D = m$

$$m^2 + a^2 = 0$$

$$m^2 = - a^2$$

$$\sqrt{m^2} = \pm \sqrt{- a^2}$$

$$m = \pm \, ia$$

$$\boxed{\begin{aligned} \boldsymbol{m} &= \mathbf{0} \pm \boldsymbol{ia} \\ \boldsymbol{m} &= \alpha \pm \boldsymbol{i}\,\beta \end{aligned}}$$

$$\alpha = 0,\ \beta = a$$

$$\text{C.F} = e^{\alpha x}\,[A \cos \beta x + B \sin \beta x]$$

$$= e^{0x}\,[A \cos ax + B \sin ax]$$

$$\text{C.F} = A \cos ax + B \sin ax \qquad \dots (2)$$

$$\text{C.F} = Af_1 + Bf_2$$

$$f_1 = \cos ax \qquad f_2 = \sin ax$$

$$f'_1 = -\,a \sin ax \qquad f'_2 = a \cos ax$$

Wronskian

$$w = \begin{vmatrix} f_1 & f'_1 \\ f_2 & f'_2 \end{vmatrix}$$

$$= \begin{vmatrix} \cos ax & -\,a \sin ax \\ \sin ax & a \cos ax \end{vmatrix}$$

$$= a \cos^2 (ax) + a \sin^2 (ax)$$

$$= a\,[\cos^2 (ax) + \sin^2 (ax)]$$

$$= a\,[1]$$

$$\boxed{\boldsymbol{w = a}}$$

$$\text{P.I} = Pf_1 + Qf_2 \qquad \dots (3)$$

$$P = -\int \frac{f_2\,R}{w}$$

$w = a,\ f_2 = -\sin ax \quad R = \tan ax$

$$= -\int \frac{\sin ax}{a} \times \tan ax\, dx$$

$$= \frac{-1}{a} \int \sin ax \cdot \frac{\sin ax}{\cos ax}\, dx \quad \left[\because \tan ax = \frac{\sin ax}{\cos ax} \right]$$

$$= \frac{-1}{a} \int \frac{\sin^2 ax}{\cos ax}\, dx$$

$\because \sin^2 \theta = 1 - \cos^2 \theta$

$$P = \frac{-1}{a} \int \frac{1 - \cos^2 ax}{\cos ax}\, dx$$

$$= \frac{-1}{a} \left[\int \frac{1}{\cos ax}\, dx - \int \frac{\cos^2 ax}{\cos ax}\, dx \right]$$

$$= \frac{-1}{a} \left[\int \sec ax\, dx - \int \cos ax\, dx \right] \quad \because \frac{1}{\cos ax} = \sec ax$$

$$\int \sec ax\, dx = \frac{\log(\sec ax + \tan ax)}{a}$$

$$P = \frac{-1}{a^2} [\log(\sec ax + \tan ax) - \sin ax]$$

$$Q = \int \frac{f_1 R}{w}\, dx$$

$f_1 = \cos ax \quad R = \tan ax,\ w = a$

$$Q = \int \frac{\cos ax \times \tan ax}{a}\, dx$$

$$= \frac{1}{a} \int \cos ax \frac{\sin ax}{\cos ax}\, dx \left[\because \tan ax = \frac{\sin ax}{\cos ax} \right]$$

$$= \frac{1}{a} \int \sin ax \, dx$$

$$= \frac{1}{a} \left[\frac{-\cos ax}{a} \right] + c$$

$$Q = \frac{-1}{a^2} \cos ax$$

$$\text{P.I} = \frac{-1}{a^2} [\log (\sec ax + \tan ax) - \sin ax] \cos ax$$

$$\frac{-1}{a^2} [\cos ax] \sin ax \qquad \dots (4)$$

$$y = \text{C.F} + \text{P.I}$$

$$(2) + (4)$$

Try: Solve $(D^2 + 3^2)\, y = \tan 3x$ in the above Replace 'a' by '3'

Example 6: Solve $(D^2 + a^2)\, y = \sec ax$ using the method of variation of parameters.

Solution:

Given: $(D^2 + a^2)\, y = \sec ax$

The auxiliary equation is $m^2 + a^2 = 0$

$\Rightarrow$ $m^2 = -a^2$

$\Rightarrow$ $m = \pm ai$

$$\text{C.F} = C_1 \cos ax + C_2 \sin ax$$

$f_1 = \cos ax$ $\quad$ $f_2 = \sin ax$ $\quad$ $X = \sec ax$

$f_1' = -a \sin ax$ $\quad$ $f_2' = a \cos ax$

$$f_1 f_2' - f_1' f_2 = a \cos^2 ax + a \sin^2 ax = a$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin ax \sec ax}{a} dx = -\frac{1}{a} \int \tan ax \, dx$$

$$= -\frac{1}{a}\left[\frac{-\log(\cos ax)}{a} \right] = \frac{1}{a^2} \log[\cos ax]$$

$$\left[\because \int \tan ax \, dx = \log \sec ax = -\log \cos ax \right]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos ax \sec ax}{a} dx$$

$$= \frac{1}{a} \int dx = \frac{1}{a} x$$

$$\mathrm{P \cdot I} = Pf_1 + Qf_2$$

$$= \frac{1}{a^2} \log[\cos ax] \cos ax + \frac{x}{a} \sin ax$$

$$y = \mathrm{C.F} + \mathrm{P.I}$$

$$= C_1 \cos ax + C_2 \sin ax + \frac{1}{a^2} \log[\cos ax] \cos ax + \frac{x}{a} \sin ax$$

Example 7: Use the method of variation of parameters to solve $(D^2 + 1)\, y = \sec x$.

Solution:

Hint: In the above Example put $a = 1$, we get

$$y = C_1 \cos x + C_2 \sin x + \cos x \log(\cos x) + x \sin x$$

Example 8: Solve $(D^2+4)\,y = \sec 2x$ by the method of variation of parameters.

Solution:

Hint: In the above Example put $a = 2$, we get

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4} \log [\cos 2x] \cos 2x + \frac{1}{2} x \sin 2x$$

Example 9: Apply the method of variation of parameters to solve $(D^2+4)\,y = \cot 2x$

Solution:

Given: $(D^2+4)\,y = \cot 2x$

The auxiliary equation is $m^2 + 4 = 0$; $m = \pm 2i$

$$\text{C.F} = e^{0x} [C_1 \cos 2x + C_2 \sin 2x]$$

$$\text{C.F} = C_1 \cos 2x + C_2 \sin 2x$$

Here, $f_1 = \cos 2x,\quad f_2 = \sin 2x \quad X = \cot 2x$

$f_1' = -2 \sin 2x,\quad f_2' = 2 \cos 2x$

$$f_1 f_2' - f_2 f_1' = 2 \cos 2x \cos 2x - \sin 2x\,(-2 \sin 2x)$$

$$= 2 [\cos^2 2x + \sin^2 2x] = 2$$

$$\text{P} \cdot \text{I} = Pf_1 + Qf_2$$

$$P = -\infty \frac{f_2 X}{f_1 f_2' - f_1' f_2}\, dx = -\infty \frac{\sin 2x \cot 2x}{2}\, dx$$

$$= -\frac{1}{2} \int \sin 2x \left(\frac{\cos 2x}{\sin 2x} \right) dx = -\frac{1}{2} \int \cos 2x\, dx$$

$$= -\frac{1}{2} \left[\frac{\sin 2x}{2} \right]$$

$$= -\frac{1}{4} \sin 2x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\, dx = \int \frac{\cos 2x \cot 2x}{2}\, dx$$

$$= \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\, dx = \frac{1}{2}\int \frac{1 - \sin^2 2x}{\sin 2x}\, dx$$

$$= \frac{1}{2}\int (\operatorname{cosec} 2x - \sin 2x)\, dx$$

$$= \frac{1}{2}\left[\frac{\log(\operatorname{cosec} 2x - \cot 2x)}{2} + \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x]$$

$$\text{P}\cdot\text{I} = Pf_1 + Qf_2$$

$$= -\frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x]\sin 2x$$

$$= \frac{1}{4}[\log(\operatorname{cosec} 2x - \cot 2x)]\sin 2x$$

$$y = \text{C}\cdot\text{F} + \text{P}\cdot\text{I}$$

$$= C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4}[\log(\operatorname{cosec} 2x - \cot 2x)]\sin 2x$$

Example 10: Solve $(D^2 + a^2)\, y = \cot ax$ by the method of variation of parameters.

Solution:

Hint: $f_1 f_2' - f_1' f_2 = a$

$$P = -\frac{1}{a^2}\sin ax,\ Q = \frac{1}{a^2}[\log[\operatorname{cosec} ax - \cot ax] + \cos ax]$$

$$y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a^2}\log[\operatorname{cosec} ax - \cot ax]\sin ax$$

5.7 METHOD OF UNDETERMINED COEFFICIENTS

In the given equation $f(D) = X$, to find the P.I, we assume a trial solution containing unknown constants which are determined by substitution in the equation.

The trial solution in each case depends on the form of X.

S.No.	Function X	Choice of P.I
1.	Ke^{px}	Ce^{px}
2.	$K(ax+b)$ (or) $K\cos(ax+b)$	$C_1 \sin(ax+b) + C_2 \cos(ax+b)$
3.	$Ke^{px}\sin(ax+b)$ (or) $Ke^{px}\cos(ax+b)$	$C_1 e^{px}\sin(ax+b) + C_2 e^{px}\cos(ax+b)$
4.	Kx^m where $m = 0, 1, 2, \ldots$	$C_0 + C_1 x + C_2 x^2 + \ldots + C_m x^m$

Case (a): Straight case

If the R.H.S function X is not a member of the solution set, then choose P.I (y_p) from the above table depending on the nature of X.

Case (b): Sum case

When the R.H.S. X is a combination (sum) of the functions in column 2 of the table, then P.I. is chosen as a combination of the corresponding functions in third column and proceed as in straight case (a).

Note: Here also as in case (a), the terms of R.H.S. X are not members of the solution set S.

Case (c): Modified case

When any term of X is a member of the solution set S, then the method fails if we choose y_p from the table. In such cases, the choice from the table should be modified as follows:

1. If a term u of X is also a term of the complementary function (i.e., $u \in S =$ solution set) then the choice from the table corresponding to u should be multiplied by
   (a) x if u corresponds to a simple root of C.F.
   (b) x^2 if u corresponds to a double root of C.F.

Case (a): Straight case

Example 1: Solve $(D^2 - 3D + 2y)\,y = 6e^{3x}$

Solution:

Given: $y'' - 3y' + 2y = 6e^{3x}$ ... (1)

The auxiliary equation is $m^2 - 3m + 2 = 0$

$$\Rightarrow \quad m = 1, m = 2$$

(C.F) $$y_c = Ae^x + Be^{2x}$$ ... (2)

Here, the solution set $S = \{\, e^x, e^{2x} \,\}$

R.H.S of (1) is not a member of S.

So, choose (P.I) $y_p = Ce^{3x}$ ... (3)

$$y'_p = 3Ce^{3x}$$

$$y''_b = 9Ce^{3x}$$

$(1) \Rightarrow 9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x} = 6e^{3x}$

$\Rightarrow\ 2Ce^{3x} = 6e^{3x}$

$\Rightarrow \quad 2C = 6$

$\Rightarrow \quad C = 3$

$(3) \ \Rightarrow y_p = 3e^{3x}$

The general solution is

$$y = y_c + y_p$$

$$\Rightarrow \qquad y = Ae^x + Be^{2x} + 3e^{3x}$$

Example 2: Solve: $(D^2 - 2D + 3)\, y = x^3 + \cos x$

Solution:

$$y'' - 2y' + 3y = x^3 + \cos x \qquad \text{... (1)}$$

The auxiliary equation is $m^2 - 2m + 3 = 0$

$$\Rightarrow \ m = 1 \pm \sqrt{2}\, i$$

(C.F) $y_c = e^x \, [A \cos \sqrt{2}x + B \sin \sqrt{2}x]$ ... (2)

Here, the solution set $S = \{ \, e^x \cos \sqrt{2}x, e^x \sin \sqrt{2}x \, \}$

R.H.S of (1) is not a member of S.

So, choose (P.I)

$$y_p = C_0 + C_1 \, x + C_2 \, x^2 + C_3 \, x^3 + C_4 \sin x + C_5 \cos x \qquad \text{... (3)}$$

Case (c): Modified case

Example 3: Solve: $\dfrac{d^2 y}{dx^2} + 9y = \cos 3x$

Solution:

Given: $y'' + 9y = \cos 3x$ ... (1)

The auxiliary equation is $m^2 + 9 = 0$

$$\Rightarrow \ m = \pm\, 3i$$

$$\text{(C.F)} \ \ y_c = A \cos 3x + B \sin 3x \qquad \text{... (2)}$$

Here, the solution set $S = \{ \, \cos 3x, \sin 3x \, \}$

Normally we choose (P.I) $y_p = C_1 \sin 3x + C_2 \cos 3x$

R.H.S of (1) is a member of S.

The corresponding terms should be multiplied by x

$$y_p = x \, [C_1 \sin 3x + C_2 \cos 3x] \qquad \text{... (3)}$$

TWO MARKS QUESTIONS AND ANSWERS

1. Solve $\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Solution:

Given: $D^2 y - 5Dy + 6y = 0$

$$(D^2 - 5D + 6)\, y = 0$$

$$\frac{d^2}{dx^2} = D^2, \frac{d}{dx} = D$$

The Auxiliary Equation is

Put $D = P$

$$P^2 - 5P + 6 = 0$$

$$P_1 = 2, P_2 = 3 \qquad \text{(Roots different)}$$

$$y = \text{C.F} = Ae^{P_1 x} + Be^{P_2 x}$$

$$\boxed{y = Ae^{2x} + Be^{3x}}$$

2. Solve $(D^2 + 6D + 9)\, y = 0$

Solution:

Given: $(D^2 + 6D + 9)\, y = 0$

The Auxiliary Equation is

Put $D = P$

$$P^2 + 6p + 9 = 0$$

$$P = -3, P = -3 \qquad \text{(Roots same)}$$

$$y = \text{C.F} = (Ax + B)\, e^{-Px}$$

$$\boxed{y = (Ax + B)\, e^{-3x}}$$

3. Solve $y'' + 2y' + y = 0$

Solution:

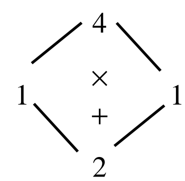
$$y'' = \frac{d^2 y}{dx^2}, \; y' = \frac{dy}{dx}$$

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$D^2 y + 2Dy + y = 0$$

$$(D^2 + 2D + 1)\, y = 0$$

The Auxiliary equation is



Put $D = P$

$$P^2 + 2P + 1 = 0$$

$$P_1 = -1, \; P_2 = -1$$

$$y = \text{C.F} = (Ax + B)\, e^{px}$$

$$\boxed{\boldsymbol{y = (Ax + B)\, e^{-1x}}}$$

4. Solve: $(D^2 - 6D + 13)\, y = 0$

Solution:

$$(D^2 - 6D + 13)\, y = 0$$

The auxiliary equation is

Put $D = P$

$$P^2 - 6P + 13 = 0$$

$$\Rightarrow \; ax^2 + 6x + c = 0$$

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, \; b = -6, \; c = 13$$

$$= \frac{6 \pm \sqrt{36 - 4 \times 13 \times 1}}{2\,(1)}$$

$$= \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2}$$

$$= \frac{2\,(3 \pm 2i)}{2}$$

$$P = 3 \pm 2i$$

$$P = \alpha \pm i\,\beta$$

$$\alpha = 3,\ \beta = 2$$

$$y = \text{C.F} = e^{\alpha x}\,(A \cos \beta x + B \sin \beta x)$$

$$\boldsymbol{y = e^{3x}\,(A \cos 2x + B \sin 2x)}$$

5. Solve: $(D^2 + D + 1)\,y = 0$

Solution:

$$(D^2 + D + 1)\,y = 0$$

The auxiliary equation is

Put $D = P$

$$P^2 + P + 1 = 0$$

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$P = \frac{-1 \pm i\sqrt{3}}{2}$$

$$P = \frac{-1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$$P = \alpha \pm i\,\beta$$

$$\alpha = \frac{-1}{2},\ \beta = \frac{\sqrt{3}}{2}$$

$$y = \text{C.F} = e^{\alpha x}\,(A\cos\beta x + B\sin\beta x)$$

$$\therefore\ y = e^{-1/2\,x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

6. Solve $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

Solution:

Given: $D^3y - 6D^2 + 11Dy - 6y = 0$

$$(D^3 - 6D^2 + 11D - 6)\,y = 0$$

The auxiliary equation is

Put $D = P$

$P^3 - 6P^2 + 11P - 6 = 0$ (Solve using calculator)

$P_1 = 1,\ P_2 = 2,\ P_3 = 3$ (Roots different)

$$y = \text{C.F} = Ae^{P_1x} + Be^{P_2x} + Ce^{P_3x}$$

$$\boldsymbol{y = Ae^{1x} + Be^{2x} + Ce^{3x}}$$

7. Solve $(D^3 - 3D^2 + 3D - 1)\,y = 0$

Solution:

The auxiliary equation is

Put $D = P$

$$(D^3 - 3D^2 + 3D - 1)\, y = 0$$

$$P^3 - 3P^2 + 3P - 1 = 0 \qquad \text{(solve using calculator)}$$

$$P_1 = 1, P_2 = 1, P_3 = 1 \qquad \text{(Roots equal)}$$

$$y = \text{C.F} = [Ax^2 + Bx + c]\, e^{Px}$$

$$y = [Ax^2 + Bx + c]\, e^{1x}$$

8. Solve: $(D^3 + D^2 + 4D + 4)\, y = 0$

Solution:

The auxiliary equation is

Put $D = P$

$$P^3 + P^2 + 4P + 4 = 0,\ P_1 = -1,\ P_2 = 2,\ P_3 = 2.$$

On solving using calculator

$$y = \text{C.F} = Ae^{-1x} + [B \cos 2x + C \sin 2x]$$

9. Solve: $(D^2 + 1)\, y = 0$

Solution:

The auxiliary equation is

Put $D = P$

$$P^2 + 1 = 0$$

$$P^2 = -1$$

$$P = \pm 1i$$

$$P = \alpha \pm i\, \beta$$

$$\alpha = 0, \beta = 1$$

$$y = \text{C.F} = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$$

$$= e^{0x} (A \cos 1x + B \sin 1x)$$

$$\boxed{\boldsymbol{y = A \cos x + B \sin x}}$$

10. Solve: $(D^2 + a^2)\,y = 0.$

Solution:

The auxiliary equation is

Put $D = P$

$$P^2 + a^2 = 0$$

$$P^2 = -a^2$$

$$P = \pm ai$$

$$P = 0 \pm ai$$

$$P = \alpha \pm i\beta$$

$$\alpha = 0\ ;\ \beta = a$$

$$y = \text{C.F} = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$y = e^{0x}(A\cos ax + B\sin ax)$$

$$\boxed{\boldsymbol{y = A\cos ax + B\sin ax}}$$

11. Find the particular integral of $(D^2 + 4D + 8)\,y = e^{2x}$.

Solution:

$$\text{P.I} = \frac{1}{D^2 + 4D + 8}\cdot e^{2x}$$

$$= \frac{e^{2x}}{4 + 8 + 8}$$

$$\boxed{\mathbf{P.I} = \frac{\boldsymbol{e^{2x}}}{\mathbf{20}}}$$

12. Solve: $(D^2 - 4)\,y = 1$

Solution:

$$(D^2 - 4)\,y = 1 \times e^{0x}$$

$$\text{P.I} = \frac{1}{D^2 - 4} \times e^{0x}$$

$$= \frac{e^{0x}}{0^2 - 4}$$

$$\boxed{\textbf{P.I} = \frac{\mathbf{1}}{\mathbf{-4}}}$$

13. Find the particular integral of $(D^2 + 1)\, y = \sin x$

Solution:

$$\text{P.I} = \frac{1}{D^2 + 1} \sin x$$

$$= \frac{\sin x}{-1 + 1}$$

$$= \frac{\sin x}{0}$$

$$\frac{1}{D^2 + 1} = \frac{x \sin x}{2D}$$

$$= \frac{x \sin x}{2D} \times \frac{D}{D}$$

Replace D^2 by $-(a^2) = -1$

$$= \frac{xD\,(\sin x)}{2D^2}$$

$$= \frac{x \cos x}{2\,(-1)}$$

$$\textbf{P.I} = \frac{\boldsymbol{x \cos x}}{\mathbf{-2}}$$

14. Find the particular integral of $(D^2+4)\,y=\sin x$

Solution:

$$\text{P.I} = \frac{1}{D^2+4}\sin 2x$$

$$= \frac{\sin 2x}{-4+4}$$

Replace D^2 by $-(a^2) = -4$

$$= \frac{\sin 2x}{0}$$

$$= \frac{xD\,(\sin 2x)}{2D \times D}$$

$$= \frac{x\,[2\cos 2x]}{2\,(-4)}$$

$$= \frac{2x\cos 2x}{-8}$$

$$\mathbf{P.I} = \frac{\boldsymbol{x\cos 2x}}{\mathbf{-4}}$$

15. Transform the equation $x^2y''+xy'=x$ into a linear differential equation with constant co-efficients.

Solution:

Given: $x^2\,y''+xy'=x$

$x^2D^2\,y+xDy=x$

$x^2D^2 = D'(D'-1)$	$x=e^z$
$xD = D'$	$z=\log x$

$$(D'(D'-1)+D')\,y=e^z$$

$$(D'^2-D'+D')\,y=e^z$$

$$\boxed{(D'^2)\,y=e^z}$$

16. Convert $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$ into linear ordinary differential equation.

Solution:

Given: $x^2 D^2 y - xDy + y = 0$

$$(x^2 D^2 - xD + 1)\, y = 0$$

Put
$$\boxed{\begin{array}{c} x = e^z \\ z = \log x \\ x^2 D^2 = D'(D'-1) \\ xD = D' \end{array}}$$

$$[D'(D'-1) - D' + 1]\, y = 0$$

$$[D'^2 - D' - D' + 1]\, y' = 0$$

$$\boxed{[D'^2 - 2D' + 1]\, y = 0}$$

17. Convert into linear ordinary differential equation.

Solution:

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$$

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0 \qquad \text{(Multiply by } x\text{)}$$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0$$

$$x^2 D^2 = D'(D'-1)$$

$$xD = D'$$

$$[D'(D'-1) + D']\, y = 0$$

$$[D'^2 - D' + D']\, y = 0$$

$$\boxed{[D'^2]\, y = 0}$$

18. Convert $x^2 y'' + 2xy' + 2y = 0$ into linear ordinary differential equations.

Solution:

Given: $x^2 D^2 y + 2xDy + 2y = 0$

$$(x^2 D^2 + 2xD + 2)\, y = 0$$

$$x^2 D^2 = D'\,(D' - 1)$$

$$xD = D'$$

$$[D'\,(D' - 1) + 2D' + 2]\, y = 0$$

$$[D'^2 - D' + 2D' + 2]\, y = 0$$

$$\boxed{[D'^2 + D' + 2]\, y = 0}$$

19. Transform $[x^2 D^2 + 3xD + 5]\, y = x \cos(\log x)$ into a linear ordinary equation.

Solution:

Given: $[x^2 D^2 + 3xD + 5]\, y = x \cos(\log x)$

Put

$$\boxed{\begin{aligned} x &= e^z \\ z &= \log x \\ x^2 D^2 &= D'\,(D' - 1) \\ xD &= D' \end{aligned}}$$

$$[D'\,(D' - 1) + 3D' + 5]\, y = e^z \cos z$$

$$[D'^2 - D' + 3D' + 5]\, y = e^z \cos z$$

$$\boxed{[D'^2 + 2D' + 5]\, y = e^z \cos z}$$

20. Transform $[x^2 D^2 + xD + 1]\, y = 0$ into ordinary differential equation.

Solution:

$$\boxed{\begin{array}{c} x = e^z \\ z = \log x \\ x^2 D^2 = D'(D' - 1) \\ xD = D' \end{array}}$$

$$[D'(D' - 1) + D' + 1]\, y = 0$$

$$[D'^2 - D' + D' + 1]\, y = 0$$

$$\boxed{[D'^2 + 1]\, y = 0}$$

21. Transform the equation

$$(2x + 3)^2 \frac{d^2 y}{dx^2} - 2\,(2x + 3)\frac{dy}{dx} - 12y = 6x$$

Solution:

Given: $(2x + 3)^2 D^2 y - 2\,(2x + 3)\, Dy - 12y = 6x$

$$[(2x + 3)^2 D^2 - 2\,(2x + 3)\, D - 12]\, y = 6x$$

$$\boxed{\begin{array}{l} \text{Put} \quad 2x + 3 = e^z \\ \qquad\qquad z = \log(2x + 3) \\ (2x + 3)^2 D^2 = 4D'(D' - 1) \\ (2x + 3)\, D = 2D' \end{array}}$$

$$[4D'[D' - 1) - 2\,(2D') - 12]\, y = 6\left[\frac{e^z - 3}{2}\right]$$

$$[4D'^2 - 4D' - 4D' - 12]\, y = \frac{6e^z}{2} - \frac{18}{2}$$

$$[4D'^2 - 8D' - 12]\, y = 3e^z - 9$$

22. Convert $[(x+1)^2 D^2 + (x+1) D + 1]\, y = 4 \cos [\log (x+1)]$

Solution:

$$\boxed{\begin{aligned} (x+1)^2 D^2 &= D' (D'-1) \\ (x+1) D &= D' \\ x+1 &= e^z \\ z &= \log (x+1) \end{aligned}}$$

$$[x^2 D^2 + xD + 1]\, y = 4 \cos [\log (x+1)]$$

$$[D' (D'-1) + D' + 1]\, y = 4 \cos z$$

$$[D'^2 - D' + D' + 1]\, y = 4 \cos z$$

$$\boxed{[D'^2 + 1]\, y = 4 \cos z}$$

23. Transform into linear ordinary differential equation $\dfrac{dx}{dt} - y = 0, \dfrac{dy}{dt} + x = 0$

Solution:

$$Dx - y = 0 \qquad \text{... (1)}$$

$$x + Dy = 0 \qquad \text{... (2)}$$

$$\Delta = \begin{vmatrix} D & -1 \\ 1 & D \end{vmatrix} = D^2 + 1$$

$$\Delta\, y = \begin{vmatrix} D & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0$$

$$\Delta\, y = 0$$

$$(D^2 + 1)\, y = 0$$

24. If $f_1 = \cos x, f_2 = \sin x$ then find the Wronskian.

Solution:

$$\text{Wronskian, } w = \begin{vmatrix} f_1 & f'_1 \\ f_2 & f'_2 \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{\mathbf{w = 1}}$$

25. In MVP, the formula for finding particular integral.

Solution:

$$\text{P.I} = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 R}{W} dx,$$

$$Q = \int \frac{f_1 R}{w} dx$$

EXERCISE

Higher order linear differential equations with constant coefficients

(a) R.H.S = 0

1. Solve $(D^2 + D + 1)\, y = 0$
2. Solve $(3D^2 + D - 14)\, y = 0$
3. Solve $\dfrac{d^2x}{dt^2} - 8\,\dfrac{dx}{dt} + 16x = 0$

(b) R.H.S $= e^{ax}$

1. Solve $(D^2 - 2D + 1)\, y = 5e^{3x} + \cosh 2x$
2. Solve $(3D^2 + D - 14)\, y = \sinh x$
3. Solve $(D^2 + 6D + 5)\, y = e^{2x}$
4. Solve $(D^3 - 12D + 6)\, y = (e^x + e^{-2x})^2$
5. Solve $(D^2 + 2D + 1)\, y = \pi$
6. Solve $(D^2 + 7D + 12)\, y = e^{2x} + 6$
7. Solve $(D^3 - 3D^2 + 4D - 2)\, y = \sinh^2 x$

(c) R.H.S $= \cos ax$ (or) $\sin ax$

1. Solve $\dfrac{d^2 y}{dx^2} + 4y = \sin 2x$
2. Solve $(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$
3. Solve $(D^2 - 3D + 2)\, y = \cos 3x \cos 2x$
4. Solve $(D^2 - 5D + 6)\, y = \sin x \sin 2x$
5. Solve $(D^2 + 4)^2\, y = \cos 2x$
6. Solve $(D^2 + 1)\, y = \cos^2 x$

(d) R.H.S $= e^{ax} + [\cos ax \text{ (or) } \sin ax]$

1. Solve $\dfrac{d^2 y}{dx^2} + 4y = e^x + \sin 2x$
2. Solve $(D^2 + 16)\, y = e^{-3x} + \cos 4x$
3. Solve $(D^2 - 4D - 5)\, y = \cos x + e^{-x}$
4. Solve $(D^3 + D^2 - D - 1)\, y = \cos 2x + 7$
5. Solve $(D^2 + 6D + 8)\, y = 11 \cos^2 x$

(e) R.H.S $= x^n$

1. Solve $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + 3y = 5x^2$
2. Solve $(D^3 + 3D^2 + 2D)\, y = x^2 + 1$
3. Solve $(D^2 + 5D + 4)\, y = x^2 + 7x + 9$
4. Solve $D^2 (D^2 + 4)\, y = 96x^2$
5. Solve $(D^2 - 5D + 6)\, y = x^2 + 3$
6. Solve $(D^4 + 8D^2 + 16)\, y = 16x + 10$

(f) R.H.S $= e^{ax}$ X type

1. Solve $(D^2 - 2D + 2)\, y = e^x \sin x$
2. Solve $(D^2 - 2D + 1)\, y = x^2 e^{3x}$
3. Find the P.I. of $(D^2 - 4D + 3)\, y = e^x \cos 2x$.
4. Solve $(D^2 + 1)\, y = x \sin hx$
5. Solve $(D^2 + 6D + 9)\, y = e^{-2x} x^3$
6. Solve $(D^2 + 5D + 4)\, y = e^{-x} \sin 2x$

7. Solve $(D^2 + D + 1)\, y = e^{-x} \sin^2 \dfrac{x}{2}$.

8. Solve $(D^2 + 2D + 5)\, y = e^x \cos 3x$

9. Solve $(D^2 - 3D + 2)\, y = xe^{3x} + \sin 2x$

(g) R.H.S. $= x^n \sin ax$ **(or)** $x^n \cos ax$

1. Solve $(D^2 - 1)\, y = x^2 \cos x$

2. Solve $(D^2 - 4D + 4)\, y = 3x^2\, e^{2x} \sin 2x$

3. Solve $(D^2 - 2D)\, y = e^x\, x^2 \cos x$

4. Solve $(D^2 + a^2)\, y = x \cos ax$

5. Solve $(D^2 + 1)^2\, y = x^2 \cos x$

Solve the following using method of variation of parameters

1. $\dfrac{d^2 y}{dx^2} + a^2 y = \sec ax$

2. $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \tan x$

3. $(D^2 + 4)\, y = \tan 2x$

4. $y'' - 2y' - 3y = 2e^{4x}$

5. $y'' - 2y' + 3y = x^3 + \sin x$

6. $y'' + 4y = 8x^2$

7. $y'' - 3y' + 2y = e^x$

8. $y''' + 2y'' - y' - 2y = e^x + x^2$

9. $\dfrac{d^2 y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{3x} + \sin x$

10. $y'' + 4y = 4\sec^2 2x$

11. $y'' - 2y' + y = \dfrac{e^x}{x}$

12. $x^2 \dfrac{d^2 y}{dx^2} + x \dfrac{dy}{dx} - y = x^2 \log x$

13. $\dfrac{d^2 y}{dx^2} + 2 \dfrac{dy}{dx} + y = \dfrac{e^{-x}}{x^2}$

14. $y'' + 7y' - 8y = e^{2x}$

Solve the following differential equations Cauchy - Euler method

1. $(x^2 D^2 - 3xD + 4)\, y = x^2 \cos (\log x)$
2. $(x^3 D^3 + 2x^2 D^2 - xD + 1)\, y = \log x$
3. $(x^2 D^2 + 4xD + 2)\, y = x^2 + \dfrac{1}{x^2}$
4. $(x^2 D^2 + xD + 1)\, y = \sin (2 \log x) \sin (\log x)$
5. $(x^2 D^2 - xD + 4)\, y = x^2 \sin (\log x)$
6. $(x^2 D^2 + 2xD - 20)\, y = (x + 1)^2.$
7. $x^2 y'' - xy' + y = x$
8. $x^2 y'' - xy' - 3y = x^2 \log x$
9. $x^3 y''' + 3x^2 y'' + xy' + y = x + \log x$
10. $x^2 y'' - 3xy' + 5y = x^2 \sin (\log x)$

Solve the following differential equations Legendre's linear differential equations

1. $(1 + 2x)^2 \dfrac{d^2 y}{dx^2} = 6 (1 + 2x) \dfrac{dy}{dx} + 16y = 8 (1 + 2x)^2$
2. $(3x + 2)^2 y'' + 3 (3x + 2) y' - 36y = 8x^2 + 4x + 1$
3. $(2x + 5)^2 y'' - 6 (2x + 5) y' + 8y = 6x$

Solve the following simultaneous differential equations

1. $(D+5)\,x+y=e^{t}\ ;\ (D+3)\,y-x=e^{2t}$
2. $\dfrac{dx}{dt}+y=\sin t+1\ ;\ \dfrac{dy}{dt}+x=\cos t$ given that $x=1,\ y=2$ at $t=0$.
3. $\dfrac{dx}{dt}+4x+3y=t,\ \dfrac{dy}{dt}+2x+5y=e^{t}$
4. $\dfrac{dx}{dt}+5x-2y=t,\ \dfrac{dy}{dt}+2x+y=0\ ;\ \ x=y=0$ when $t=0$
5. $\dfrac{dx}{dt}=3x+8y,\ \dfrac{dy}{dt}=-x-3y,\ x(0)=6,\ y(0)=-2$
6. $(D+2)\,x+y=0,\ 3x+(D+2)\,y=2e^{2t}$
7. $\dfrac{dx}{dt}+x+2y=t,$

   $\dfrac{dy}{dt}+2xy+y=0$
8. $\dfrac{dx}{dt}-y=t,$

   $\dfrac{dy}{dt}+x=\sin t$
9. $3\,(1-D)\,x+4y=3t+1$

   $3\,(D+1)\,y+2x=e^{t}$
10. $\dfrac{dx}{dt}+x+2y=t,$

    $\dfrac{dy}{dt}+2x+y=0$
11. $\dfrac{dx}{dt}-y=t,$

    $\dfrac{dy}{dt}+x=\sin t$
12. $3\,(1-D)\,x+4y=3t+1$

    $3\,(D+1)\,y+2x=e^{t}$

U23MAT12-MATRICES AND CALCULUS (MODEL QP-1)

Time: Three hours | Maximum: 100 marks

Answer ALL questions

PART – A $(10 \times 2 = 20)$

1. Find the Nature of the Quadratic form $x_1^2 + 5x_2^2 + 4x_3^2 + 2x_2 x_3 + 6x_3 x_1 + 2x_1 x_2$

2. If $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$ then find Eigen values of $A^{-1}, 2A^{-1}, A^3$

3. Find the critical point of the function $f(x) = 2x^3 + 3x^2 - 36x$

4. Find the value of $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$

5. If $x = u(1-v)$ and $y = 2uv$ then find $\dfrac{\partial(x, y)}{\partial(u, v)}$

6. If $x = r\cos\theta$ and $y = r\sin\theta$ then find $\dfrac{\partial r}{\partial x}$.

7. Evaluate $\displaystyle\int_{\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} d\theta\, dr$.

8. Evaluate $\displaystyle\int_{0}^{1}\int_{1}^{2} x(x+y)\, dx\, dy$.

9. Find the particular integral $(D^2 + 4)\, y = \cos 2x$

10. Transform the equation

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x)) \text{ into a}$$

differential equation with constant coefficients.

Part - B ($5 \times 16 = 80$)

11. (a) Reduce the quadratic form $2x_1 x_2 + 2x_2 x_3 + 2x_3 x_1$ into a canonical form by an orthogonal reduction. Also find the Rank, Index, Signature and the Nature of the Quadratic form.

(OR)

(b) (i) Verify Cayely's Hamilton theorem and use it to find A^4 and A^{-1} if $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$

(ii) Diagonalize the matrix by Orthogonal Transformation and also prove that Eigen vector are orthogonal for $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

12. (a) (i) For the function $f(x) = 2 + 2x^2 - x^4$ find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and inflection point

(ii) Determine the value of 'λ' for which the following function is continuous at $x = -1$

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1}; & x \neq -1 \\ \lambda; & x = -1 \end{cases}$$

(OR)

(b) (i) If $y = x^4 + 2x^2 - x$, then find $\dfrac{dy}{dx}$ and also find an equation of the tangent and normal line to the curve at the point (1, 2).

(ii) Find the absolute maximum and minimum values of function $f(x) = 2x^3 - 3x^2 - 12x + 1$; $[-2, 3]$

13. (a) (i) Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

(ii) Find the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ of the transformation

$y = r \sin\theta \sin\phi$, $x = r \sin\theta \cos\phi$

$z = r \cos\theta$,

(OR)

(b) (i) Find the Taylor's series expansion of $\sin x \cos y$ in powers of x and y.

(ii) Obtain the maximum and minimum values of $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

14. (a) (i) Express in polar co-ordinates and then evaluate it.

$$\int_0^a \int_y^a \frac{x^2}{(x^2 + y^2)^{3/2}}\, dx\, dy.$$

(ii) Find the area enclosed by the curves $y = x^2$ and $x + y = 2$

(OR)

(b) (i) Change the order of integration in

$$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$$ and hence evaluate it.

(ii) Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ using triple integration.

15. (a) Solve $\frac{d^2 y}{dx^2} + a^2 y = \tan ax$ by the method of variation of parameters.

(b) Solve $(x^2 D^2 - xD + 1)\, y = \sin(\log x)$

(OR)

(c) Solve $\frac{dx}{dt} - y = t, \frac{dx}{dt} + x = t^2$.

(d) Solve $(1 + x^2) \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 4 \cos(\log(1 + x))$

U23MAT12-MATRICES AND CALCULUS (MODEL QP-2)

Time: Three hours Maximum: 100 marks

Answer ALL questions

PART – A $(10 \times 2 = 20)$

1. Find the Nature of the Quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1 x_2 - 6x_1 x_3 + 6x_2 x_3$

2. If $2, -1, -3$ are the Eigen values of a matrix "A" then find the Eigen values of matrix $A^2, A^3, 2A - 3I$

3. Find $\frac{dy}{dx}$ if $x^3 + y^3 + 6xy$ by using implicit differentiation.

4. Find $\lim\limits_{x \to 1} \frac{x^2 - 4x}{x^2 - 3x - 4}$

5. If $u = \frac{2x - y}{2}$ and $v = \frac{y}{z}$ then find $\frac{\partial (u, v)}{\partial (x, y)}$

6. Find the stationary points of the function $f = x^2 + y^2 + 6x + 12$.

7. Evaluate $\int\limits_{\pi/2}^{\pi/2} \int\limits_{0}^{2 \cos \theta} d\,\theta\, dr$

8. Evaluate $\int\limits_{0}^{1} \int\limits_{0}^{2} \int\limits_{0}^{3} xy^2 z\, dx\, dy\, dz$

9. Transform the equation $(2x + 3)^2 \frac{d^2 y}{dx^2} - 2\,(2x + 3) \frac{dy}{dx} - 12y = 6x$ into a differential equation with constant co-efficient.

10. Transform into linear ode $\frac{dx}{dt} + 2y = -\sin t, \frac{dy}{dt} - 2x = \cos t$

Part - B $(5 \times 16 = 80)$

11. (a) Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2 x_3 - 10x_3 x_1 - 4x_1 x_2$ into a canonical form by an orthogonal reduction. Also find the Rank, Index, Signature and the Nature of the Quadratic form.

(OR)

(b) (i) Verify Cayley's Hamilton theorem and use it to find A^4 and A^{-1} if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

(ii) Diagonalize the matrix by Orthogonal Transformation and also prove that Eigen vector are orthogonal for $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

12. (a) (i) For the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. Find the intervals of increase or decrease, local maximum an minimum values, the intervals of concavity and inflection points.

(ii) Find the domain where the function f is continuous. Also find the numbers at which the function f is continuous where

$$f(x) = \begin{cases} 4 - x^2 ; & x \leq 0 \\ x - 5 ; & 0 \leq x \leq 1 \\ 4x^2 - 9 ; & 1 \leq x \leq 2 \\ 3x + 4 ; & x \geq 2 \end{cases}$$

(OR)

(b) (i) If $x^2 + 2xy - y^2 + x = 2$, then find $\dfrac{dy}{dx}$ and also find an equation of the tangent and normal line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point (1,2)

(ii) Find the absolute maximum and minimum values of function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 1 \; ; \; -2 \leq x \leq 3$$

13. (a) (i) A rectangular box open at top is to have a volume 32 CC. Find the dimension of the box requiring the least material for the construction.

(ii) Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3

If $y_1 = \dfrac{x_2 x_3}{x_1}, y_2 = \dfrac{x_3 x_1}{x_2}, y_3 = \dfrac{x_1 x_2}{x_3}$

(OR)

(b) (i) Find the Taylor's series expansion of $\sin xy$ about $(x-1)$ and $(y-\pi/2)$ upto second degree terms.

(ii) Obtain the maximum and minimum values of $f(x, y) = x^3 + y^3 - 3xy$

14. (a) (i) Transform the integral $\int\limits_0^2 \int\limits_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ into polar-coordinates and hence evaluate it.

(ii) Find the volume of the tetrahedron bounded by the coordinate's planes $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$

(OR)

(b) (i) Change the order of integration $\int\limits_0^{\infty} \int\limits_x^{\infty} \dfrac{e^{-y}}{y}\, dx\, dy$ and hence evaluate it.

(ii) Evaluate $\iiint (x+y+z)\, dx\, dy\, dz$ over V, where V is the volume of the rectangular parallelopiped bounded by $x = 0, x = 4, y = 0, y = 1, z = 0, z = 1.$

15. (a) (i) Solve $(x^2 D^2 - xD + 1)\, y = \left(\dfrac{\log x}{x} \right)^2$.

(ii) Solve the system of equations $\dfrac{dx}{dt} + 2y = -\sin t$, $\dfrac{dy}{dt} - 2x = \cos t$

(OR)

(b) (i) Solve $\dfrac{d^2 y}{dx^2} + 4y = \operatorname{cosec} 2x$ by method of variation of parameters.

(ii) Solve: $(D^2 - 2D + 5)\, y = \sin x$

U23MAT12-MATRICES AND CALCULUS (MODEL QP-3)

Time: Three hours Maximum: 100 marks

Answer ALL questions

PART – A $(10 \times 2 = 20)$

1. Find the Nature of the Quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$

2. If 3, 6 are the Eigen values of $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ write down all the Eigen values of A^{-1}

3. If $x^2 + y^2 = 1$ find $\frac{dy}{dx}$.

4. Find $\lim\limits_{x \to 0} \frac{\sin 5x}{\sin 3x}$

5. If $x = uv$ and $y = \frac{u}{v}$ then find $\frac{\partial (x, y)}{\partial (u, v)}$

6. Find the Stationary points of the function $f = x^2 + 3xy^2 - 15x - 12y$.

7. Change the order of integration in $\int_0^1 \int_{y^2}^{y} f(x, y)\, dx\, dy$

8. Evaluate $\int_1^{lnx} \int_0^{lny} e^{x+y}\, dx\, dy$

9. Find the particular integral of $(D^2 - 36)\, y = e^{-6x}$

10. Solve the equation $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$

Part - B $(5 \times 16 = 80)$

11. (a) Reduce the quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$ into a canonical form by an orthogonal reduction. Also find the Rank, Index, Signature and the Nature of the Quadratic form.

(OR)

(b) (i) Verify Cayely's Hamilton theorem and use it to find A^4 and A^{-1} if $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

(ii) Diagonalize the matrix by Orthogonal Transformation and also prove that Eigen vector are orthogonal for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

12. (a) (i) For the function $f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$. Find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and inflection point

(ii) Find the values of a a and b that makes "f" continuous on $f(x) = \begin{cases} 2x - 2 & x < -1 \\ ax + b, & -1 \le x \le 1 \\ 5x + 7 & x > 1 \end{cases}$

(OR)

(b) (i) If $x^2 + y^2 = 25$, then find $\frac{dy}{dx}$ and also find an equation of the tangent and normal line to the curve $x^2 + y^2 = 25$ at the point (3,4)

(ii) Find the absolute maximum and minimum values of function.

$f(x) = x^3 - 3x^2 + 1; \ -\frac{1}{2} \le x \le 4$

13. (a) (i) The temperature at any point (x, y, z) in space is given by $T = 400xyz^2$. Find the maximum temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

(ii) Find the Jacobian of (u, v, w) with respect to (x, y, z) If $u = \frac{yz}{x}, v = \frac{xz}{y}, w = \frac{xy}{z}$

(OR)

(b) (i) Find the Taylor's series expansion of $x^2 y^2 + 2x^2 y + 3xy^2$ in powers of $(x + 2)$ and $(y - 1)$

(ii) Obtain the maximum and minimum values of $f(x, y) = 3x^2 - y^2 + x^3$

14. (a) (i) Express in polar co-ordinates and the evaluate it.
$$\int_0^a \int_y^a \frac{x}{(x^2 + y^2)}\, dx\, dy.$$

(ii) Evaluate $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$

(OR)

(b) (i) Change the order of integration in

$\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy$ and hence evaluate it.

(ii) Evaluate $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx$

15. (a) Solve $(1+x)^2 \dfrac{d^2 y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin(\log(1+x))$

(ii) Solve $(D^2 + 5D + 4)\, y = e^{-x} \sin 2x$

(OR)

(b) (i) Solve $(x^2 D^2 - 3xD + 4)\, y = x^2 \cos(\log x)$

(ii) Solve $(D^2 + a^2)\, y = \sec ax$ using the method of variation of parameters.

U23MAT12-MATRICES AND CALCULUS (MODEL QP-4)

Time: Three hours Maximum: 100 marks

Answer ALL questions

PART – A $(10 \times 2 = 20)$

1. Determine the nature, index and signature of the quadratic for $2x_2 x_3 + 2x_3 x_1 + 2x_1 x_2$

2. If $A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 6 \end{pmatrix}$ then find Eigen values of $A^{-1}, 2A^{-1}, A^3$.

3. Find the derivative of the following function
   (i) $y = \dfrac{x^3 - 2x^2 + 5}{x^2}$ (ii) $y = (4x^2 - 3)(2x + 1)$

4. Find $\dfrac{dy}{dx}$ for $ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$

5. If $u = f(x - y, y - z, z - x)$ then show that
   $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

6. State Euler's Theorem for homogeneous function.

7. Sketch the region of the integration in
   $\displaystyle\int_0^1 \int_x^1 f(x, y)\, dy\, dx.$

8. Evaluate $\displaystyle\int_1^a \int_1^a \frac{dx\, dy}{xy}$.

9. Solve $(D^2 + D + 1)\, y = \sin x$

10. Transform the equation $(2x + 7)^2 y'' - 6(2x + 7) y' + 8y = 8x$ into a differential equation with constant co-efficient.

Part - B $(5 \times 16 = 80)$

11. (a) Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$ into a canonical form by an orthogonal reduction. Also find the Rank, Index, Signature and the Nature of the Quadratic form.

(OR)

(b) (i) Verify Cayely's Hamilton theorem and use it to find A^4 and A^{-1} if $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

(ii) Diagonalize the matrix by Orthogonal Transformation and also prove that Eigen vector are orthogonal for $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

12. (a) (i) For the function $f(x) = 2x^3 + 3x^2 - 36x$. Find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and inflection point

(ii) Find the values of a and b that makes "f" continuous on $f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & x < 2 \\ ax^2 - bx + 3\,, & 2 \le x \le 3 \\ 2x - a + b & x \ge 3 \end{cases}$

(OR)

(b) (i) Differentiate the function $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x, the graph of $f(x)$ has a horizontal tangents?

(ii) Find the absolute maximum and minimum values of function $f(x) = x^3 - 6x^2 + 5$ on interval $[-3, 5]$.

13. (a) (i) Find the dimension of the rectangular box without a top of maximum capacity, whose surface area is 432 sq.cm.

(ii) Examine $x + y + z = u,\ y + z = uv,\ z = uvw$ then prove that $\dfrac{\partial (x, y, z)}{\partial (u, v, w)} = u^2 v$

(OR)

(b) (i) Obtain the Taylor's series expansion of $e^x \cos y$ in of power of x and y upto third degree terms.

(ii) Obtain the maximum and minimum values of $f(x, y) = x^3 + y^3 - 12x - 3y + 20$

14. (a) (i) Express in polar co-ordinates and the evaluate it $\displaystyle\iint \frac{x^2 y^2}{x^2 + y^2}\, dx\, dy.\ x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2, b > a$

(ii) Change the order of integration in $\displaystyle\int_0^a \int_0^{2\sqrt{ax}} x^2\, dy\, dx$ and hence evaluate it.

(OR)

(b) (i) Find the area of the parabola $y = x^2$ and the straight line $y = x$

(ii) Evaluate $\displaystyle\int_0^{2a} \int_0^{x} \int_y^{x} xyz\, dx\, dy\, dz$

15. (a) (i) Solve $(D^2 + a^2)\, y = \cot ax$ by the method of variation of parameters.

(ii) Solve $(D^2 + 10D + 25)\, y = x$

(OR)

(b) (i) Solve $(2x + 3)\, 2y'' - 2\,(2x + 3)\, y' - 12y = 6x$

(ii) Solve $(x^2 D^2 - xD + 4)\, y = x^2 \sin(\log x) + \dfrac{33}{x^3}$

(iii) Solve the system of equations $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x = \sin 2t$ and $\dfrac{dx}{dt} - \dfrac{dy}{dt} + 2y = \cos 2t$

U23MAT12-MATRICES AND CALCULUS (MODEL QP-5)

Time: Three hours Maximum: 100 marks

Answer ALL questions

PART – A $(10 \times 2 = 20)$

1. Determine the nature, index and signature of the quadratic for $2x^2 - 2y^2 + 4z^2 + 2xy - 6yz + 6zx$.

2. If $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 6 \end{pmatrix}$ then find Eigen values of A^{-1}, $2A^{-1}$, A^3.

3. Evaluate the $\lim\limits_{x \to 1} \left[\dfrac{x^2 - 4x}{x^2 - 3x - 4} \right]$.

4. Find critical point of $y = 5x^2 - 6x$.

5. If $u = x^3 + y^3$ and $x = a \cos t, y = b \sin t$ find $\dfrac{du}{dt}$.

6. Prove $\dfrac{\partial^2 f}{\partial x\, \partial y} = \dfrac{\partial^2 f}{\partial y\, \partial x}$ if $f = x^3 + y^3 + z^3 - 3xyz$

7. Find the limits of integration $\iint\limits_R f(x, y)\, dx\, dy$ where R is the triangle bounded by $x = 0, y = 0, x + y = 2$.

8. Evaluate $\displaystyle\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dy\, dx$.

9. Transform into differential equation
$x^2 \dfrac{d^2 y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = \cos(\log x)$

10. Solve $(D^2 + 10D + 25)\, y = e^{2x}$

Part - B $(5 \times 16 = 80)$

11. (a) Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$ into a canonical form by an orthogonal reduction. Also find the Rank, Index, Signature and the Nature of the Quadratic form.

(OR)

(b) (i) Verify Cayely's Hamilton theorem and use it to find A^4 and A^{-1} if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(ii) Diagonalize the matrix by Orthogonal Transformation and also prove that Eigen vector are orthogonal for $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{pmatrix}$

12. (a) (i) For the function $f(x) = x^3 - 3x^2 - 12x$. Find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and inflection point

(ii) Find the values of a and b that makes "f" continuous on

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \le 2 \\ ax^2 - bx - 3 & 2 < x < 3 \\ 2x - a + b & x \ge 3 \end{cases}$$

(OR)

(b) (i) Find the equation tangent and normal line to the curve $y = x^4 + 2x^2 - x$ at the point (1,2)

(ii) Find the absolute maximum and minimum values of function $f(x) = -x^3 + 12x + 5$ on interval $[-3, 3]$

13. (a) (i) Find the dimensions of the rectangular box without a top of maximum capacity, whose surface area is 108 sq.cm.

(ii) Examine $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$ are functionally dependent.

(OR)

(b) (i) Obtain the Taylor's series expansion of $e^x \sin y$ of power of $(x - 1)$ and $\left(y - \frac{\pi}{2}\right)$ upto third degree.

(ii) Obtain the maximum and minimum values of $f(x, y) = x^3 y^3 (6 - x - y)$.

14. (a) (i) Express in polar co-ordinates and the evaluate it $\int_0^\infty \int_0^\infty e^{-(x^2 + y^2)}\, dx\, dy$.

(ii) Change the order of integration in $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\, dx\, dy$ and hence evaluate it.

(OR)

(b) (i) Find the area of the cardioid $r = a(1 + \cos\theta)$

(ii) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1 - x^2 - y^2 - z^2}}$.

15. (a) (i) Solve $(3x + 2)^2 \frac{d^2 y}{dx^2} + 3(3x + 2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

(ii) Solve $(D^2 - 4D + 3)\, y = e^{-x} \sin x + \cos 4x$

(OR)

(b) (i) Solve $\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 12\frac{\log x}{x^2}$

(ii) Solve $(D^2 + 1^2)\, y = \cot x$ by the method of variation of parameters.