Statistics And Numerical Methods

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SYLLABUS

U23MAT22	STATISTICS AND NUMERICAL METHODS	L	T	P	C
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UNIT I: TESTING OF HYPOTHESIS 12

Introduction – Sampling distributions – Tests for single mean, proportion and difference of means (Large and small samples) – Tests for single variance and equality of variances – Chi square test for goodness of fit – Independence of attributes.

UNIT II: DESIGN OF EXPERIMENTS 12

Introduction – Analysis of variance – One way and two way classifications – Completely randomized design – Randomized block design – Latin square design.

UNIT III: SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS 12

Solution of algebraic and transcendental equations – Fixed point iteration method – Newton Raphson method – Solution of linear system of equations – Gauss elimination method – Gauss Jordan method – Iterative methods of Gauss Jacobi and Gauss Seidel – Eigen Value of a matrices by power method and jacobi's method for symmetric matrices.

UNIT IV: INTERPOLATION, NUMERICAL DIFFERENTIATION AND INTEGRATION 12

Lagrange's and Newton's divided difference interpolations – Newton's forward and backward difference interpolation – Approximation of derivatives using interpolation polynomials – Numerical single and double integrations using Trapezoidal and Simpson's 1/3 rules.

UNIT V: NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS 12

Single step methods: Taylor's series method – Euler's method – Modified Euler's method – Fourth order Runge – Kutta method for solving first order differential equations – Multi step methods: Milne's and Adams Bashforth predictor corrector methods for solving first order differential equations.

TOTAL: 60 PERIODS

CONTENTS

STATISTICS AND NUMERICAL METHODS

FORMULA

UNIT - I: TESTING OF HYPOTHESIS

χ^2 - Test (or) Chi Square Test

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

where $E_i = \dfrac{(\text{Row total} \times \text{Column total})}{\text{Total}}$

For 2×2 table

a	b
c	d

$$\chi^2 = \frac{(a+b+c+d)\,(ad-bc)^2}{(a+c)\,(b+d)\,(a+b)\,(c+d)}$$

***F*-Test [Variance test]**

Test statistics

$$F = \frac{S_1^2}{S_2^2} \quad \text{(or)} \quad F = \frac{S_2^2}{S_1^2}$$

Here $S_1^2 > S_2^2$

d.f

$(n_1 - 1 \; ; \; n_2 - 1)$

Here $S_2^2 > S_1^2$

d.f

$(n_2 - 1 \; ; \; n_1 - 1)$

Case 1: Standard deviation is given

$$S_1^2 = \frac{n_1\, s_1^2}{n_1 - 1}$$

$$S_2^2 = \frac{n_2\, s_2^2}{n_2 - 1}$$

Case 2: Sum of squares of deviations are given

$$S_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_1 - 1}$$

$$S_2^2 = \frac{\Sigma (x_2 - \bar{x}_2)^2}{n_2 - 1}$$

Case 3: Sum of squares are given

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2 \Rightarrow S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2 \Rightarrow S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

***t*-test – (Mean test or Average test)**

Type 1: Sample size is large ($n \geq 30$)

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Standard Table Value

At 5% LOS $z = 1.96$

At 1% LOS $z = 2.56$

Type 2: Sample size is small ($n < 30$)

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

d.f $= (n_1 + n_2 - 2)$

Case 1: Standard deviations are given

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

Case 2: Sum of squares of deviations are given

$$S^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

Case 3: Sum of squares are given

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

Type 3: Student *t*-test (single mean test)

$t = \dfrac{\bar{X} - \mu}{\left(\dfrac{s}{\sqrt{n-1}}\right)}$	(or)	$t = \dfrac{\bar{X} - \mu}{\left(\dfrac{s}{\sqrt{n}}\right)}$
(Standard deviation is given)		(Standard deviation is not given)

Proportion test

Single proportion test	**Double proportion test**
$Z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}}$	$z = \dfrac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

UNIT - II: DESIGNS OF EXPERIMENTS

ONE - WAY ANOVA

Source of variations	Sum of squares	Degrees of freedom	Mean square value	*F*-ratio
Between Columns	SSC	$C-1$	$MSC = \frac{SSC}{C-1}$	
Error	SSE	$N-C$	$MSE = \frac{SSE}{N-C}$	$F = \frac{\text{Greater variance}}{\text{Smaller variance}}$
Total	SST	$N-1$		

TWO WAY ANOVA

Source of variations	Sum of squares	Degrees of freedom	Mean square value	*F*-ratio
Between columns	SSC	$C-1$	$MSC = \frac{SSC}{C-1}$	$F_C = \frac{\text{Greatervariance}}{\text{Smaller variance}}$
Between rows	SSR	$r-1$	$MSR = \frac{SSR}{r-1}$	
Between error	SSE	$(r-1)(c-1)$	$MSE = \frac{SSE}{(r-1)(c-1)}$	$F_R = \frac{\text{Greater variance}}{\text{Smaller variance}}$
Total	SST			

To find FC, FR and FV.

Note: Compare MSC with MSE and MSR with MSE and MSV with MSe.

Three-Way ANOVA or Latin square design

Source of variations	Sum of squares	Degrees of freedom	Mean square value	*F*-ratios
Between columns	SSC	$C - 1$	$MSC = \frac{SSC}{C-1}$	$F_C = \frac{\text{Greater variance}}{\text{Smaller variance}}$
Between rows	SSR	$R - 1$	$MSR = \frac{SSR}{R-1}$	
Between treatments	SSV	$v - 1$	$MSV = \frac{SSV}{v-1}$	
Error	SSE	$(v - 1)(v - 2)$	$MSE = \frac{SSE}{(v-1)(v-2)}$	$F_v = \frac{\text{Greater variance}}{\text{Smaller variance}}$
Total	SST			

UNIT - III: SOLUTIONS OF EQUATIONS AND EIGEN VALUE PROBLEMS

- Newton Raphson Iterative Formula

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)} \right]$$

- Newton's iterative formula for finding $\sqrt{N}$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n} \right]$$

- Newton's Iterative formula for finding $\frac{1}{N}$

$$x_{n+1} = x_n \, [2 - N x_n]$$

- **Order of convergence for Newton Raphson Method is 2**.

- **Convergence condition** for Newton Raphson Method is

$$|f(x) f''(x)| < |f'(x)|^2$$

Fixed point iteration method

Formula:	$x = \phi(x)$
Order of convergence	$= 1$
Convergence condition	$\lvert\phi'(x)\rvert < 1$

- **Gauss Elimination Method:**

  Convert the Augmented matrix into **upper triangular matrix**.

- **Gauss Jordan Method:**

  Convert the Augmented matrix into **diagonal** or **unit matrix**

- **Condition for Gauss Jacobi / Gauss Seidal method**

The system should be **diagonally dominant** (ie) coefficient matrix should be diagonally dominant.

- Jordan principle for finding inverse of a matrix

$$(A, I) \longrightarrow (I, A^{-1})$$

- **Power Method:**

Power method is used to find the dominant eigen value (Numerically largest Eigen value) of a given matrix.

Gauss Jacobi Method (All Eigen values)

For 2×2 Matrix

$$S_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}}\right)$$

For 3×3 Matrix

$$S = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11} - a_{33}}\right)$$

$$S = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}}\right)$$

Note: Choose S according to your problem.

UNIT - IV: INTERPOLATION AND NUMERICAL DIFFERENTIATION NUMERICAL INTEGRATION

Interpolation

- For Equal Intervals [Newton's Forward difference formula]
- For Unequal Intervals [Newton's Divided difference formula]
- For both equal and Unequal Intervals [Lagrange's formula]
- **Newton's forward formula:**

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

where $p = \dfrac{x - x_0}{h}$

Newton's Backward formula

$$y = y_n + q\,\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \ldots\ldots$$

$$q = \frac{x - x_n}{h}$$

Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)\,\Delta x_0 + (x - x_0)(x - x_1)\,\Delta^2 x_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 x_0 + \ldots\ldots$$

Lagrange's formula

For three given points (x_0, y_0), (x_1, y_1) and (x_2, y_2)

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

For 4 points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and (x_3, y_3)

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 +$$

$$\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1 +$$

$$\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 +$$

$$\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

Numerical integration

Single integration

(i) Trapezoidal rule

[Any number of intervals]

$$\int_a^b f(x)\, dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

(or)

$$\int_a^b f(x)\, dx = \frac{h}{2} \begin{bmatrix} \text{(Sum of first and last values)} \\ + 2 \text{ (Remaining values)} \end{bmatrix}$$

(i) Simpson's 1/3$^{\text{rd}}$ Rule (or) Simpson's Rule [Even no. of Intervals]

$$\int_a^b f(x)\, dx = \frac{h}{3} [(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \dots) + 4(y_1 + y_3 + \dots)]$$

(or)

$$\int_a^b f(x)\, dx = \frac{h}{3} \begin{bmatrix} \text{(Sum of first last values)} + 2 \text{ (Sum of even values)} \\ + 4 \text{ (sum of odd values)} \end{bmatrix}$$

Simpson's 3/8th Rule (Number of Intervals must be multiples of 3)

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[\begin{array}{l}(y_0 + y_n) + 2\,(y_3 + y_6 + y_9 + \ldots) \\ + 3\,(y_1 + y_2 + y_4 + y_5 + \ldots)\end{array}\right]$$

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[\begin{array}{l}\text{(Sum of first and last values)} \\ + 2\,\text{(Sum of Multiples of 3 values)} \\ + 3\,\text{(Sum of Remaining values)}\end{array}\right]$$

Double integration

(i) Trapezoidal rule

(ii) Simpson's rule

Trapezoidal rule for double integration

$$\int_a^b \int_c^d f(x, y)\,dx\,dy$$

$$= \frac{hk}{4}\left[\begin{array}{l}\text{(Sum of } f \text{ at corners)} \\ + 2\,\text{(Sum of } f \text{ at first boundary except corner)} \\ + 4\,\text{(Sum of Remaining values)}\end{array}\right]$$

Simpson's rule for double integration

$$\int_a^b \int_c^d f(x, y)\,dx\,dy$$

$$= \frac{hk}{9}\left[\begin{array}{l}\text{(Sum of } f \text{ at corner)} \\ + 4\,\text{(Sum of } f \text{ at first boundary except corner)} \\ + 16\,\text{(Sum of Remaining values)}\end{array}\right]$$

NUMERICAL DIFFERENTIATION

At $x = x_0$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \ldots\right]$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \dots \right]$$

At $x \neq x_0$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{(3p^2 - 6p + 2)}{6}\Delta^3 y_0 + \dots \right]$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{(6p^2 - 18p + 11)}{12}\Delta^4 y_0 + \dots \right]$$

$$\boxed{p = \frac{x - x_0}{h}}$$

At $x = x_n$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \dots \right]$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots \right]$$

At $x \neq x_n$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2q+1)}{2}\nabla^2 y_n + \frac{(3q^2 + 6q + 2)}{6}\nabla^3 y_n + \dots \right]$$

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (q+1)\nabla^3 y_n + \frac{(6q^2 + 18q + 11)}{12}\nabla^4 y_n + \dots \right]$$

$$\boxed{q = \frac{x - x_n}{h}}$$

UNIT - V: NUMERICAL SOLUTIONS OF ODE

Taylor Series Method (Single step Method)

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \ldots$$

Put $n = 0$,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

Put $n = 1$,

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

Euler's Method (Single Step Method)

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Put $n = 0$, $y_1 = y_0 + hf(x_0, y_0)$

Put $n = 1$, $y_2 = y_1 + hf(x_1, y_1)$

Modified Euler's Method (Single Step Method)

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n) \right]$$

Put $n = 0$, $y_1 = y_0 + hf\left[x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0) \right]$

Put $n = 1$, $y_2 = y_1 + hf\left[x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1) \right]$

Runge Kutta Method of Order 4 (R-K method) (Single Step Method)

$$y_1 = y_0 + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}\,;\; y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}\,;\; y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\,(x_0 + h\,;\; y_0 + k_3)$$

Milne's Method: (Multi-step method)

Predictor Formula: $y_{4,\,p} = y_0 + \frac{4h}{3}\,[2y_1' - y_2' + 2y_3']$

Corrector Formula: $y_{4,\,c} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4']$

Adam's Method: (Multi-step method)

Predictor Formula

$$y_{4,\,p} = y_3 + \frac{h}{24}\,[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

Corrector Formula

$$y_{4,\,c} = y_3 + \frac{h}{24}\,[9y_4' + 19y_3' - 5y_2' + y_1']$$

- The solve single step method problems the need only one prior (Initial) value.
- To solve multi step method problems the need four prior (Initial) values.

UNIT - I

Testing of Hypothesis

1.0 INTRODUCTION

In engineering, many decision making problems involve evaluating a hypothesis about a parameter, a process known as hypothesis testing. This technique is crucial for statistical inference and is widely applicable in experiments and tests within the field. Hypothesis testing is closely related to confidence intervals, both of which are essential for analyzing data in comparative experiments. Before discussing sampling, it's important to first define the concept of population.

1.1 BASIC DEFINITIONS

(a) Statistics

A collection of quantitative data is called a statistics (or)

A collection or organisation, analysis, interpretation and presentation of a data is called a statistics.

There are 2 types of statistics

1. Descriptive statistics
2. Inferential statistics

(b) Application of statistics

- Mathematical science
- Environmental science
- Bio-statistics
- Quality control
- Machine learning

(c) Population

A population consists of collection of individual units, which may be persons or experimental outcomes, whose characteristics are to be studied.

(d) Sample

A sample is a portion of the population that is studied to learn about the characteristics of the population.

(e) Random sample

A random sample is one in which each item of a population has an equal chance of being selected.

(f) Sampling

The process of drawing a sample from the population is called sampling.

(g) Sample size

- The no. of items selected in a sample is called the sample size and it is denoted by 'n'.
- If $n \geq 30$ the sample is called large sample and if $n \leq 30$ the sample is called small sample

(h) Parameter

Parameter are the measures (such as mean, median, standard deviation) which describe a population and the word statistic is used to indicate various measures relating to the sample.

(i) Sampling distribution

Consider all possible samples of size 'n' drawn from a given population at random. We calculate mean values of these samples. If we group these different means according to their frequencies, the frequency distribution so formed is called sampling distribution.

(j) Standard error

The standard deviation of the sampling distribution is called the standard error.

Notation

	Population	**Sample**
Mean	μ	$\bar{x}$
S.D	σ	s
Proportion	ϕ	p

(k) Statistical hypothesis

Sometimes we make assumption about population on the basis of sample observation, we make assumption about population which is not necessarily true, are called statistical hypothesis.

Types of hypothesis

There are 2 types of hypothesis

1. Null hypothesis (H_0)

2. Alternative hypothesis (H_1)

(l) Null hypothesis (H_0)

The hypothesis tested for possible rejection under the assumption that it is true is called null hypothesis. The null hypothesis is a hypothesis which reflects no change or no difference. It is H_1) usually denoted by H_0.

(m) Alternative hypothesis (H_1)

The converse of the null hypothesis is called alternative hypothesis. It is denoted by H_1.

Example

If $H_0 : \mu_1 = \mu_2$ (There is no difference between the means)

The alternative Hypothesiswill be

(i) $H_1 : \mu_1 \neq \mu_2$ [Two tailed alternative Hypothesis]

either $H_1 : \mu_1 < \mu_2$ (or) $\mu_1 > \mu_2$[Right-tailed (or) Left tailed]

(n) Level of significance

It is the probability level below which the null hypothesis is rejected.

Generally, 5% (or) 1% is the level of significance.

Type - I & Type - II error

Type-I	**Type-II**
Rejected $H_0 \rightarrow$ when it is true.	Accepted $H_0 \rightarrow$ when it is false.

(o) Critical region *[A.U Tvli. M/J 2011] [A.U N/D 2017 R-8]*

A region, corresponding to a statistic *t*, in the sample space *S* which amounts to rejection of the null hypothesis H_0 is called as critical region or region of rejection.

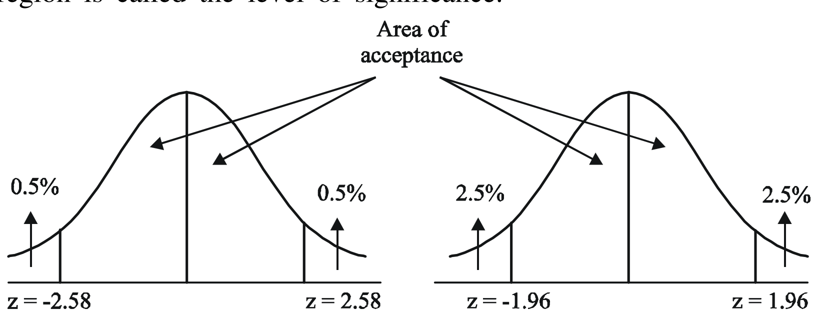
The region of the sample space *S* which amounts to the acceptance of H_0 is called acceptance region.

(p) Critical value or significant value

The value of the test statistic which separates the critical region from the acceptance region is called the critical value or significant value.

(q) Level of significance *[A.U. N/D 2013]*

The probability that the value of the statistics lies in the critical region is called the level of significance.



In general, these levels are chosen as 0.01 to 0.05, called 1% level and 5% level of significance respectively.

(r) Errors *[A.U N/D 2011, A.U CBT A/M 2011]* *[A.U A/M 2015 R-8] [A.U M/J 2016 (R13)]*

In sampling theory to draw valid inferences about the population parameter on the basis of the sample results, we decide to accept or to reject H_0 after examining a sample from it.

1.1. (a) Sampling distribution of means

Let $f(x)$ be the probability distribution of some given population from which we draw a sample of size n. Then it is natural to look for the probability distribution of the sample statistic $\overline{X}$, which is called the sampling distribution for the sample mean, or the sampling distributions of means.

Result

The mean of the sampling distribution of means, denoted by μ given by $E[\overline{X}] = \mu, \overline{x} = \mu$ where μ is the mean of the population.

Note

General procedure for hypothesis tests *[A.U N/D 2014]*

1. From the problem context, identify the parameter of intersect.
2. State the null hypothesis, H_0.
3. Specify an appropriate alternative hypothesis, H_1
4. Choose a significance level α.
5. Determine an appropriate test statistic.
6. State the rejection for the statistic.
7. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute the value.
8. **Conclusion:** Decide whether or not, H_0 should be rejected and report that in the problem context.

1.2 SAMPLING DISTRIBUTIONS

The probability distribution of a sample statistic is often called the sampling distribution of the statistic.

Alternatively we can consider all possible samples of size n that can be drawn from the population, and for each sample we composite the statistic. In this manner we obtain the distribution of the statistic, which is its sampling distribution.

The sample mean

Let $X_1, X_2, X_3, \ldots, X_n$ denote the independent, identically distributed, random variables for a random sample of size n.

Then the mean of the sample mean in a random variable defined by

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

If $x_1, x_2, \ldots, x_n$ denote values obtained in a particular sample of size n, then the mean for that sample is denoted by

$$\overline{x} = \frac{x_1, x_2 + \ldots + x_n}{n}$$

Example: If a sample of size 4 results in the sample values 7, 1, 6, 2, then the sample mean is

$$\overline{x} = \frac{7+1+6+2}{4} = \frac{16}{4} = 4$$

1.3 PARAMETRIC TEST

- Student t-test [single mean test] [For small and large sample]
- t-test [Two sample t-test for small and large sample]
- F-test [Variance test between two samples.

1.3.1 One sample t-test [Both small and large samples]

For single mean is given, (Both small and large samples)

Large sample	Small sample
$Z = \dfrac{\bar{x} - \mu}{\left(\dfrac{s}{\sqrt{n}}\right)}$	$t = \dfrac{\bar{x} - \mu}{\left(\dfrac{s}{\sqrt{n-1}}\right)}$

where, $\bar{x} =$ sample mean

$\mu =$ population mean

$s =$ standard deviation = S.D

$n =$ population size

Degrees of freedom $= n - 1$

Note:

(i) Large sample single mean

$$Z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} \text{ (or) } Z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n-1}}}$$

(ii) Variance $= s^2 = \sigma^2$

$$\text{S.D} = \sqrt{var} = \sqrt{s^2} = s = \sigma$$

Hypothesis setting:

(iii) H_0 : (population means are same)

H_1 : (population means are not same).

Applications

- To test the significance of the single mean.
- To test the significance of the difference between 2 sample mean.
- To test the significance of the coefficient of correlation.

Assumption of student *t*-test

- The parent population from which the sample is drawn is normal.
- The sample observations are independent.
- The population standard deviation, σ is unknown.

WORKED EXAMPLES

Example 1: A sample of 900 members has a mean 3.4 cm and standard deviation 2.61 cm. Is the sample from a large population of mean 3.25 cms and standard deviation of 2.61 cms? (Test at 5% level of significance. The value of z at 5% level is $|z_2| < 1.96$) *[A.U M/J 2010]*

[A.U A/M 2010 (R-08)] [A.U N/D 2016 (R13)]

Solution:

Given

$$n = 900,\ \mu = 3.25,\ s = 2.61,\ \bar{x} = 3.4,\ \alpha = 5\%$$

1. $H_0 : \bar{x} = \mu$
2. $H_1 : \bar{x} \neq \mu$ [Use two-tailed test]
3. $\alpha = 5\%$
4. Table value $|Z|$

   $|Z| = 1.96$

5. Calculation:

$$Z = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)} = \frac{3.4 - 3.25}{\left(\frac{2.61}{\sqrt{900}}\right)} = 1.724$$

6. Conclusion:

   Here, Cal $Z <$ Table Z

   i.e., 1.724 < 1.96,

   Hence we accept H_0 at 5% level of significance.

7. 95% confidence limits are

$$\bar{x} \pm 1.96 \frac{s}{\sqrt{n}} = 3.4 \pm (1.96)\left(\frac{2.61}{\sqrt{900}}\right)$$

$$= 3.4 \pm 0.1705 = 3.575 \text{ and } 3.2295$$

Example 2: The mean life time of a sample of 100 light bulbs produced by a company is computed to be 1570 hours with a standard deviation of 120 hours. It μ is the mean life time of all the bulbs produced by the company, test the hypothesis $\mu \neq 1600$ hours with $\alpha = 0.05$ and 0.01. *[A.U A/M 2003]*

[A.U N/D 2020 & 2021 (R-17)]

Solution:

Given

$n = 100, \mu = 1600, s = 120, \bar{x} = 1570$

$\alpha = 0.05$ and $\alpha = 0.01$

1. $H_0 : \bar{x} = \mu$
2. $H_1 : \bar{x} \neq \mu$ [Use two-tailed test]
3. (i) $\alpha = 5\%$; (ii) $\alpha = 1\%$
4. Table value

   (i) $\alpha = 5\%$, $|Z| = 1.96$, (ii) $\alpha = 1\%$ $|Z| = 2.58$
5. Calculate: $Z = \dfrac{\bar{x} - \mu}{\left(\dfrac{s}{\sqrt{n}}\right)} = \dfrac{1570 - 1600}{\left(\dfrac{120}{\sqrt{100}}\right)} = -2.5$ i.e., $|Z| = 2.5$
6. Conclusion:

   (i) Cal $Z \nless$ table Z (or) Cal $Z >$ table Z

   i.e., 2.5 > 1.96

   So, we reject H_0 at 5% level of significance.

   (ii) Cal $Z <$ table Z

   i.e., 2.5 < 2.58

   Hence we accept H_0 at 1% level of significance.

Example 3: A machinist is making engine parts with axle diameters of 0.700 inch. A random sample of 10 parts shows a mean diameter of 0.742 inch with a standard deviation of 0.04 inch. Test whether the work is meeting the specification at 5% level.

Solution:

$$\text{Formula } t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$$

H_0 : Population means are same

H_1 : Population means are not same

$\bar{x} = 0.742,\ \mu = 0.700,\ n = 10,\ s = 0.04$

$$t = \frac{0.742 - 0.700}{\left(\frac{0.40}{\sqrt{10-1}}\right)} = \frac{0.04}{\frac{0.04}{3}}$$

$$t = 3.15$$

Calculated value $= 3.15$.

Conclusion

Table value level of significance at 5%, $= 2.262$

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Conclusion: The populations means are not same.

Example 4: A random sample of nine men from a large city gave a mean height 68 inches and the unbiased estimate of the population variance from the sample was 4.5 inches. Test whether the mean height of men in the city is 68.5 inches at 5% level.

Solution:

$$\text{Formula, } t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$$

H_0 : Population means are same

H_1 : Population means are not same

$$\bar{x} = 68, \mu = 68.5, n = 9$$

$$\text{var} = s^2 = 4.5 \Rightarrow s = \sqrt{4.5}$$

$$s = 2.12$$

$$t = \frac{68 - 68.5}{\left(\frac{2.12}{\sqrt{9-1}}\right)} = \frac{0.5}{\left(\frac{2.12}{\sqrt{8}}\right)}$$

$$t = \frac{0.5}{0.749}$$

$$\boldsymbol{t = 0.667}$$

Conclusion

Calculated value $= 0.667$.

Table value level of significance at 5% level

$$\text{d.f} = n - 1 = 9 - 1 = 8$$

$$\text{Table value} = 2.306$$

Since table value is greater than calculated value H_0 is accepted.

Conclusion: The population means are same.

Example 5: From a large population of unemployed youths, a random sample of 25 is selected and an intelligence test given to them. From the test data, it was found that average I.Q is 97 with a standard deviation of 12. Are these data consistent with the hypothesis that unemployed youths were selected from a population of average intelligence (i.e) a population with an I.Q of 100?

Solution:

$$\text{Formula, } t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$$

H_0 : Population means are same

H_1 : Population means are not same

$\bar{x} = 97, \mu = 100, s = 12, n = 25$

$$t = \frac{97 - 100}{\left(\frac{12}{\sqrt{25-1}}\right)} = \frac{97 - 100}{\left(\frac{12}{\sqrt{24}}\right)}$$

$$t = \frac{-3}{2.44}$$

$$t = -1.22 \Rightarrow |t| = 1.22$$

Calculated value = 1.22.

Conclusion

Table value level of significance at 5% level,

$$\text{d.f} = n - 1 = 25 - 1 = 24$$

$$\text{Table value} = 2.064$$

Since table value is greater than calculated value H_0 is accepted.

Conclusion: Population means are same.

Example 6: The mean weekly sale of the ice cream bar was 146.3 bars. After an advertising campaign the mean weekly sale in 22 shops for a typical week increased to 153.7 & showed a S.D 17.2. Is this evidence that the advertising was successful?

Solution:

$$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$$

H_0 : Population means are same

H_1 : Population means are not same

$\bar{x} = 153.7, \mu = 146.3, s = 17.2, n = 22$

$$t = \frac{153.7 - 146.3}{\left(\frac{17.2}{\sqrt{22-1}}\right)} = \frac{7.4}{\left(\frac{17.2}{\sqrt{21}}\right)}$$

$$t = \frac{7.4}{3.753} = 1.973$$

Conclusion

Calculated value = 1.973

Table value level of significance at 5%

$$\text{d.f} = n - 1 = 22 - 1 = 21$$

$$\text{Table value} = 2.080$$

Since table value is greater than calculated value H_0 is accepted.

Conclusion: Population means are same.

Example 7: A sample of 20 items has mean 42 units and S.D 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.

Solution:

$$\text{Formula, } t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$$

H_0 : Population means are same.

H_1 : Population means are not same.

$$\bar{x} = 42, \mu = 45, s = 5, n = 20$$

$$t = \frac{42 - 45}{\left(\frac{5}{\sqrt{19}}\right)} = -2.616$$

$$t = -2.616 \Rightarrow |t| = 2.616$$

Conclusion

Calculated = 2.616

Table value level of significance at 5%

$$\text{d.f} = n - 1 = 20 - 1 = 19$$

$$\text{Table value} = 2.093$$

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Conclusion: Population means are not same.

Example 8: The mean life time of a sample of 25 bulbs is found as 1550 h with an S.D of 120 h. The company manufacturing the bulbs claims that the average life of their bulbs is 1600 h. Is the claim acceptable at 5% level?

Solution:

$$\text{Formula } t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$$

H_0 : Population means are same.

H_1 : Population means are not same.

$\bar{x} = 1550, \mu = 1600, s = 120, n = 25$

$$t = \frac{1550 - 1600}{\left(\frac{120}{\sqrt{25-1}}\right)}$$

$$t = \frac{50}{\left(\frac{120}{\sqrt{24}}\right)} = \frac{50}{24.495}$$

$$\boxed{t = 2.041}$$

Conclusion

Calculated value = 2.041

Table value level of significance at 5%.

$$\text{d.f} = n - 1 = 25 - 1 = 24$$

Table value = 2.064

Since table value is greater than calculated value H_0 is accepted.

Conclusion: Population means are same.

Student *t*-test (Raw data)

$$\text{Formula } t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

n = sample size, μ = population mean

$\bar{x} = ?, s = ?$

where $\bar{x} = \frac{\Sigma x}{n}$

$$s = \sqrt{\frac{\Sigma (x - \bar{x})^2}{n - 1}}$$

$\Sigma (x - \bar{x})^2 \rightarrow$ sum of squares of deviation.

$$\text{d.f} = n - 1$$

Level of significance at 5% or 1%

Example 9: From a population of college students 10 students are randomly selected their weekly pocket money was observed as 20, 22, 21, 15, 25, 19, 18, 20, 21, 22. Test whether the sample supports that on a average the student get Rs.25 as pocket money.

Solution:

$$n = 10, \mu = 25, \bar{x} = 20.3$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{203}{10}$$

$$\bar{x} = 20.3$$

x	$(x - \bar{x})$	$(x - \bar{x})^2$
20	− 0.300	0.090
22	1.70	2.89
21	0.7	0.49
15	− 5.3	28.09
25	4.7	22.09
19	− 1.3	1.69
18	− 2.3	5.29
20	− 0.300	0.09
21	0.7	0.49
22	1.7	2.89
$\Sigma x = 203$	$\Sigma (x - \bar{x})$	64.100

$$s = \sqrt{\frac{64.1}{9}}$$

$$s = \frac{8.0062}{3}$$

$$s = 2.6687$$

H_0 : Population means are same

H_1 : Population means are not same

Calculation

$$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

$$= \frac{20.3 - 25}{\frac{2.6687}{\sqrt{10}}} = \frac{-4.7}{0.8489}$$

$$t = -5.569$$

$$|t| = 5.569$$

Conclusion

Calculated value = 5.569

Table value level of significance at 5%,

$$\text{d.f} = n - 1 = 10 - 1 = 9$$

Table value = 2.262

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Conclusion: Population means are not same.

Example 10: A certain stimulus administered to each of 12 patients resulted in following increases of blood pressures: 5, 2, 8, – 1, 3, 0, 6, – 2, 1, 2, 5, 0, 4. Can it be concluded that the stimulus will be in general accompanied by an increase in blood pressure.

Solution.

$n = 12, \mu = 0$ (blood pressure at initial),

$$\bar{x} = \frac{\Sigma x}{n} = \frac{31}{12} = 2.58$$

x	$(x-\bar{x})$	$(x-\bar{x})^2$
5	2.420	5.856
2	– 0.580	0.336
8	5.420	29.376
– 1	– 3.580	12.816
3	0.420	0.176
0	– 2.580	6.656
6	3.420	11.696
– 2	– 4.580	20.976
1	– 1.580	2.496
5	2.420	5.856
0	– 2.580	6.656
4	1.420	2.016
Σx 31	$\Sigma (x-\bar{x})$	$\Sigma (x-\bar{x})^2 = 104.912$

$$s = \frac{\sqrt{\Sigma (x-\bar{x})^2}}{\sqrt{n-1}}$$

$$s = \frac{\sqrt{104.912}}{\sqrt{12-1}} = \frac{\sqrt{104.912}}{\sqrt{11}} = \frac{10.243}{3.317} = 3.088$$

H_0 : Population means are same.

H_1 : Population means are not same.

Calculation

$$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

$$= \frac{2.58 - 0}{\left(\frac{3.088}{\sqrt{12}}\right)} = \frac{2.58}{0.891}$$

$$t = 2.896$$

Conclusion

Calculated value = 2.896

Table value level of significance at 5%

$$\text{d.f} = n - 1 = 12 - 1 = 11$$

Table value = 2.201

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Conclusion: Population means are not same.

Example 11: A random sample of 16 values from a normal population showed a mean of 41.5 inches & sum of squares of deviations from this mean equal to 135 square inches and the assumption of a mean of 43.5 for the population is not reasonable.

Solution:

Given data: $n = 16, \Sigma (x - \bar{x})^2 = 135, \bar{x} = 41.5, \mu = 43.5, s = ?$

$$s^2 = \frac{\Sigma (x - \bar{x})^2}{n - 1} = \frac{135}{15} = 9 \Rightarrow s^2 = 9, \ s = 3$$

H_0 : Population means are same.

H_1 : Population means are not same.

$$\text{Formula } t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

$$= \frac{41.5 - 43.5}{\left(\frac{3}{\sqrt{16}}\right)} = \frac{-2}{0.750}$$

$$t = -2.667 \quad |t| = 2.667$$

Conclusion

Calculated value = 2.667

Table value level of significance at 5%

$$\text{d.f} = n - 1 = 16 - 1 = 15$$

$$\text{Table value} = 2.131$$

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Conclusion: Population means are not same.

Example 12: Ten individuals are chosen at random from a population 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of this data, discuss the suggestion that the mean height in the universe is 66 inches.

Solution:

$$n = 10, \mu = 66$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{678}{10} = 67.8$$

x	$(x-\bar{x})$	$(x-\bar{x})^2$
63	− 4.800	23.04
63	− 4.800	23.04
66	− 1.800	3.24
67	− 0.800	0.640
68	0.200	0.040
69	1.200	1.440
70	2.200	4.840
70	2.200	4.840
71	3.200	10.240
71	3.200	10.240
$\Sigma x = 678$		$\Sigma (x-\bar{x})^2 = 81.6$

$$s = \frac{\sqrt{\Sigma (x-\bar{x})^2}}{\sqrt{n-1}} = \sqrt{\frac{81.6}{9}} = \frac{9.033}{3} \quad s = 3.011$$

H_0 : Population means are same.

H_1 : Population means are not same.

$$\text{Formula } t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n}}\right)} = \frac{67.8-66}{\left(\frac{3.0}{\sqrt{10}}\right)} = \frac{1.8}{0.949} = 1.897$$

Calcualted value $= 1.897$

Table value level of significance at 5%

$$\text{d.f} = n-1 = 10-1 = 9$$

$$\text{Table value} = 2.262$$

Since table value is greater than calculated value H_0 is accepted.

Conclusion: Population means are same.

Example 13: A random sample of 10 boys had the following I.Q's 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do these data support the assumption of a population mean I.Q of 100?

Solution:

Given data

$$n = 10, \mu = 100$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{972}{10} = 97.2$$

x	$(x - \bar{x})$	$(x - \bar{x})^2$
70	− 27.200	739.84
120	22.800	519.84
110	12.800	163.84
101	3.800	14.44
88	− 9.200	84.64
83	− 14.200	201.64
95	− 2.200	4.84
98	0.800	0.64
107	9.800	96.04
100	2.800	7.84
$\Sigma x = 972$		$\Sigma (x - \bar{x})^2 = 188.36$

$$S = \frac{\sqrt{\Sigma (x - \bar{x})^2}}{\sqrt{n - 1}} = \frac{\sqrt{1883.6}}{\sqrt{10.1}} = 14.274$$

H_0 : Population means are same.

H_1 : Population means are not same.

Calculation

$$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)} = \frac{97.2 - 100}{\left(\frac{14.274}{\sqrt{10}}\right)} = \frac{-2.8}{\left(\frac{14.274}{3.162}\right)}$$

$$t = \frac{-2.8}{4.514} = -0.6202, \; |t| = 0.6202$$

Conclusion

Calculated value = 0.6202

Table value level of significance at 5%

$$\text{d.f} = n - 1 = 10 - 1 = 9$$

Table value = 2.262

Since table value is greater than calculated value H_0 is accepted.

Conclusion: Population means are same.

Example 14: Suppose that the annual rainfall x at a certain place is normally distributed with $E(x) = 30$. Data for the past 8 years are given as follows.

year:	1983	1984	1985	1986	1987	1988	1989	1990
x:	34.1	33.7	27.4	31.1	30.9	35.2	28.4	32.1

We want to test $H_0 : \mu = 30$ $H_1 : \mu \neq 30$

Are we to reject H_0 at 5% level?

Solution:

$$n = 8, \mu = 30$$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{252.9}{8} = 31.618$$

x	$(x-\bar{x})$	$(x-\bar{x})^2$
34.1	2.5	6.25
33.7	2.1	4.41
27.4	− 4.2	17.64
31.1	− 0.5	0.25
30.9	− 0.7	0.49
35.2	3.6	12.96
28.4	− 3.2	10.24
32.1	0.5	0.25
$\Sigma x = 252.9$		$\Sigma (x-\bar{x})^2 = 52.49$

$$s \frac{\sqrt{\Sigma (x-x)^2}}{\sqrt{n-1}} = \frac{\sqrt{52.49}}{\sqrt{7}} = \frac{7.245}{2.646} = 2.738$$

H_0 : Population means are same.

H_1 : Population means are not same.

$$\text{Calculated value } t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n}}\right)} = \frac{31.61-30}{\left(\frac{2.738}{\sqrt{8}}\right)} = \frac{1.610}{\left(\frac{2.738}{2.828}\right)}$$

$$t = \frac{1.610}{0.968} = 1.663$$

$$t = 1.663$$

Conclusion: Calculated value = 1.663

Table value at 5% level of significance d.f $= n-1 = 8-1 = 7$

Table value = 2.365

Since table value is greater than calculated value H_0 is accepted

Conclusion: Population means are same.

Example 15: Sandal powder is packed into packets by a machine. A random sample of 12 packets to drawn & their weight are found to be (in kg) 0.49, 0.48, 0.47, 0.48, 0.49, 0.50, 0.51, 0.49, 0.48, 0.50, 0.51 & 0.48. Test if the average weight of the packing can be taken as 0.5 kg at 5% level of significance.

Solution:

Given data $n = 12, \mu = 0.5, \bar{x} = ?, s = ?$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{5.88}{12} = 0.49$$

x	$(x - \bar{x})$	$(x - \bar{x})^2$
0.49	0.0	0
0.48	− 0.01	0.0001
0.47	− 0.02	0.0004
0.48	− 0.01	0.0001
0.49	0	0
0.50	0.01	0.0001
0.51	0.02	0.0004
0.49	0	0
0.48	− 0.01	0.001
0.50	0.01	0.0001
0.51	0.02	0.0004
0.48	− 0.01	0.0001
$\Sigma x = 5.88$		$\Sigma (x - \bar{x})^2 = 0.0018$

$$s = \frac{\sqrt{\Sigma (x - \bar{x})^2}}{\sqrt{n-1}} = \frac{\sqrt{0.0016}}{\sqrt{11}} = \frac{0.542}{3.317} = 0.013$$

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{0.49 - 0.5}{\frac{0.013}{\sqrt{12}}}$$

$$t = \frac{-0.01}{\frac{0.013}{3.364}} = \frac{-0.01}{0.004} = -2.5 \quad |t| = 2.5$$

Conclusion

Calculated value $= 2.5$

Table value level of significance at 5%

$$\text{Table value} = 2.179$$

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Conclusion: Population means are not same.

Example 16: Ten specimens of copper wires drawn from a larger lot have the following breakingStrength (in Kg.unit) 578, 572, 570, 568, 572, 578, 570, 572, 569, 548 Test whether the mean breaking strengths of the lot may be taken to be 578 kg weight.

Solution:

Given data $n = 10, \mu = 578, \bar{x} = ?, s = ?$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{5697}{10} = 569.7$$

x	$(x-\bar{x})$	$(x-\bar{x})^2$
578	8.3	68.89
572	2.3	5.29
570	0.3	0.09
568	− 1.7	2.89
572	2.3	5.29
578	8.3	68.89
570	0.3	0.09
572	2.3	5.29
569	− 0.7	5.29
548	− 21.7	470.89
$\Sigma x = 5697$		$\Sigma (x-\bar{x})^2 = 628.1$

$$s = \frac{\sqrt{\Sigma (x-\bar{x})^2}}{\sqrt{n-1}},\ s = \frac{\sqrt{628.1}}{\sqrt{9}} = \frac{25.06}{3} = 8.353$$

Calculation

$$t = \frac{\bar{x}-\mu}{\left(\frac{s}{\sqrt{n}}\right)} = \frac{569.7-578}{\left(\frac{8.353}{\sqrt{10}}\right)}$$

$$= \frac{-8.3}{\left(\frac{8.353}{3.162}\right)} = \frac{8.3}{2.641} = -5.142$$

$$|t| = 3.142$$

Conclusion

Calculated value $= 3.142$

Table value level of significance at 5%

$$d.f = n - 1 = 10 - 1 = 9$$

$$\text{Table value} = 2.262$$

Since table value is less than calculated value, H_0 is rejected, H_1 is accepted.

Conclusion: Population means are not same.

1.4 DIFFERENCE OF MEANS [SMALL AND LARGE SAMPLES]

t-test is used to test the significance difference between the two sample means (average).

Sample size:	n_1, n_2
Sample mean:	$\bar{x}_1, \bar{x}_2$
S.D	S_1, S_2

Applications

- To test the significance of single mean (student *t*-test).
- To test the significance of difference between sample means

Note:

(i) Large sample : $n \geq 30$

(ii) Small sample : $n < 30$

Large sample

For $n \geq 30$

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_1^2}{n_2}}}$$

Small sample

For $n < 30$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Case 1:

S.D $\rightarrow \sigma, s$ (Given) (Standard derviation given)

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

Case 2:

Sum of squares is given

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

Case 3:

Sum of squares of deviation given

$$S^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

Degrees of freedom

$$\text{d.f} = n_1 + n_2 - 2$$

Large sample t-test

Table value at 5% Los = 1.960

at 1% los = 2.572.

WORKED EXAMPLES

Example 1: In a random sample of size 500, the mean is found to be 20. In another independent sample of size 400, the mean is 15. Could the samples have been drawn from the same population with S.D 4? *[A.U A/M 2022 (R-21)]*

Solution:

Given

$n_1 = 500, \quad \overline{x_1} = 20, \quad \sigma_1 = \sigma_2 = 4 \text{ (or) } S_1 = S_2 = 4$

$n_2 = 400, \quad \overline{x_2} = 15,$

1. $H_0 : \mu_1 = \mu_2$ (or) $\overline{x_1} = \overline{x_2}$

2. $H_1 : \mu_1 \neq \mu_2$ (or) $\overline{x_1} \neq \overline{x_2}$ [Use two-tailed test]

3. $\alpha = 1\%$

4. Table value $|Z| = 2.58$

5. Calculation

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{20 - 15}{\sqrt{\frac{(4)^2}{500} + \frac{(4)^2}{400}}} = 18.6$$

6. Conclusion:

   Here Cal $Z >$ table Z

   i.e., 18.6 > 2.58

   Hence we reject H_0 at 1% level of significance.

Example 2: A sample of heights of 6400 Englishmen has a mean of 67.85 inches and a S.D of 2.56 inches, while a sample of heights of 1600 Australians has a mean of 68.55 inches and a S.D. of 2.52 inches. Do the data indicate that Australians are on the average taller than Englishmen?

[A.U. N/D 2007] [A.U. N/D 2019 R-17]

Solution:

$$n_1 = 6400\,, \quad \overline{x_1} = 67.85, \quad s_1 = 2.56$$

$$n_2 = 1600\,, \quad \overline{x_2} = 68.85, \quad s_2 = 2.52$$

$\mu_1 \rightarrow$ mean height of the population of Englishmen

$\mu_2 \rightarrow$ mean height of the population of Australian

1. $H_0 : \mu_1 = \mu_2$ (or) $\overline{x_1} = \overline{x_2}$ [No significant difference]
2. $H_1 : \mu_1 < \mu_2$ (or) $\overline{x_1} < \overline{x_2}$ [one-tailed test (left)]
3. $\alpha = 5\%$ and $\alpha = 1\%$
4. Table value

   If $\alpha = 1\%$ then $|Z| = 2.33$

   If $\alpha = 1\%$ then $|Z| = 1.645$
5. Calculate:

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{67.85 - 68.55}{\sqrt{\frac{(2.52)^2}{6400} + \frac{(2.52)^2}{1600}}} = -9.906$$

$$|Z| = 1 - 9.9061$$

$$Z = 9.906$$

6. Conclusion:

   (i) Here Cal $Z >$ table Z

   i.e., 9.90 > 1.645

Hence we reject H_0 at 5% level of significance.

(ii) Here Cal $Z >$ table Z

$$\text{i.e., } 9.906 > 2.33$$

So, we reject H_0 at 1% level of significance.

Example 3: A sample of students were drawn from 2 universities and from their weight (in kg), the *t*-test means and the standard deviation of calculator. Test the significance of the difference between the means of the two sample.

	Sample mean	S.D	Sample size
University *A*	**55**	**10**	**400**
University *B*	**57**	**15**	**100**

Solution:

$n \geq 30 \rightarrow$ Large sample

H_0 : Means are same

H_1 : Means are not same

Given data

$n_1 = 400$ $s_1 = 10$ $\bar{x}_1 = 55$

$n_2 = 100$ $s_2 = 15$ $\bar{x}_2 = 57$

Calculation

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_1^2}{n_2}}}$$

$$= \frac{55 - 57}{\sqrt{\frac{(10)^2}{400} + \frac{(15)^2}{100}}}$$

$$= \frac{-2}{\sqrt{0.25 + 2.25}} = \frac{-2}{1.58}$$

$$Z = -1.26$$

$$|Z| = 1.26$$

Conclusion

Calculated value = 1.26

Table value level of significance at 5%

Table value = 1.96

Since table value is greater than calculated value H_0 is accepted.

Therefore means are same.

Example 4: Test the significance of the difference between means of the sample drawn from 2 normal population with the samples used in the following data.

	Sample mean	S.D	Sample size
Sample 1	**61**	**4**	**400**
Sample 2	**63**	**6**	**200**

Solution:

H_0 : means are same.

H_1 : means are not same.

$n \geq 30 \rightarrow$ Large sample

Given data

$n_1 = 100$ $s_1 = 4$ $\bar{x}_1 = 61$

$n_2 = 200$ $s_2 = 6$ $\bar{x}_2 = 63$

Calculation

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_1^2}{n_2}}}$$

$$= \frac{61 - 63}{\sqrt{\frac{(4)^2}{100} + \frac{(6)^2}{200}}}$$

$$= \frac{-2}{\sqrt{0.16 + 0.18}}$$

$$= \frac{-2}{\sqrt{0.34}} = \frac{-2}{0.583}$$

$$Z = -3.43 \rightarrow |Z| = 3.43$$

Conclusion

Calculated value = 3.43

Table value level of significance at 5%

Table value = 1.96

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Means are not same.

Example 5: A buyer of electric bulbs purchases 400 bulbs, 200 bulbs of each brand upon testing there bulbs. He found that brand *A* has average 1225 hrs with standard deviation of 42 hrs. Whereas brand *B* its mean life of 1265 hours with standard deviation of 60 hrs can the buyer be contained that *B* is superior than *A* is quickly.

Given data

	Sample mean	**S.D**	**Sample size**
Brand *A*	1225	42 hrs	400
Brand *B*	**1265**	60 hrs	200

Solution:

$n \geq 30 \rightarrow$ Large sample.

Given data

$n_1 = 400 \qquad s_1 = 42 \qquad \bar{x}_1 = 1225$

$n_2 = 200 \qquad s_2 = 60 \qquad \bar{x}_2 = 1265$

Calculation

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_1^2}{n_2}}}$$

$$= \frac{1225 - 1265}{\sqrt{\frac{(42)^2}{400} + \frac{(60)^2}{200}}}$$

$$Z = \frac{-40}{\sqrt{\frac{1764}{400} + \frac{3600}{200}}} = \frac{-40}{\sqrt{4.41 + 18}} = -8.457$$

$$|Z| = 8.457$$

Conclusion:

Calculation value = = 8.457

Table value Level of significance at 5%

$$\text{Table value} = 1.96$$

Since table value is less than calculated value H_0 is rejected.

Means are not same.

Example 6: A test of the breaking strength of difference type of cables was conducted using sample of *100 pieces of each type* of cable.

Cable 1	Cable 2
$\bar{x}_1 = 1925$	$\bar{x}_2 = 1905$
$\sigma_1 = 40$	$\sigma_2 = 30$

Do the data provide sufficient evidence to indicate the difference between the mean breaking strength of two cable.

Solution

H_0 : means are same

H_1 : means are not same

t-test; $n \geq 30 \rightarrow$ Large sample

$n_1 = 100$	$\bar{x}_1 = 1925$	$\sigma_1 = 40$
$n_2 = 100$	$\bar{x}_2 = 1905$	$\sigma_2 = 30$

Calculation

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_1^2}{n_2}}}$$

$$= \frac{1925 - 1905}{\sqrt{\frac{(40)^2}{100} + \frac{(30)^2}{100}}}$$

$$Z = \frac{20}{\sqrt{16+9}} = \frac{20}{\sqrt{25}} = \frac{20}{5}$$

$$Z = 4$$

Conclusion

Calculated value = 4

Table value level of significance at 5%

Table value = 1.96

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Means are not same.

Small sample [$n < 30$]

$$\text{d.f} = n_1 + n_2 - 2$$

Example 7: Two independent samples from normal population with equal variances gave the following results.

Sample	Size	Mean	S.D
1	16	23.4	2.5
2	12	24.9	2.8

Solution:

Test for the equality of means t-test

***t*-test: $n < 30$ (small sample):**

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{16\,(2.5)^2 + 12\,(2.8)^2}{16 + 12 - 2}$$

$$= \frac{194.08}{26}$$

$$S^2 = 7.46$$

$$\boldsymbol{S = 2.73}$$

$$t = \frac{23.4 - 24.9}{2.73\sqrt{\frac{1}{16} + \frac{1}{12}}} = \frac{1.5}{1.04}$$

$$t = -\,1.442 \Rightarrow |t| = 1.442$$

Conclusion

Calculated value = 1.442

Table value level of significance at 5%

d.f = $n_1 + n_2 - 2 = 16 + 12 - 2$

d.f = 26

Tablevalue = 2.056

Since table valueis greater than calculated value H_0 is accepted.

Means are same.

Example 8: The means of 2 random sample of sizes 9 and 7 respectively 196.42 and 192.82. The sum of squares of the deviations from the means are 26.94 & 18.73 respectively, can the samples be considered to have been drawn from the same normal population?

Sample	**Size**	**Mean**	$\Sigma\,(x - \bar{x})^2$
1	**9**	**196.42**	**26.94**
2	**7**	**198.82**	**18.73**

$n < 30$ (small sample).

H_0 : means are same.

H_1 : means are not same.

Solution:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$S^2 = \frac{\Sigma\,(x_1 - \bar{x}_1)^2 + \Sigma\,(x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{26.94 + 18.73}{9 + 7 - 2}$$

$$S^2 = \frac{45.67}{14}$$

$$S^2 = 3.262$$

$$\boxed{\mathbf{S = 1.806}}$$

Calculation

$$t = \frac{196.42 - 192.82}{1.806\sqrt{\frac{1}{9} + \frac{1}{7}}}$$

$$t = \frac{3.6}{0.908}$$

$$t = 3.964$$

Conclusion

Calculated value $= 3.964$

Table value level of significance at 5%

$$\text{d.f} = 14$$

$$\text{Table value} = 2.145$$

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Means are not same.

Example 9: The heights of 6 randomly chosen sailors (in inches) are 72, 71, 69, 68, 65, 63. The heights of 10 randomly selected soldiers are 73, 72, 71, 70, 69, 69, 66, 65, 62, 61. Discuss in the light of the data, that soldiers are on the average shorter than sailors.

Solution:

$n_1 = 6, n_2 = 10$ $(n < 30 \rightarrow \text{ small sample})$

H_0 : Means are same.

H_1 : Means are not same.

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{408}{6} = 68$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{678}{10} = 67.8$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

x_1	x_2	x_1^2	x_2^2
72	73	5184	5329
71	72	5041	5184
69	71	4761	5041
68	70	4624	4900
65	69	4225	4761

x_1	x_2	x_1^2	x_2^2
63	69	3969	4761
	66		4356
	65		4225
	62		3844
	61		3721
$\Sigma x_1 = 408$	$\Sigma x_2 = 678$	$\Sigma x_1^2 = 27804$	$\Sigma x_2^2 = 46122$

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} = \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$= \frac{27804}{6} - \left(\frac{408}{6}\right)^2$$

$$= 4634 - 4624$$

$$\mathbf{s_1^2 = 10}$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$$

$$= \frac{46122}{10} - \left(\frac{678}{10}\right)^2$$

$$= 4612.2 - 2596.84$$

$$\mathbf{s_2^2 = 15.36}$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{6\,(10) + 10\,(15.36)}{6 + 10 - 2}$$

$$s^2 = 15.25$$

$$\mathbf{S = 3.905}$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$t = \frac{68 - 67.8}{3.905\sqrt{\frac{1}{6} + \frac{1}{10}}} = \frac{0.2}{2.016}$$

$$t = 0.09$$

Conclusion

Calculated value $= 0.09$

Table value level of significance at 5%

$$\text{d.f} = n_1 + n_2 - 2$$

$$= 6 + 10 - 2$$

$$\text{d.f} = 14$$

Table value $= 2.145$

Since table value is greater than calculated value H_0 is accepted.

Means are same.

Example 10: Samples of 2 types of electric tubes were tested for length of life and the following were obtained.

	Number	Mean	S.D
Type I:	**10**	**1240 hrs**	**37 hrs**
Type II:	**8**	**1042 hrs**	**40 hrs**

Solution:

Given data

$$n_1 = 10, n_2 = 8$$

$$\overline{x_1} = 1240 \quad \overline{x_2} = 1042$$

$$s_1 = 37 \quad s_2 = 40$$

***t*-test: *n* < 30 (small sample)**

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{10\,(37)^2 + 8\,(40)^2}{10 + 8 - 2}$$

$$= \frac{26{,}490}{16}$$

$$S^2 = 1655.6 \Rightarrow s = \sqrt{1655.6}$$

$$\mathbf{S = 40.68}$$

$$t = \frac{1240 - 1042}{40.68\sqrt{\frac{1}{8} + \frac{1}{8}}}$$

$$= \frac{198}{40.68\sqrt{0.225}} = \frac{198}{19.28}$$

$$t = 10.27$$

Conclusion

Calculated value = 10.27

Table value level of significance at 5%

$$\text{d.f} = n_1 + n_2 = 2 = 10 + 8 - 2 = 16$$

Table value = 2.120

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Conclusion: Means are not same.

Example 11: An experiment was conduct on a set of nine plants treated with hormones & another set of 10 plants without treatment. The number of pots per plant was noted and the data summarized are given below. Examine whether hormone treatment exercises real effect on the number of pots per plant.

	Size	Mean number of pots	Sum of squares deviation
Sample I	9	22.3	112.7
Sample II	10	19.3	98.6

Solution:

H_0 : Means are same.

H_1 : Means are not same.

Given data

$$n_1 = 9,\ n_2 = 10$$

$$\overline{x_1} = 22.3,\ \overline{x_2} = 19.3$$

$$\Sigma\,(n_1 - \overline{x_1})^2 = 112.7 \quad \Sigma\,(x_2 - \overline{x_2})^2 = 98.6$$

$$t = \frac{\overline{x}_1 - \overline{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$S^2 = \frac{\Sigma\,(x_1 - \overline{x_1})^2 + \Sigma\,(x_2 - \overline{x_2})}{n_1 + n_2 - 2}$$

$$S^2 = \frac{112.7 + 98.6}{1.9 + 10 - 2} = \frac{211.3}{17}$$

$$S^2 = 12.429 \Rightarrow \mathbf{S = 3.52}$$

$$t = \frac{22.3 - 19.3}{3.52\sqrt{\frac{1}{9} + \frac{1}{10}}} = \frac{3}{3.52\sqrt{0.211}}$$

$$t = \frac{3}{3.52\,(0.459)} = \frac{3}{1.616}$$

$$t = 1.85$$

Conclusion

Calculated value = 1.85

Table value level of significance at 5%

$$\text{d.f} = n_1 + n_2 = 2 = 17$$

Table value = 2.110

Since table value is greater than calculated value H_0 is accepted.

Means are same.

Example 12: Samples of types of electric bulbs were tested for length of life and the following data were obtained.

	Size	**Mean**	**S.D**
Sample 1	**8**	**1234*h***	**36*h***
Sample 2	**7**	**1036*h***	**40*h***

Given data

$n_1 = 8,\ n_2 = 7\quad \overline{x_1} = 1234\quad \overline{x_2} = 1036$

$s_1 = 36\quad s_2 = 40$

Solution:

t-test; $n < 30$ (small sample)

H_0 : Means are same

H_1 : Means are not same.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$S^2 = \frac{n_1\, s_1^2 + n_2\, s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{8\,(36)^2 + 7\,(40)^2}{8 + 7 - 2}$$

$$S^2 = \frac{21568}{13} = 1659.07, \quad S = 40.73$$

$$t = \frac{1234 - 1036}{40.73\sqrt{\frac{1}{8} + \frac{1}{7}}}$$

$$t = \frac{198}{40.73\sqrt{0.268}} = \frac{198}{21.098}$$

$$t = 9.385$$

Conclusion

Calculated value $= 9.385$

Table value level of significance at 5%

$$\text{d.f} = n_1 + n_2 - 2 = 8 + 7 - 2 = 15 - 2 = 13$$

Table value $= 2.160$

Since table value is lessthan calculated value H_0 is rejected, H_1 is a accepted.

Conclusion: Means are not same.

Example 13: In a test examination given to 2 groups of students, the marks obtained were as follows:

I group:	**18**	**20**	**36**	**50**	**49**	**36**	**34**	**49**
II group:	**29**	**28**	**26**	**35**	**30**	**44**	**46**	

Examine the significance of difference between average marks secured by the students of the above 2 groups.

Solution:

t-test ($n < 30 \rightarrow$ small sample)

H_0 : Means are same.

H_1 : Means are not same.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

x_1	x_2	$(x_1 - 37)^2$	$(x_2 - 34)^2$
18	29	361	35
20	28	289	36
36	26	1	64
50	35	169	1
49	30	144	16
36	44	1	100
34	46	9	144
49		144	
41		16	
$\Sigma x_1 = 333$	$\Sigma x_2 = 238$	$\Sigma (x_1 - \bar{x}_2)^2 = 1134$	$\Sigma (x_2 - \bar{x}_2)^2 = 386$

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{333}{9} = 37 \; ; \; \bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{238}{7} = 34$$

$$S^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - x_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{1134 + 386}{9 + 7 - 2}$$

$$S^2 = 108.571$$

$$\mathbf{S = 10.42}$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{37 - 34}{10.42\sqrt{\frac{1}{9} + \frac{1}{7}}}$$

$$t = \frac{3}{5.25}$$

$$t = 0.575$$

Conclusion

Calculated value = 0.575

Table value level of significance at 5%

$$\text{d.f} = n_1 + n_2 - 2 = 14$$

Table value = 2.145

Since table value is greater than calculated value H_0 is accepted.

Means are same.

1.5 PROPORTION TEST

Large sample test for single proportion

1. Test for a single proportion

If X is the number of successes in n independent trials with constant probability P of success for each trial.

$$E(X) = nP \text{ and } V(X) = nPQ$$

where $O = 1 - P$, is the probability of failure.

It has been proved that for large n, the binomial distribution tends to normal distribution. Hence for large n, $X - N(nP, nPQ)$ i.e.,

$$Z = \frac{X - E(X)}{\sqrt{V}(X)} = \frac{X - nP}{\sqrt{nPQ}} = N(0, 1)$$

and we can apply the normal test.

WORKED EXAMPLE

Example 1: In a sample of 1,000 people in Maharashtra, 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance.

Solution:

Given

$$n = 1{,}000, X = 540, p = \frac{X}{n} = \frac{540}{1000} = 0.54$$

$P =$ Population proportion of rice eaters in Maharastra $= \frac{1}{2} = 0.5$

$$O = 1 - P = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$$

1. $H_0 : P = 0.5$ [Both rice and wheat eaters are equally popular in the state]
2. $H_1 : P \neq 0.5$

3. $\alpha = 0.01$
4. Table value $|Z| = 2.58$
5. Calculate:

$$Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.54-0.50}{\sqrt{\frac{(0.5)\,(0.5)}{1000}}} = 2.53$$

6. Conclusion:

Here, Cal $Z <$ table Z

i.e., $2.53 < 2.58$

So, we accept H_0 at 1% level of significance.

We may conclude that rice and wheat eaters are equally popular in Maharashtra state.

Example 2: Twenty people were attacked by a disease and only 18 survived. Will you reject the hypothesis that the survival rate, if attached by this disease, is 85% in favour of the hypothesis that it is more, at 5% level. (use large sample test).

[A.U. N/D 2013)]

Solution:

$X =$ number of person who survived after attack by a disease $= 18$

$$p = \frac{X}{20} = \frac{18}{20} = 0.90$$

$$P = 0.85$$

$$Q = 1 - 0.85 = 0.15$$

1. $H_0 : P = 0.85$ [no significant difference]
2. $H_1 : P > 0.85$ [One tail test]
3. $\alpha = 0.05$
4. Table value $|Z| = 1.645$

5. Calculate: $Z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{0.90 - 0.85}{\sqrt{\dfrac{(0.85)\,(0.15)}{20}}} = 0.626$

6. Conclusion:

Here, Cal $Z <$ table Z

i.e., 0.626 ~< 1.645

Hence we accept H_0 at 5% level of significance.

Example 3: A manufacturer claimed that at least 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipments revealed that 18 were faulty. Test his claim at a significance level of (i) 5% (ii) 1%.

Solution:

Given $n = 200$

$X =$ Number of pieces conforming to specifications in the samples

$= 200 - 18 = 182$

$p =$ Sample proportion conforming to specifications $= \dfrac{182}{200} = 0.91$

$P = 0.95, \quad Q = 1 - P = 1 - 0.95 = 0.05$

1. $H_0 : P = 0.95$ [The proportion of pieces conforming to specification in the population is 95%]
2. $H_1 : P < 0.95$ [at least 95% conformed to the specification]
3. (i) $\alpha = 0.05$, (ii) $\alpha = 0.01$
4. Table value

   (i) $|Z| > 1.645$ at 5% level. (ii) $|Z| > 2.33$ at 1% level.
5. Calculate: $Z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}} = \dfrac{0.91 - 0.95}{\sqrt{\dfrac{(0.95)\,(0.05)}{200}}} = -2.6$

6. Conclusion:

(i) Here, Cal $Z >$ table Z

i.e., $2.6 > 1.645$

So, we reject H_0 at 5% level of significance.

(ii) Here, Cal $Z >$ table Z

i.e., $2.6 > 2.33$

So, we reject H_0 1% level of significance.

Large sample test for difference of proportions

Suppose we want to compare 2 district populations with regard possession of an attribute. Let a sample of size n_1 be chosen from the first population and another sample of size n_2 be chosen from the second population. Let X_1 be the number of persons possessing the attribute A in the first sample and let X_2 be the number of persons possessing the same attribute in the second sample.

Then $p_1 = \frac{X_1}{n_1}, p_2 = \frac{X_2}{n_2}$

As before $E(p_1) = P_1$ and $E(p_2) = P_2$ where P_1 and P_2 are the proportions in the populations.

$$V(p_1) = \frac{P_1 Q_1}{n_1} \text{ and } V(p_2) = \frac{P_2 Q_2}{n_2}$$

Since for large samples p_1 and p_2 are asymptotically normally distribution $p_1 - p_2$ is also normally distributed, and also

$$E(p_1 - p_2) = E(p_1) - E(p_2) = P_1 - P_2 \text{ and}$$
$$V(p_1 - p_2) = V(p_1) + V(p_2).$$

$$\therefore \quad Z = \frac{(p_1 - p_2) - E(p_1 - p_2)}{\sqrt{V(p_1 - p_2)}}$$

$$= \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_2 Q_2}{n_2}}} \sim N(0, 1)$$

Under the null hypothesis $H_0 : P_1 = P_2 = P_2 = P$

(Hence $Q_1 = Q_2 = Q$) test statistic becomes

$$Z = \frac{p_1 - p_2}{\sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_2 Q_2}{n_2}}} \sim N(0, 1)$$

$$\therefore \quad Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

An unbiased estimate of population proportion P based on both the samples is given by $P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$. Suppose the population proportions P_1 and P_2 are given to be different. (i.e.,) $P_1 \neq P_2$. If we want to test whether the difference $P_1 - P_2$ is significant then the test statistic becomes

$$\therefore Z = \frac{(p_1 - p_2) - E(p_1 - p_2)}{S.E(p_1 - p_2)} = \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_1 Q_2}{n_2}}} \sim N(0, 1)$$

Here sample proportions are not given. If we set up a null hypothesis $H_0 : p_1 = p_2$ then test statistic becomes

$$|Z| = \frac{|p_1 - p_2|}{\sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_2 Q_2}{n_2}}} \sim N(0, 1)$$

WORKED EXAMPLES

Example 1: During a country wide investigation the incidence of TB was found to be 1%. In a college of 400 strength 5 were reported to be affected whereas in another college of 1200 strength 10 were reported to be affected. Does this indicate any significant difference.

Solution:

$$P = 1\% = \frac{1}{100} = 0.01 \; ; \quad Q = 1 - 0.01 = 0.99$$

$$n_1 = 400 \; ; \; n_2 = 1200 \; ; \quad p_1 = \frac{5}{400} = 0.0125 \; ; \; p_2 = \frac{10}{1200} = 0.0083$$

1. $H_0 : P_1 = P_2$ [no significant difference]
2. $H_1 : P_1 \neq P_2$
3. $\alpha = 0.05$
4. Table value

   $|Z| = 1.96$ at 5% level.

5. Calculate:

$$Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0.0125 - 0.0083}{\sqrt{(0.01)\,(0.99)\left(\frac{1}{400} + \frac{1}{1200}\right)}}$$

6. Conclusion:

   Here, Cal Z < table Z

   i.e., 0.7368 < 1.96

   So, we accept H_0 at 5% level of significance.

   Hence the difference is not significant.

Example 2: Random samples of 400 men and 600 women asked whether they would like to have a flyover near their residence. 200 men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and women infavour of the proposal, are same against that they are not, at 5% level. *[A.U Tvli M/J 2011]*

Solution:

Given:

$n_1 = 400,\quad X_1 =$ No. of men favouring the proposal $= 200$

$n_2 = 600,\quad X_1 =$ No. of women favouring the proposal $= 325$

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{X_1 + X_n}{n_1 + n_2}$$

$$Q = 1 - P$$

$$P = \frac{X_1 + X_2}{n_1 + n_2} = \frac{200 + 325}{400 + 600} = \frac{525}{1000} = 0.525$$

$$Q = 1 - P = 1 - 0.525 = 0.475$$

1. $H_0 : P_1 = P_2$ [there is no significant, difference]
2. $H_1 : P_1 \neq P_2$
3. $\alpha = 0.05$
4. Table value $|Z| = 1.96$
5. Calculate: $Z = \dfrac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

$$= \frac{0.500 - 0.541}{\sqrt{(0.525)\,(0.475)\left(\dfrac{1}{400} + \dfrac{1}{600}\right)}} = -\,1.269$$

$$\therefore\ |Z| = 1.269$$

6. Conclusion:

   Here, Cal Z < table Z

   i.e., 1.269 < 1.96

   So, we accept H_0 at 5% level of significance.

Hence we may conclude that men and women do not differ significantly as regards proposal of flyover is concerned.

1.6 F-DISTRIBUTION [TEST FOR EQUALITY OF VARIANCES] [SNEDECOR'S F-DISTRIBUTION]

1.6.1 The *F*-Distribution

Suppose that two independent normal populations are of interest, when the population means and variances, say μ_1, μ_2, σ_1^2 and σ_2^2, are unknown. We wish to test hypothesis about the equality of the two variances, say, H_0, $\sigma_1^2 = \sigma_2^2$. Assume that two random samples of size n_1 from population 1 and of size n_2 from population 2 are available, and let S_1^2 and S_2^2 be the sample variances. We wish to test the hypothesis.

$$H_0 : \sigma_1^2 = \sigma_2^2$$

$$H_1 : \sigma_1^2 \neq \sigma_2^2$$

The development of a test procedure for these hypothesis requires a new probability distribution called F distribution.

If s_1^2 and s_2^2 are the variances of two samples of sizes n_1 and n_2 respectively, the estimates of the population variance based on these samples are respectively $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$. The quantities $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ are called the degrees of freedom of these estimates. We want to test if these estimates S_1^2 and S_2^2 are significantly different or if the samples may be regarded as drawn from the same population or from two populations with same variance σ^2.

$$\text{Let } F = \frac{S_1^2}{S_2^2} = \frac{\dfrac{n_1 s_1^2}{n_1 - 1}}{\dfrac{n_2 s_2^2}{n_2 - 1}} \qquad \ldots (1)$$

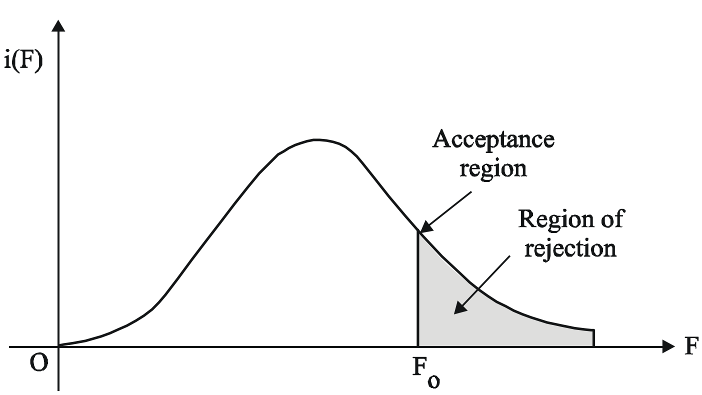
The sampling distribution of F is of the form.

$$\{y = f(F) = K \frac{F^{\frac{v_1 - 2}{2}}}{(v_1 F + v_2)^{\frac{v_1 + v_2}{2}}} \qquad \ldots (2)$$

where v_1, v_2 are the degrees of freedom of two estimates and K is got by

$$\int_0^\infty f(F)\, dF = 1$$

A rough sketch of this curve is given below:



If $S_1^2 = S_2^2$, then $F = 1$. Hence, our object is to find how far any observed value of F differs from unity, consistent with our assumption of the equality of the population variances.

The area of the curve $y = f(F)$ to the right of F_0 gives the probability that $F > F_0$. Here, F_0 is the critical value of F. If $F > F_0$ the difference of variances is significant. If $F < F_0$, the difference is not significant.

The critical values of F, say F_0, is got from F-table corresponding to degrees of freedom (v_1, v_2) at a level of significance.

In setting $F = \frac{S_1^2}{S_2^2}$, the numerator is greater than denominator

i.e., $S_1^2 > S_2^2$, so that $F > 1$.

$F = \frac{S_1^2}{S_2^2} = \frac{n_1 s_1^2 (n_2 - 1)}{(n_1 - 1)}$ where s_1^2, s_2^2 are variances of two samples and

$$S_1^2 = \frac{\Sigma (x - \bar{x})^2}{n_1 - 1}, S_2^2 = \frac{\Sigma (x - \bar{x})^2}{n_2 - 1} \text{ and } S_1^2 > S_2^2$$

If $F < F_0$, critical value, the difference is not significant and if $F > F_0$, the difference is significant.

Note

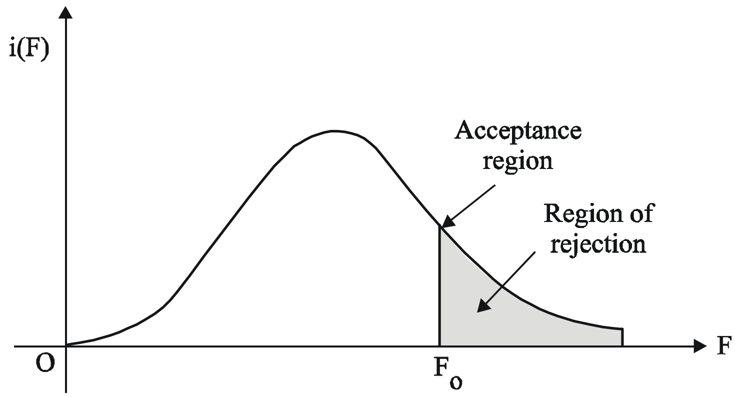
$$P > 0 \text{ always.}$$

Applications

(i) F-test is used to test (i) whether two independent samples have been drawn from the normal populations with the same variances σ^2, or

(ii) Whether the two independent estimates of the population variation are homogeneous or not.

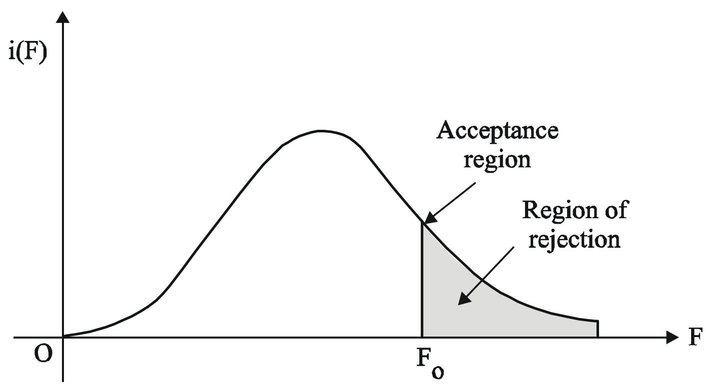
$$F_\alpha (v_1, v_2)$$



Here, $P\,(F > F_{\alpha}\,(v_1, v_2)) = \alpha$

Refer to tables for critical $F_{\alpha}\,(v_1, v_2)$

1.6.2 Properties of the F-distribution



1. The probability curve of the *F*-distribution is roughly sketched below.
2. The square of the *t*-variate with *n* degrees of freedom follow *F*-distribution with *l* and *n* degrees of freedom.
3. The mean of the *F*-distribution is $\dfrac{v_2}{v_2 - 2}\,(v_2 > 2)$
4. The variance of the *F*-distribution is

$$\frac{(2v_2^2\,(v_1 + v_2 - 2)}{v_1\,(v_2 - 2)^2\,(v_2 - 4)}\,(v_2 > 4)$$

***F*-Test**

F-test is used to test the significant difference between 2 estimates of **population variance** (or) to test if the 2 samples have come from the **same population**.

Formula

Test statistics:

$$F = \frac{S_1^2}{S_2^2} \text{ if } S_1^2 > S_2^2 \qquad\qquad F = \frac{S_2^2}{S_1^2} \text{ if } S_2^2 > S_1^2$$

$$\text{d.f} = (n_1 - 1, n_2 - 1) \qquad\qquad \text{d.f} = (n_2 - 1, n_1 - 1)$$

Case 1: S.D (s) / Variance (S^2) is given

$$S_1^2 = \frac{n_1\, s_1^2}{n_1 - 1}\ ;\ s_2^2 = \frac{n_2\, s_2^2}{n_2 - 1}$$

Case 2: Sum of squares of deviations is given

$$S_1^2 = \frac{\Sigma\,(x_1 - \bar{x}_1)^2}{n_1 - 1}$$

$$S_2^2 = \frac{\Sigma\,(x_2 - \bar{x}_2)^2}{n_2 - 1}$$

Case 3: Sum of squares is given

$$s_1^2 = \frac{\Sigma\, x_1^2}{n_1} - \left(\frac{\Sigma\, x_1}{n_1}\right)^2$$

$$s_2^2 = \frac{\Sigma\, x_2^2}{n_2} - \left(\frac{\Sigma\, x_2}{n_2}\right)^2$$

$$S_1^2 = \frac{n_1\, s_1^2}{n_1 - 1}\ ;\ S_2^2 = \frac{n_2\, s_2^2}{n_2 - 1}$$

WORKED EXAMPLES

Example 1: It is known that the mean diameter of rivets produced by 2 firms *A* and *B* are practically the same but the standard deviations may differ. For 22 rivets produced by *A*, the standard deviation is 2.9 m, while for 16 rivets manufactured by *B*, the standard deviation is 3.8. Test whether the products of *A* have the same variability as those *B*.

Solution:

Given data

$$n_1 = 22 \qquad s_1 = 2.9$$

$$n_2 = 16 \qquad s_2 = 3.8$$

H_0 : variances are same

H_1 : variances are not same

$$F = \frac{S_1^2}{S_2^2} \text{ (or) } \frac{S_2^2}{S_1^2}$$

$$S_1^2 = \frac{n_1\, s_1^2}{n_1 - 1} = \frac{22\,(2.9)^2}{22 - 1}$$

$$\boxed{\mathbf{S_1^2 = 8.810}}$$

$$S_2^2 = \frac{n_2\, s_2^2}{n_2 - 1} = \frac{16\,(3.8)^2}{16 - 1}$$

$$S_2^2 = 15.40$$

$$\boxed{\mathbf{S_2^2 > S_1^2}}$$

$$F = \frac{S_2^2}{S_1^2} = \frac{15.40}{8.810}$$

$$\boxed{\mathbf{F = 1.748}}$$

Conclusion

Calculated value = 1.748

Table valu level of significance at 5%

$$\text{d.f} = (n_2 - 1, n_1 - 1) \Rightarrow \text{d.f } (16 - 1, 22 - 1)$$

$$\text{d.f} = (15{,}21)$$

Table value = 2.18

Since table value is greater than calculated value H_0 is accepted.

$\therefore$ Variances are same.

Example 2: Two random samples of sizes 8 & 11, drawn from a normal populations, are characterized as follows:

	Sample size	**Sum of observations**	**Sum of square observations**
Sample 1	8	9.6	61.52
Sample 2	11	16.5	73.26

Solution:

$n_1 = 8$ $\quad\quad \Sigma (x_1) = 9.6 \quad\quad \Sigma (x_1)^2 = 61.52$

$n_2 = 11$ $\quad\quad \Sigma (x_2) = 16.5 \quad\quad \Sigma (x_2)^2 = 73.26$

H_0 : variances are same

H_1 : variances are not same

$$F = \frac{S_1^2}{S_2^2} \text{ (or) } \frac{S_2^2}{S_1^2}$$

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$= \frac{61.52}{8} - \left(\frac{9.6}{8}\right)^2$$

$$= 7.690 - 1.440$$

$$\mathbf{s_1^2 = 6.250}$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$$

$$= \frac{73.26}{11} - \left(\frac{16.5}{11}\right)^2$$

$$= 6.660 - 2.250$$

$$\mathbf{s_2^2 = 4.410}$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{8\,(6.25)}{7} = 7.143$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{11\,(4.41)}{10} = 4.851$$

Since, $S_1^2 > S_2^2$ $F = \frac{S_1^2}{S_2^2} = \frac{7.143}{4.851} = 1.472$

Conclusion

Calculated value = 1.472

Table value level of significance at 5%

d.f $(n_1 - 1, n_2 - 1) = (8 - 1,\ \ 11 - 1)$

d.f = (7, 10)

Table value = 3.14

Since table value is greater than calculated value H_0 is accepted.

Variances are same.

Example 3: In one sample of 8 observations, the sum of the squares of the deviations of the sample values from the sample mean was 84.4 and in other sample of 10 observations it was 102.6. Test whether this difference is a significant at 5% level.

Solution:

Given data

F-test (Sum of squares of derivation given)

$$n_1 = 8 \qquad \Sigma (x_1 - \bar{x}_1)^2 = 84.4$$

$$n_2 = 10 \qquad \Sigma (x_2 - \bar{x}_2)^2 = 102.6$$

H_0 : variances are same.

H_1 : variances are not same.

$$F = \frac{S_1^2}{S_2^2} \text{ (or) } \frac{S_2^2}{S_1^2}$$

$$S_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_1 - 1} = \frac{84.4}{7} = 12.05$$

$$S_2^2 = \frac{\Sigma (x_2 - \bar{x}_2)^2}{n_2 - 1} = \frac{102.6}{9} = 11.4$$

Since $S_1^2 > S_2^2$,

$$F = \frac{S_1^2}{S_2^2} = \frac{12.057}{11.4} = 1.058$$

Conclusion

Calculated value = 1.058

Table value level of significance at 5%

d.f = $(n_1 - 1, n_2 - 1) = (7, 9)$

Tablevalue = 3.29

Since table value is greater than calculated value. H_0 is accepted.

Variances are same.

Example 4: 2 samples of sizes 9 & 8 give the sum of the squares of deviations from their respective means equal to 160 & 91 respectively. Can the samples be regarded as drawn from the same normal population?

Solution:

F-test (Sum of squares of deviation given)

$$n_1 = 9 \qquad \Sigma (x_1 - \bar{x}_1)^2 = 160$$

$$n_2 = 8 \qquad \Sigma (x_2 - \bar{x}_2)^2 = 91$$

H_0 : variances are same.

H_1 : variances are not same.

$$F = \frac{S_1^2}{S_2^2} \text{ (or) } \frac{S_2^2}{S_1^2}$$

$$S_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_1 - 1} = \frac{160}{8} = 20$$

$$S_2^2 = \frac{\Sigma (x_2 - x_2)^2}{n_2 - 1} = \frac{91}{7} = 13$$

Since $S_1^2 > S_2^2$

$$F = \frac{S_1^2}{S_2^2} = \frac{20}{13} = 1.538$$

Conclusion

Calculatd value = 1.538

Table value level of significance at 5%

d.f = $(n_1 - 1, n_2 - 1) = (8, 9)$

Table value = 3.73

Since table value is greater than calculated value H_0 is accepted.

Variances are same.

Example 5: Two random samples gave the following results

Sample	Size	Sample mean	Sum of the squares of deviation from mean
1	**10**	**15**	**90**
2	**12**	**14**	**108**

Examine whether the samples have come from the same normal population.

Solution:

(i) *F*-test

$$n_1 = 10, n_2 = 12 ;$$

$$\bar{x}_1 = 15, \bar{x}_2 = 14$$

$$\Sigma (x_1 - \bar{x}_1)^2 = 90, \Sigma (x_2 - \bar{x}_2)^2 = 108$$

$$F = \frac{S_1^2}{S_2^2} \text{ (or) } \frac{S_2^2}{S_1^2}$$

$$S_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_1 - 1} = \frac{90}{10 - 1} = \frac{90}{9} = 10$$

$$S_2^2 = \frac{\Sigma (x_2 - \bar{x}_2)^2}{n_2 - 1} = \frac{108}{12 - 1} = \frac{100}{11} = 9.81$$

Since $S_1^2 > S_2^2$,

$$F = \frac{S_1^2}{S_2^2} = \frac{10}{9.81} = 1.019$$

Conclusion

Calculated value = 1.019

Table value level of significance at 5%

$$\text{d.f} = (n_1 - 1, n_2 - 1) = (9, 11)$$

Table value = 2.90

Since table value is greater than calculated value H_0 is accepted.

Variance are same.

(ii) *t*-test

H_0 : means are same.

H_1 : means are not same.

$n_1 = 10 \qquad \bar{x}_1 = 15 \qquad \Sigma (x_1 - \bar{x}_1)^2 = 90$

$n_2 = 12 \qquad \bar{x}_2 = 14 \qquad \Sigma (x_2 - \bar{x}_2)^2 = 108$

$$S^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2 + \Sigma (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{90 + 108}{10112 - 2} = \frac{198}{20}$$

$$S^2 = 9.9, \quad S = 3.146$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{15 - 14}{3.146\sqrt{\frac{1}{10} + \frac{1}{12}}}$$

$$t = \frac{1}{3.146\sqrt{0.1 + 0.083}} = \frac{1}{3.146\sqrt{0.183}}$$

$$t = \frac{1}{3.146\,(0.428)} = \frac{1}{1.346} = 0.743$$

Conclusion

Calculated value $= 0.743$

Table value level of significance at 5%

$\text{d.f} = n_1 + n_2 - 2 = 10 + 12 - 2 = 20$

Table value $= 2.086$

Since table value is greater than calculated value H_0 is accepted.

Means are same.

Example 6: A group of 10 rats fed on diet *A* and another group of 8 rats on a difference diet *B*, recorded the following increase in weight (gms).

Diet *A*	**5**	**6**	**8**	**1**	**12**	**4**	**3**	**9**	**6**	**10**
Diet *B*	**2**	**3**	**6**	**8**	**10**	**1**	**2**	**8**		

Solution:

F-test (variance)

H_0 : variances are same.

H_1 : variances are not same.

$$n_1 = 10, n_2 = 8$$

$$F = \frac{S_1^2}{S_2^2} \text{ (or) } \frac{S_2^2}{S_1^2}$$

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{64}{10} = 6.4$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{40}{8} = 0.5$$

x_1	x_2	x_1^2	x_2^2
5	2	25	4
6	3	36	9
8	6	64	36
1	8	1	64
12	10	144	100
4	1	16	1
3	2	9	4
9	8	81	64
6		36	
10		100	
$\Sigma x_1 = 64$	$\Sigma x_2 = 40$	$\Sigma x_1^2 = 512$	$\Sigma x_2^2 = 282$

$\Sigma x_1 = 64 \;\; \Sigma x_2 = 40 \;\; \Sigma x_1^2 = 512 \;\; \Sigma x_2^2 = 282$

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} = \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$= \frac{512}{10} - \left(\frac{64}{10}\right)^2 = 51.2 - 40.76$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2 \qquad s_1^2 = 10.24$$

$$= \frac{282}{8} - \left(\frac{40}{8}\right)^2$$

$$= 35.25 - 25$$

$$s_2^2 = 10.25$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{10\,(10.24)}{10 - 1} = \frac{102.4}{9} = 11.3$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{8\,(10.25)}{8 - 1} = \frac{82}{7} = 11.71$$

Since $S_2^2 > S_1^2$,

$$F = \frac{S_2^2}{S_1^2} = \frac{11.71}{11.31} = 1.035$$

Conclusion

Calculated value = 1.035

Table value level of significance at 5%

d.f = $(n_2 - 1, n_1 - 1) = (8 - 1, 10 - 1) = (7, 9)$

Table value = 3.29

Since table value is greater than calculated value H_0 is accepted.

Variances are same.

Example 7: 2 independent samples of size 9 & 7 from a normal population had the following values of the variables.

Sample I:	**18**	**13**	**12**	**15**	**12**	**14**	**16**	**14**	**15**
Sample II:	**16**	**19**	**13**	**16**	**18**	**13**	**15**		

Do the estimates of the population variances differ significantly at 5% level.

Solution:

***F*-test**

H_0 : variances are same.

H_1 : variances are not same.

$$F = \frac{S_1^2}{S_2^2} \text{ (or) } \frac{S_2^2}{S_1^2}$$

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{129}{9} = 14.33$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{110}{7} = 15.7$$

x_1	x_2	x_1^2	x_2^2
18	16	324	266
13	19	169	361
12	13	144	169
15	16	225	256
12	18	144	324
14	13	196	169
16	15	256	225
14		196	
15		225	
$\Sigma x_1 = 129$	$\Sigma x_2 = 110$	$\Sigma x_1^2 = 1879$	$\Sigma x_2^2 = 1760$

$\Sigma x_1 = 129 \;\; \Sigma x_2 = 110 \;\; \Sigma x_1^2 = 1879 \;\; \Sigma x_2^2 = 1760$

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$= \frac{1879}{9} - \left(\frac{129}{9}\right)^2 = 208.7 - 205.3$$

$$s_1^2 = 3.4$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$$

$$= \frac{1760}{7} - \left(\frac{110}{7}\right)^2 = 251.4 - 246.8$$

$$s_2^2 = 4.6$$

$$s_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{9\,(3.4)}{9-1} = \frac{30.6}{8} = 3.825$$

$$s_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{7\,(4.6)}{7-1} = \frac{32.2}{6} = 5.367$$

Since $s_2^2 > S_1^2$

$$F = \frac{s_2^2}{s_1^2} = \frac{5.367}{3.825} = 1.403$$

Conclusion

Calculated value = 1.403

Table value level of significance at 5%

d.f $(n_2 - 1, n_1 - 1) = (7 - 1, 9 - 1) = (6, 8)$

Table value = 3.58

Since table value is greater than calculated value H_0 is accepted.

Variances are same.

There are 2 types of test.

1. Parametric test (parameter is given)
2. Non-parametric test (Parameter is not given)

1.7 NON-PARAMETRIC TEST

χ^2 test [Chi - Square test]

χ^2 test: It is the very powerful test for testing significance difference between theory and experimental and it is also very easiest test. To test between the **attributes.**

Formula

To find calculated value

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$O_i \Rightarrow$ Observed frequency (given data)

$E_i \Rightarrow$ Expected frequency (need to find out).

Application of χ^2 test

- To test the goodness of fit.
- To test the Independence of attributes.
- To test the homogeneityof independentestimates of the population variance.
- To test if the hypothetical value of the population variance is σ^2.

Conditions for the applications of ψ^2 test.

1. The sample observations should be independnet.
2. Constraints on the cellf requencies, if any must be linear.
3. N, the total frequency, should be at last 50.

Note

- For the independence of attributes to find the table (r-number of rows, c-number of colums) value degrees of freedom $= (r-1)\,(c-1)$
- Expected frequency E_i,

$$E_i = \frac{\text{row total} \times \text{column total}}{\text{Grand total}}$$

Grand total $\rightarrow$ sum of row total or column total (both row total and column total same).

Contingency table for χ^2 test

2×2 matrix $\rightarrow$

a	b
c	d

$$\chi^2 = \frac{(a+b+c+d)\,(ad-bc)^2}{(a+d)\,(b+c)\,(a+b)\,(c+d)}$$

1.7.1 Problems on chi-square test of Independence of Attributes

Example 1: Test whether the general ability and mathematical ability are associated at 5% level of significance from the following table.

		General ability		
Mathematical ability		**Good**	**Fair**	**Poor**
	Good	**44**	**22**	**7**
	Fair	**265**	**257**	**178**
	Poor	**41**	**91**	**98**

Solution:

Since attributes are given. It is a χ^2 test of Independence of Attributes.

Step 1: Given data

Mathematical ability		**General ability**			**Row Total**
		Good	Fair	Poor	
	Good	44	22	7	73
	Fair	265	257	178	700
	Poor	41	91	98	230
Column total		350	370	283	1003

↓
Grand Total

Step 2: Formula & setup a hypothesis

$$\text{Formula } \chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$\text{where } E_i = \frac{(\text{Row Total}) \times (\text{ColumnTotal})}{\text{Grand Total}}$$

Setup a hypothesis

H_0 : The two Attributes (General & Mathematical Ability) are Independent.

H_1 : The two Attributes (General & Mathematical Ability) are dependent.

Calculate expected frequency

$$E\,(44) = \frac{73 \times 350}{1003},\ E\,(22) = \frac{73 \times 370}{1003},\ E\,(7) = \frac{73 \times 283}{1003}$$

$$E\,(265) = \frac{700 \times 350}{1003},\ E\,(257) = \frac{700 \times 370}{1003},\ E\,(178) = \frac{700 \times 283}{1003}$$

$$E\,(41) = \frac{230 \times 350}{1003},\ E\,(91) = \frac{230 \times 370}{1003},\ E\,(98) = \frac{230 \times 283}{1003}$$

Step 3: Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
44	25.47	13.48
22	26.92	0.899
257	288.22	0.005
178	197.50	1.93
41	80.25	19.19
91	84.84	0.44
98	84.89	16.89
		$\Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right] = 63.54$

Step 4: Conclusion

Calculated value $= 63.54$

Table value at 5% LOS

degree of freedom $= (r-1)\,(c-1)$

$= (3-1)\,(3-1)$

d.f $= 4$

Table value $= 9.488$

Since table value < Calculated value

H_0 is rejected. H_1 is accepted.

$\therefore$ The two attributes are dependent.

Example 2: From the following table, test the association between the eye colours of fathers and sons.

		Eye colour in sons	
Eye colour in fathers	**Not light**	**230**	**148**
	light	**151**	**471**

Solution:

Step 1: Given data

			Row Total	
	230	148	378	
	151	471	622	
Column Total	381	619	1000	Grand Total

Step 2: Formula and set up a hypothesis

$$\text{Formula } \chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$E_i = \frac{\text{Row total} \times \text{Column total}}{\text{Grand total}}$$

H_0 : Two Attributes (Eye color in sons & father) are Independent.

H_1 : Two Attributes (Eye color in sons & father are dependent).

Step 3: Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
230	144.01	51.345
148	233.98	31.595
151	236.98	31.195
471	385.01	19.205
		$\chi^2 = 133.34$

Step 4: Conclusion

Calculated value = 133.34

Table value at 5% loss

$$df = (r-1)\,(c-1)$$

$$= (2-1)\,(2-1)$$

$$df = 1$$

Table value = 3.841

Since Table value is less than calculated value.

H_0 is rejected, H_1 is accepted.

Two Attributes [Eye color in sons & father] are dependent.

Example 3: Test whether the shift has any association with good or bad parts.

	Day shift	**Evening shift**	**Night shift**
Good parts:	**960**	**940**	**950**
Bad parts:	**40**	**50**	**45**

Solution

Step 1: Given data

				Row Total	
	960	940	950	2850	
	40	50	45	135	
Column Total	1000	990	995	2985	Grand Total

Step 2: Formula

$$\text{Formula } \chi^2 = \Sigma\left(\frac{(O_i - E_i)^2}{E_i}\right)$$

$$E_i = \frac{\text{Row total} \times \text{Column total}}{\text{Grand total}}$$

H_0 : There is no relationship between no. of good & bad parts.

H_1 : There is some relationship between no. of good & bad parts.

Step 3: Calculation

$$E\,[960] = \frac{2850 \times 1000}{2985}$$

$$E\,[940] = \frac{2850 \times 990}{2985}$$

$$E\,[950] = \frac{2850 \times 995}{2985}$$

$$E\,[40] = \frac{135 \times 1000}{2985}$$

$$E\,[50] = \frac{135 \times 990}{2985}$$

$$E\,[45] = \frac{135 \times 995}{2985}$$

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
960	954.77	0.028
940	945.22	0.028
950	950.00	0.000
40	45.22	0.602
50	44.77	0.610
45	45.00	0.000
		$\chi^2 = 1.268$

Step 4: Conclusion

Calculated value= 1.268

$$\chi^2 = d.f = (r-1)\ (c-1) = (2-1)\ (3-1)$$

$$d.f = 2$$

Table value level of significance at 5%,

Table value = 5.991

Since Table value is greater than calculated value, H_0 is accepted.

Conclusion: There is no relationship between no. of good parts and bad parts.

Example 4: The following table gives the classification of 100 workers according to sex and the nature of work. Test whether nature of work is *independent* of the sex of the worker.

		Skilled	**Unskilled**
Sex	**Male**	**40**	**20**
	Female	**10**	**30**

Solution:

Step 1: Given data

			Row Total	
	40	20	60	
	10	30	40	
Column Total	50	50	100	Grand Total

Step 2: Formula and Set up a hypothesis:

H_0 : work nature is independent of the sex of the worker.

H_1 : work nature is dependent on the sex of the worker.

$$\text{Formula } \chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$E_i = \frac{\text{Row total} \times \text{Column total}}{\text{Grand total}}$$

$$E\ (40) = \frac{60 \times 50}{100}$$

$$E\ (20) = \frac{60 \times 50}{100}$$

$$E\ (10) = \frac{40 \times 50}{100}$$

$$E\ (30) = \frac{40 \times 50}{100}$$

Step 3: Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
40	30	3.333
20	30	3.333
10	20	3.000
30	20	5.000
		$\chi^2 = 16.666$

Step 4: Conclusion

Calculated value = 16.666

$$\chi^2 = d.f = (r - 1)\ (c - 1) = (2 - 1)\ (2 - 1)$$

$$d.f = 1$$

Table value level of significance at 5%

Table value = 3.841

Since table value is less than calculated value, H_0 is rejected, H_1 is accepted.

Conclusion: The work nature is dependent on the sex of the worker.

Example 5: In a locality 100 persons were randomly selected asked about their educational achievements. The results are given as below.

Sex	**Education**		
	Middle	**High School**	**College**
Male	**10**	**15**	**25**
Female	**25**	**10**	**15**

Solution:

Step 1: Set up a Hypothesis

H_0 : There is no relationship between the educational achievement and sex.

H_1 : There is some relationship between the educational achievements and sex.

Step 2: Formula

$$\chi^2 = \Sigma \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

$$E_i = \frac{\text{Row total} \times \text{Column total}}{\text{Grand total}}$$

Given data

				Row Total	
	10	15	25	50	
	25	10	15	50	
Column Total	35	25	40	100	Grand Total

Step 3: Calculation

$$E\,(10) = \frac{50 \times 35}{100} \qquad E\,(25) = \frac{50 \times 35}{100}$$

$$E\,(15) = \frac{50 \times 25}{100} \qquad E\,(10) = \frac{50 \times 25}{100}$$

$$E\,(25) = \frac{50 \times 40}{100} \qquad E\,(15) = \frac{50 \times 40}{106}$$

O_i	E_i	$(O_i - E_i)^2/E_i$
10	19.5	3.21
15	12.5	0.51
25	20	1.25
25	17.5	3.21
10	12.5	0.5
15	20	1.25
		$\chi^2 = 9.92$

Step 4: Conclusion

Calculated value $= 9.92$

$$\text{d.f} = (r-1)\,(c-1) = (2-1)\,(3-1) = 2$$

Table value level of significance at 5% $= 5.991$

Since calculated value greater than table value.

$\therefore H_0$ is rejected, H_1 is accepted.

Conclusion: There is some relationship between the educational achievements and sex.

Example 6: A certain drug was administrated to 2500 people out of a total of 800 included in the sample to test its efficiency against hyphoid the results are given below.

	Typhoid	No. typhoid
Drug	**200**	**300**
No. Drug	**280**	**20**

On the basis of this data can it be concluded the drug is effective in preventing typhoid.

Solution:

Step 1: Set up a hypothesis

Null hypothesis: $\boldsymbol{H_0}$ **:** Two attributes are independent

Alternative hypothesis: $\boldsymbol{H_1}$ **:** The two attributes are dependent.

Step 2: Test statistic

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

O_i – Observed frequency

E_i – Expected frequency

Step 3: Given Data

	Typhoid	No. Typhoid	Row total	
Drug	200	300	500	
No drug	230	20	300	
Column total	430	320	800	Grand Total

Step 4: Calculate expected frequency

$$E_i = \frac{\text{Row total} \times \text{Column total}}{\text{Grand total}}$$

Step 5: Calculate χ^2 value

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
200	200	33.333
300	200	50
200	180	55.555
20	120	83.333
$\Sigma \frac{(O_i - E_i)^2}{E_i} = 222.221$		

Step 6: Conclusion

Calculated value $= 222.221$

Table value at 5% LOS

Degree of freedom $= (r-1) \times (c-1) = 1 \times 1 = 1$

Table value $= 3.841$

Since the table value is less than the calculated vale

H_0 is rejected. H_1 is accepted

The two attributes are dependent.

Example 7: 1000 families were selected at random in city to test there belief that high income families usually sent there children to public schools and there low income families often sent there children to government schools the following results these obtained.

Income	School	
	Public	Govt
Low	370	430
High	130	70

Test whether income and type of school in are independent.

Solution:

Step 1: Test statistic

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

O_i – Observed frequency

E_i – Expected frequency

Step 2: Given Data

Income/School	Public	Govt	Row total
Low	370	430	800
High	130	70	200
Column total	500	500	1000

Calculate expected frequency (E_i)

$$E_i = \frac{\text{(Row column total)} \times \text{(column total)}}{\text{Grand total}}$$

Step 3: Calculate χ^2 value:

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
370	400	2.25
430	400	2.25
130	100	9
70	100	9
$\Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right] = 22.50$		

Calculated value $= 22.50$

Step 4: Conclusion

Calculated value $= 22.50$

Table value at 5% LOS

Degrees of freedom $= (r-1)\times(c-1) = (2-1)\times(2-1) = 1$

Table value $= 3.841$

Since the table is less than the calculated value.

H_0 is rejected,

H_1 is accepted

The Attributes are dependent.

Example 8: Two researches adopted different sampling technqiues while investigating the same group of studnets to find the number of students falling into different intelligence leve. The results are as follows.

Researcher	Below average	Average	Above average	Excellence	Row total
X	**86**	**50**	**44**	**10**	**200**
Y	**40**	**33**	**25**	**2**	**100**
Column total	**126**	**93**	**69**	**12**	**300**

Would you say that the sampling technique adopted by the 2 researches are significantly different?

Step 1: Set up of a hypothesis

H_0 : The two attributes are independent.

H_1 : The two attributes are dependent.

Step 2: Test statistic

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

O_i – Observed frequency. E_i – Expected frequency.

Step 3: Given Data

Researcher	Below average	Average	Above average	Excellence	Row total
X	86	50	44	10	200
Y	40	33	25	2	100
Column total	126	93	69	12	300

Step 4: Calculated expected frequency (E_i)

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$E_i = \frac{\text{Row total} \times \text{column total}}{\text{Grand total}} = \frac{\text{Row total} \times \text{column total}}{300}$$

Step 5: Calculated χ^2 value

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
86	84	0.047
60	62	0.064
44	46	0.086
10	8	0.5
40	42	0.095
33	31	0.095
25	23	0.129
2	4	0.173
	$\Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right] = 2.094$	

Step 6: Conclusion

Calculated value $= 2.094$

Table value at 5% LOS

d.f $= (r-1)\times(c-1) = (2-1)\times(4-1) = 3$

Table value $= 7.115$

Since the table value is greater than the calculated value

H_0 is accepted,

The two attributes are independent

Example 2: A sample analysis of examination results of 1000 students were made and It was found that 260 failed 110 first class 420 second class and rest obtained third class. Do these data support the general examination result in the ratio 2:1:4:3.

Solution:

Step 1: Set up a hypothesis

H_0 : The general results are four categories are in the ratio 2:1:4:3

H_1 : The general results are four categories are not in the ratio 2:1:4:3

Step 2: Test statistic

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

O_i – Observed frequency

E_i – Expected frequency.

$$\frac{2}{10}\times 1000 = 200$$

$$\frac{1}{10}\times 1000 = 100$$

$$\frac{4}{10}\times 1000 = 400$$

$$\frac{3}{10}\times 1000 = 300$$

Step 3: Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
260	200	18
110	100	1
400	400	1
210	300	27

$$\Sigma \frac{(O_i - E_i)^2}{E_i} = 47$$

Step 4: Conclusion

Calculated value $= 47$

Table value at 5% LOS

degree of freedom $= 4 - 1 = 3$

d.f $= 7.115$, Table value $= 7.115$

Since table value less than calculated value H_0 is rejected, H_1 is accepted.

1.7.2 Goodness of fit

Uniformly (or) Equal (or) Binomial distribution	Poisson distribution	Normal distribution
d.f $= n - 1$	d.f $= n - 2$	d.f $= n - 3$

$$\chi^2 = \Sigma \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

$$E_i = \frac{\text{Grand Total}}{n}$$

($n \rightarrow$ No. of items)

Example 1: The following table gives the no. of aircraft accidents that occurred during the various days of the week. Test whether the accidents are uniformly distributed over the week.

Days	Mon	Tue	Wed	Thu	Fri	Sat	Total
No. of accidents	14	18	12	11	15	14	84

Solution:

Step 1: Setup a Hypothesis

H_0 : The accidents are uniformly distributed over the week.

H_1 : The accidents are not uniformly distributed over the week.

Step 2: Formula

Formula $\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$

$$E_i = \frac{\text{Grand Total}}{N} \quad E_i = \frac{84}{6} = 14$$

Step 3: Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
14	14	0
18	14	1.142
12	14	0.285
11	14	0.643
15	14	0.071
14	14	0
		$\chi^2 = 2.141$

Step 4: Conclusion

Table value level of significance at 5%,

$$d.f = n - 1$$

$$d.f = 6 - 1$$

$$d.f = 5$$

Table value $= 11.070$

Since table value is greater than calculated value.

$\therefore$ H_0 is accepted.

Conclusion: The accidents are uniformly distributed over the week.

Example 3: The theory predicts the population of beans in 4 groups *A*, *B*, *C* and *D* should be 9:3:3:1. In an experiment among 1600 beans, the no. of group were 882, 313, 287 & 118. Does the experimental result support the theory?

Solution:

Step 1: Set up a Hypothesis

H_0 : The Experimental result support the theory.

H_1 : The Experimental result does not support the theory.

Step 2: Formula

$$\chi^2 = \Sigma \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

$$E_i = \frac{\text{Grand Total}}{N}$$

Step 3: Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
882	$\frac{9}{16} \times 1600 = 900$	0.36
313	$\frac{3}{16} \times 1600 = 300$	0.56
287	$\frac{3}{16} \times 1600 = 300$	0.56
118	$\frac{1}{16} \times 1600 = 100$	3.24
		$\chi^2 = 4.72$

Step 4: Conclusion

Calculated value $= 4.72$

Table value level of significance at 5%,

$$\text{d.f} = n - 1 = 4 - 1 = 3$$

$$\text{Table value} = 7.115$$

Since table value is greater than calculated value.

$\therefore$ H_0 is accepted.

Conclusion: The experimental result support the theory.

Example 4: A die is thrown 264 times with the following results.

No. appeared on die	1	2	3	4	5	6
Frequency	**40**	**32**	**20**	**58**	**54**	**60**

Show that the die is biased.

Solution:

Step 1: Set up a Hypothesis

H_0 : The die is unbiased.

H_1 : The die is biased.

Step 2: Formula

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$E_i = \frac{\text{Grand Total}}{N} = \frac{264}{6} = 44$$

No. appeared on the die	1	2	3	4	5	6	Grand Total
Frequency	40	32	20	58	54	60	264

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
40	44	0.364
32	44	3.272
20	44	13.090
58	44	4.454
54	44	2.273
60	44	5.813
		$\chi^2 = 29.266$

Step 3: Conclusion

Calculated value $= 29.266$

Table value level of significance at 5%.

$$\text{d.f} = (n - 1) = 6 - 1 = 5$$

Table value $= 11.070$

Calculated value, since table value is less than.

$\therefore$ H_0 is rejected. $\Rightarrow H_1$ is accepted.

Conclusion: The die is biased.

Example 5: Children having one parent of blood type *M* and the other type *N* will always be one of the 3 types *M*, *MN*, *N* and average proportions of these will be 1:2:1. Out of 300 children having one *M* parent one *N* parent, 30% were found to be of type *M*, 45% of the type *MN* and the remaining of type *N*. Test the hypothesis by using χ^2 test.

Solution:

Step 1: Formula

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$E_i = \frac{\text{Grand Total}}{N}$$

Step 2: Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
$\frac{30}{100} \times 300 = 90$	$\frac{1}{4} \times 300 = 75$	3
$\frac{45}{100} \times 300 = 135$	$\frac{2}{4} \times 300 = 150$	1.5
$\frac{25}{100} \times 300 = 75$	$\frac{1}{4} \times 300 = 75$	0
		$\chi^2 = 4.5$

Step 3: Conclusion

Calculated value $= 4.5$

Table value level of significance at 5%,

$$\text{d.f} = n - 1 = 3 - 1 = 2$$

$$\text{Table value} = 5.991$$

Since table value is greater than calculated value H_0 is accepted.

Example 6: 150 digits are chosen at random from a set of tables. The frequencies of the digits were as given below:

Digits:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Frequencies:	**15**	**19**	**12**	**14**	**10**	**15**	**18**	**16**	**14**	**17**

Test the correctness of the hypothesis that the digits were distributed in equal numbers in the set of tables.

Solution:

Step 1: Set up a Hypothesis

H_0 : The digits were distributed in equal numbers in set of tables.

H_1 : The digits were not distributed in equal numbers in set of tables.

Digits	0	1	2	3	4	5	6	7	8	9	Grand Total
Frequencies	15	19	12	14	10	15	18	16	14	17	150

Step 2: Formula

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$E_i = \frac{\text{Grand Total}}{N} = \frac{150}{10} = 15$$

Calculation

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
15	15	0
19	15	1.067
12	15	0.60
14	15	0.067
10	15	1.667
15	15	0
18	15	0.60
16	15	0.067
14	15	0.067
17	15	0.267
		$\chi^2 = 4.402$

Step 4: Conclusion

Calculated value $= 4.402$

Tale value level of significance at 5%,

$$\text{d.f} = n - 1 = 10 - 1 = 9$$

$$\text{Table value} = 16.919$$

Since table value is greater than calculated value H_0 is accepted.

Conclusion: The digits were distributed in equal numbers in set of tables.

Example 7: Test the correctness of the Hypothesis that the digits were distributed in the set of tables.

Digits	0	1	2	3	4	5	6	7	8	9
Frequency	1026	1107	997	966	075	933	1107	978	964	853

Solution:

Step 1: Set up a hypothesis

H_0 : The digits may be taken to occur equally frequently in the directory.

H_1 : The digits may not taken to occur equally frequently in the directory

Step 2: Test statistic

$$\chi^2 = \Sigma \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

O_i – Observed frequency

E_i – Expected frequency

Step 3: Calculate expected frequency

$$E_i = \frac{\text{Grand total}}{\text{N}} = \frac{1000}{10} = 1000$$

Step 4: Calculation

Calculate O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
1026	1000	0.676
1107	1000	11.449
997	1000	0.009
966	1000	1.156
1075	1000	5.025
933	1000	4.489
1107	1000	11.449
972	1000	0.784
964	1000	1.296
853	1000	21.609
	$\Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right] = 58.542$	

Step 5

Calculated value $= 58.542$

Table value at 5% LOS

degrees of freedom $= n - 1 = 10 - 1 = 9$

$$\text{d.f} = 9$$

Table value $= 16.919$

Since the table value is less than the calculated

H_0 is rejected

H_1 is accepted

The digits may not taken to occur equally frequently in the directory.

MISCELLANEOUS PROBLEMS

Example 1: In a certain sample of 2000 families 1400 families were consumers of tea out of 1300 Hindu families 1236 families consume tea use χ^2-test and test and state whether there is any significant difference between the consumption of tea among hindu and non-hindu families.

Solution:

	Hindu Family	**Non-Hindu**	**Row total**
Consumer of tea	1236	164	1400
Non consumption	504	36	600
Column total	1800	200	2000

Example 2: Use χ^2-test and state whether there is any significant difference between the two attributes given below.

	Favourable	**Non-favourable**	**Row total**
Conventional	**40**	**70**	**110**
New	**60**	**30**	**90**
Column total	**100**	**100**	**200**

Solution:

Step 1: Set up a hypothesis

H_0 Null: Two Attributes are Independent.

H_1 Alternative: The two Alternative are dependent.

Step 2: Test statistic

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i}$$

O_i – Observed frequency

E_i – Expected frequency

Step 3: Given table

	Favourable	**Non-favourable**	**Row total**
Conventional	40	70	110
New	60	30	90
Column total	100	100	200

Step 4:

Calculated Expected frequency (E_i)

$$E_i = \frac{\text{Row total} \times \text{Column total}}{\text{Grand total}}$$

Step 5: Calculate χ^2 value

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
40	55	4.090
70	55	4.090
60	45	5
30	45	5

$$\Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right] = 18.18$$

Step 6: Conclusion

Calculated value $= 18.18$

Table value at 5% LOS

degree of freedom: $(r-1)\times(c-1) = (2-1)\times(2-1)$

$d.f = 1$

Table value $= 3.841$

Since the table value is less than the calculated value

H_0 is rejected

H_1 is accepted

The two attributes are dependent.

Example 3: In 20 throws a single die the following distribution of face was observed.

Face	**1**	**2**	**3**	**4**	**5**	**6**
Frequency	**30**	**25**	**18**	**10**	**22**	**15**

Show that the die is biased.

Solution:

Step 1: Given table

Face	1	2	3	4	5	6
Frequency	30	25	18	10	22	15

Step 1: Set up a hypothesis

H_0 The die is unbiased.

H_1 The die is biased.

Step 2: Test statistic

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i}$$

O_i – Observed frequency

E_i – Expected frequency

Step 3: Calculate expected frequency

Step 4: Calculated χ^2 value

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
30	20	5
25	20	1.25
18	20	0.2
10	20	5
22	20	0.2
15	20	1.25

$$\Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right] = 12.9$$

Step 5: Conclusion

Calculated value $= 12.9$

Table value at 5% LOS

degrees of freedom $= n - 1 = 6 - 1 = 5$

Table value $= 11.070$

Since the table value is less than the calculated value

H_0 is rejected,

H_1 is accepted

The die is biased.

Example 4: A sample analysis of examination results of 500 students was made. It was found that 220 students have failed, 170 have secured a third class, 90 have secured a second class and the rest, a first class. So do these figures support the general belief that the above categories are in the ratio 4:3:2:1 respectively?

Solution:

The variable of interest is the results in the four categories.

1. H_0 : The results in the four categories are in the ratio 4:3:2:1

2. H_1 : The results in the four categories are not in the ratio 4:3:2:1.

3. On the assumption H_0, the expected frequencies of the 4 classes are

$$\frac{4}{10} \times 500, \frac{3}{10} \times 500, \frac{2}{10} \times 500, \frac{1}{10} \times 500$$

i.e., 200, 150, 100, 50

$O \rightarrow$ Observed frequency, $E \rightarrow$ Expected frequency

	O	***E***	$\boldsymbol{\frac{(O-E)^2}{E}}$	**d.f.**	**Cal** χ^2	**Table** χ^2 **5% level**
Fail	220	200	2	$\therefore v = n - 1$ $= 4 - 1$ $= 3$	$\chi^2 = \Sigma \frac{(O-E)^2}{E}$ $= 23.667$	$\chi^2_{0.05}$ $= 7.815$
III	170	150	2.667			
II	90	100	1			
I	20	50	18			
	$\Sigma O =$ 500	$\Sigma E =$ 500	$\Sigma \frac{(O-E)^2}{E}$ $= 23.667$		$\Sigma E = \Sigma O$	

d.f $= V = n - 1$, since $\Sigma E = (= \Sigma O)$ has been found using the sample data.

4. Conclusion:

Here, Cal $\chi^2 >$ table χ^2

i.e., 23.667 > 7.815

So, we reject H_0 at 5% level of significance, H_1 is accepted.

$\therefore$ The results of the four categories are not in the ratio 4:3:2:1.

Example 5: A sample analysis of examination results of 1000 students were made and it was found that 260 fail, 110 first class, 420 second class and rest obtained third class. Do these data support the general examination result in the ratio 2:1:4:3.

[A.U N/D 2019 (R-17)]

Solution:

Given

$$n = 4, 2 + 1 + 4 + 3 = 10$$

The variable of interest is the results in the four categories.

1. H_0 : The results in the four categories are in the ratio 2:1:4:3
2. H_1 : The results in the four categories are not in the ratio 2:1:4:3
3. On the assumption H_0, the expected frequencies of the 4 classes are

$\frac{2}{10} \times 1000, \frac{1}{10} \times 1000, \frac{3}{10} \times 1000$ i.e., 200, 100, 400, 300

$O \rightarrow$ Observed frequency, $E \rightarrow$ Expected frequency

	O	**E**	$\frac{(O-E)^2}{E}$	**d.f.**	**Cal χ^2**	**Table χ^2 5% level**
Fail	260	200	18	$\therefore v = n-1$	$\chi^2 =$	$\chi^2_{0.05}$
III	110	100	1	$= 4-1$ $= 3$	$\Sigma \frac{(O-E)^2}{E}$ $= 47$	$= 7.815$
II	420	400	1			
III	210	300	27			
	$\Sigma O =$ 1000	ΣE $= 1000$	$\Sigma \frac{(O-E)^2}{E}$ $= 47$		$\Sigma E = \Sigma O$	

Conclusion

Since calculated value is greater than table value H_0 is rejected, H_1 is accepted.

Example 6: 4 coins were tossed 160 times and the following results were obtained.

No. of heads:	**0**	**1**	**2**	**3**	**4**
Observed frequencies:	**17**	**52**	**54**	**31**	**6**

Under the assumption that the coins unbiased, find the expected frequencies of getting 0, 1, 2, 3, 4 heads and test the goodness of fit. *[A.U A/M 2011]*

Solution:

1. H_0 : The coins are unbiased
2. H_1 : The coins are biased
3. Probability of getting head $= p = \frac{1}{2}$

   Probability of getting tail $= q = \frac{1}{2}$

Then the expected frequencies are

$$NP(x) = N(nC_x)\,p^x\,q^{n-x},$$

$$N = \Sigma\,O = 160$$

x_1	O	$E = NnC_x\,p^x\,q^{n-x}$ $= 1604C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$	$\frac{(O-E)^2}{E}$	d.f	Cal χ^2	Table χ^2 5% level
0	17	$(160)\,(4C_0)\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 = 10$	4.9	$n-1$ $= 5-1$ $= 4$	$\chi^2 = \Sigma\frac{(O-E)^2}{E}$ $= 12.73$	$\chi^2_{0.05}$ $= 9.488$
1	52	$(160)\,(4C_1)\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = 40$	3.6			
2	54	$(160)\,(4C_1)\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = 40$	0.6			
3	31	$(160)\,(4C_2)\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 = 40$	2.03			
4	6	$(160)\,(4C_4)\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = 10$	1.6			
	$N = \Sigma\,O = 160$	$\Sigma\,E = 160$	$\Sigma\frac{(O-E)^2}{E}$ $= 12.73$			

d.f $= v = n - 1$, since only $\Sigma\,E\,(= \Sigma\,O)$ has been found using the sample data. The values of p and q have not been found by using the sample data.

Example 7: Out of 8000 graduates in a town 800 are females, out of 1600 graduate employees 120 are females. Use χ^2 to determine if any distinction is made in appointment on the basis of sex. Value of χ^2 at 5% level for one degree of freedom is 3.84. *[A/M 2010]*

	Male	Female	Total
Graduates in a town	7200	800	8000
Graduate employees	1480	120	1600
Total	8680	920	9600

The Reader can solve the problem.

Example 8: Mechanical engineers testing a new era welding technique, classified weld both with respect to appearance and an X-ray inspection.

	Appearance *X*ray	**Bad**	**Normal**
Bad	**20**	**7**	**3**
Normal	**13**	**51**	**16**
Good	**7**	**12**	**21**
Total	**40**	**70**	**40**

Test for independence using 0.05 level of significance.

[A.U A/M 2018 R-13] [A.U N/D 2013 R-21]

The reader can solve the problem.

Poisson distribution

$$\chi^2 = \Sigma \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$E_i = NP[X = x]$$

$$E_i = \frac{Ne^{-\lambda} \lambda^x}{x!} \qquad \text{where } x = 0, 1, 2 \ldots$$

Example 9: The following table shows the distribution of goaling the foot ball match.

No. of goals	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
No. of mistakes	**95**	**158**	**108**	**63**	**40**	**9**	**5**	**2**

Fit the poisson distribution and test the goodness of fit.

Solution:

χ^2 test (Poisson distribution)

H_0 : Fit is good

H_1 : Fit is not good

$$E_i = \frac{Ne^{-\lambda} \lambda^x}{x!}$$

x	**f**	**fx**
0	95	0
1	158	158
2	108	216
3	63	189
4	40	160
5	9	45
6	5	30
7	2	14
	$\Sigma f = 480$	$\Sigma f_x = 812$

$$\lambda = \frac{\Sigma fx}{\Sigma f} = \frac{812}{480} \Rightarrow \lambda = 1.7$$

$$E_i = \frac{480e^{-1.7}(1.7)^x}{n!}$$

$$E_i = \frac{87.68\,(1.7)^x}{x!}$$

$$\Sigma f = 480 \quad \Sigma fx = 812$$

x	O_i	E_i	$(O_i - E_i)^2/E_i$
0	95	87.68	0.61
1	158	149	0.543
2	108	126.69	2.702
3	63	71.79	1.078
4	40	30.51	2.944
5	9	10.37	0.180
6	5	2.43	1.462
7	2	0.71	2.343
			$\Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right] = 15.238$

Conclusion

Calculatedvalue = 15.258

Table value level of significance at 5%

$$\text{d.f} = n - 2 = 8 - 2 = 6$$

Table value = 12.596

Since table value is less than calculated value H_0 is rejected, H_1 is accepted.

Fit is not good.

Example 10: The no. of defects per unit in a sample of 330 units of a manufactured products was found as follows.

No. of defects	**0**	**1**	**2**	**3**	**4**
No. of units	**214**	**92**	**20**	**3**	**1**

Fit a poisson distribution and test the goodness of fit.

Solution:

χ^2-test (Poisson distribution)

H_0 : The fit is good

H_1 : Fit is not good

x	f	fx
0	214	0
1	92	92
2	20	40
3	3	9
4	1	4

$$\lambda = \frac{\Sigma fx}{\Sigma f}$$

$$= \frac{145}{330}$$

$$\boxed{\lambda = 0.44}$$

$$E_i = \frac{Ne^{-\lambda}\lambda^x}{x!}$$

$$= \frac{330e^{-0.44} - (0.44)^2}{x!}$$

$$E_i = \frac{212.53\ (0.44)^x}{x!}$$

x	O_i	E_i	$(O_i - E_i)^2/E_i$
0	214	212.53	0.010
1	92	93.513	0.024
2	20	20.573	0.016
3	3	3.017	0
4	1	0.332	1.344
			$\Sigma\left[\dfrac{(O_i - E_i)^2}{E_i}\right] = 1.394$

Conclusion

Calculated value = 1.394

Table value level of significance at 5%

$$\text{d.f} = n - 2 = 5 - 2 = 3$$

Table value = 7.45

Since table value is greater than calculated value H_0 is accepted.

Fit is good.

Binomial distribution

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$E_i = NP\,(X = x)$$

$$E_i = N\,[nC_x\, p^x\, q^{n-x}]$$

$$p + q = 1$$

$$q = 1 - p$$

$$P\,[X = x] = nC_x\, p^x\, q^{n-x}$$

$p \rightarrow$ probability of success.

$q \rightarrow$ probability of failure.

Example 11: 4 coins were tossed 160 times and the following results were obtained.

No. of heads	**0**	**1**	**2**	**3**	**4**
Frequency	**19**	**50**	**52**	**30**	**9**

Test the goodness of fit with the help of χ^2 on the assumption that the coins are unbiased.

Solution:

χ^2 test (Binomial distribution)

H_0 : the conins are unbiased.

H_1 : the conins are biased.

$n = 4, N = 160, p = 1/2 \Rightarrow q = 1 - p = 1/2$

$E = [nC_x\, p^x\, q^{n-x}]$

$= 160\,[4C_x\,(1/2)^x\,(1/2)^{4-x}]$

$= 160\,[4C_x\,(1/2)^{x+4-x}$

$= 160\,[4C_x\,(1/2)^4]$

$= 160\,[4C_x\,(1/16)]$

$E_i = 10\,(4C_x)$

x	O_i	E_i	$(O_i - E_i)^2/E_i$
0	19	10	8.1
1	50	40	2.5
2	52	60	10.6
3	30	40	2.5
4	9	10	0.1
			$\Sigma\left[\dfrac{(O_i - E_i)^2}{E_i}\right] = 14.26$

Conclusion

$$\text{Calculated value} = 14.26$$

Table value level of significance at 5%

$$\text{d.f} = n - 1 = 5 - 1 = 4$$

$$\text{Table value} = 9.488$$

Since table value is less than calculated value H_0 is rejected, H_1 is aceepted.

Than the coins are biased.

Example 12: A survey of 320 families with 5 children each yielded the following distribution.

No. of girls:	0	1	2	3	4	5
No. of boys	5	4	3	2	1	0
No. of families	12	40	88	110	56	14

Is this result consistent with the hypothesis that male and female births are equally probable.

Solution:

H_0 : Male and Female births are equally probable.

H_1 : Male and Female births are not equally probable.

$n = 5, N = 320, p = 1/2, q = 1 - p = 1/2$

$$E_i = N\,[nC_x\,p^x\,q^{n-x}]$$

$$= 320\,[5C_x\,(1/2)^x\,(1/2)^{5-x}]$$

$$= 320\,[5C_x\,(1/2)^{x+5-x}]$$

$$= 320\,[5C_x\,(1/2)^5] = \frac{320}{32}\,[5C_x]$$

$$E_i = 10\,[5C_x]$$

x	O_i	E_i	$(O_i - E_i)^2/E_i$
0	12	10	0.4
1	40	50	2.0
2	88	100	1.44
3	110	100	1.00
4	56	50	0.7200
5	14	10	1.60
			$\chi^2 = 7.160$

Conclusion

Calculated value $= 7.160$

Table value level of significance at 5%

$$\text{d.f} = n - 1 = 6 - 1 = 5$$

Table value = 9.488

Since table value is greater than calculated value H_0 is accepted.

∴ Male and Female births are equally probable.

EXERCISES

1. A random sample of 100 students gave a mean weight of 58 kg with a S.D. of 4 kg. Test the hypothesis that the man weight in the population is 60 kg.

2. On an examination given 60 students at large number or different schools, the mean grade was 74.5 and S.D. was 89.0. At one particular school, where 200 students took examination, the mean grade was 75.9. Discuss the significance of this result at 5% level.

3. A random sample of 400 items is drawn from a normal population whose mean is 5 and whose variance is 4. If the sample mean is 4.45, can the sample be regarded as truly random sample?

4. The average breaking strength of steel rods is specified as 20. In random sample of 100, mean was 19.9 and S.D. 0.4. Are the rods upto specifications? Test at 1% level of significance.

5. The average number of defective articles in a certain factory is claimed to be less than the average for all factories. The average for all the factories is 30.5. A random sample 100 defectives gives the following distribution.

Classes	16-20	21-25	26-30	31-35	36-40
Frequency	12	22	20	30	16

   Test the claim of the factory.

6. A sample of 450 items is taken from a population whose S.D. is 20. The mean of the sample is 30. Test whether the sample has come from a population with mean 29.

7. A sample of 400 students is fould to have a mean height of 171.38 cm. Can it be reasonably regarded as a sample from a large population with mean 171.17 cm and the standard deviation 3.3 cm.

8. The mean breaking strength of cables supplied by a manufacturer is 1800 with a S.D. of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of cables has increased. In order to test this claim, a sample of 50 cables is tested. It is found that the mean breaking strength is 1850. Can we support the claim at 1% level of significance?

9. The waves of a factory's workers are assumed to be normally distributed with mean $\mu = 49$ and variance 25. A random sample of 50 workers gives the total wages is Rs.2500. Test the hypothesis $\mu = 52$ against $\mu = 49$ at 1% level of significance.

10. Two salesmen A and B are working in a certain area. From sa sample survey conducted by the central office, the following information was obtained.

Sales man	No. of sales	Average sales (Rs.)	S.D. (R.s)
A	200	170	20
B	180	205	25

Test whether there is any significant difference in the average sales between the two salesmen at 5% level.

11. A random sample of 1000 men from a city shows that their mean wage is Rs. 5 per day with a S.D. of Rs.1.50. A random sample of 1500 men from another city gives a mean wage of Rs.4.50 per day with a S.D. of Rs.2.0 Does the mean rate of wages varies as between the two cities?

12. Intelligence tests of two groups of boys and girls gave the following information

	Girls	Boys
Sample size	100	120
Mean	86	83
S.D.	12	10

Examine if the difference in means is significant? Test at 5% level.

13. A sample of 400 items has a mean of 11.3 and another sample of 900 items has a mean of 10.1. Can the samples be regarded as having been drawn from a common population of S.D.1.2.

14. Two machines A and B produced 200 and 250 articles on the average per day with a standard deviation of 20 and 25 articles respectively on the basis of records of 50 days production. Can you regard both machines equally efficient at 1% level of significance?

15. In a sample of 900, the mean is found to be 18.5 and s.d. 2.5. In another sample of 800, the mean is 19.2 and s.d. 2.7. Assuming that the samples are independent, discuss whether two samples have come from the population which has the sames 1% level of significance.

ANSWERS

1. $Z = 5.0$; H_0 is rejected.
2. $Z = 2.45$, significant at 5% level.
3. $Z = 5.5$; H_0 is rejected. No.
4. $Z = 2.5$; Yes at and level.
5. $|Z| = 2.68$; reject H_0. The claim of the factory is valid.
6. $Z = 1.06$; Accept H_0 at 5% level.
7. $Z = 1.272$, H_0 is accepted at 5% level.
8. $Z = 3.535$, H_0 is rejected. Manufacturer's claim is accepted.

9. $|Z| = 2.828$; reject H_0. Accept the hypothesis $\mu = 49$ at % level.

10. $|Z| = 14.962$, reject H_0

11. $Z = 7.13$; reject H_0

12. $Z = 1.99$; Not significant

13. $Z = 16.67$; reject H_0. No

14. $Z = 11.03$; No. Reject H_0.

15. $Z = 2.22$, Accept H_0; Yes at 1% level.

EXERCISES

1. A machine which produces mica insulating washers of use in electric devices is set to turn out washers having a thickness of 10 mms. A sample of 10 washers has an average thickness of 9.52 mm with a S.D. 0.60 mm. Test whether the product meets the specification at 5% level.

2. Ten individuals are chosen at random from a population and their heights are found to be in inches, 63, 63, 64, 65, 66, 69, 69, 70, 70, 1. Discuss the suggestion that for 9 d.f. the value of the student's that 5% level of significance is 2.262.

3. The nine items of a sample had the following values: 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of the nine items differ significantly from the assumed population mean of 47.5?

4. A random sample of size 16 has 53 as mean and the sum of squares of the deviations taken from mean is 150. Can this sample be regarded as taken from the population having 56 as mean.

5. The mean weekly sales of soap bars in department stores was 146.3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a topical week increased to 153.7 and a standard deviation 17.2. Was the advertising campaign successful?

6. The heights of 10 males of a given locality are found to be 70, 67, 62, 6,8, 61, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable to believe that the average height is greater than 64 inches.

7. A random sample of 16 values from a normal population showed a mean of 41.5 inches and sum of squares of deviation from this mean equal to 135 square inches. Show that the assumption of a mean of 43.5 inches for the population is not reasonable.

8. Two working designs are under consideration for adoption is a plant. A time and motion study shows that 12 workers using design A have an average assembly time of 300 seconds with S.D of 12 seconds and that of 15 workers using B have an average assembly time of 335 seconds and a S.D of 15 seconds. Is there a significant difference in the average assembly time between two working designs?

9. The intelligent quotient of 16 students from one area of a city showed a mean of 107 with a S.D of 10 while the IQ of 14 students from another area of the city showed a mean of 112 with a S.D. of 8. Is there a significant difference between mean I.Q's of the two groups at 5% level?

10. Samples of two types of electric tubes were tested for length of life and the following data were obtained.

	Type I	Type II
Size	8	7
Mean	1234 hrs	1036 hrs
S.D	36 hrs	40 hrs

Is the difference in the means sufficient to warrant that type I is superior to type II regarding of life?

11. A random sample of 12 boys were fed on diet A and the gains in weight in pounds are given below:

    25, 32, 30, 34, 24, 25, 14, 32, 24, 30, 31, 35

    Another random sample of 15 boys were fed on diet B and the gains in weight in pounds are given below:

    44, 34, 22, 10, 47, 31, 40, 30, 32, 35, 18, 21, 35, 29, 22

12. In an experiment to compare two types of pig - food A and B, the following results of increase in weights were observed.

Pig Number	:	1	2	3	4	5	6	7	8
Increase in weight in pounds									
Food A	:	49	53	51	52	47	50	53	53
Food B	:	52	55	52	53	51	54	54	53

Assuming that the two samples of pigs are independent, can we conclude that food B is better than food A?

ANSWERS

1. $t = 2.258$; H_0 is rejected.
2. $t = 2.02$; $H_0 : \mu = 65$ is accepted
3. $t = 1.85$; $H_0 : \mu = 47.5$ is rejected
4. $t = 3.8$; $H_0 : \mu = 56$ is rejected
5. $t = 9.03$; advertising campaign was successful
6. $t = 2$; Reject H_0. We conclude that the mean height is greater than 64 inches
7. $t = 2.667$, Reject H_0
8. $t = 6.319$, Reject H_0 difference is significant.
9. $|t| = 1.44$; H_0 is accepted
10. $|t| = 9.39$, H_0 is rejected (Type I is superior to Type II)
11. $t = 0.61, H_0$ is accepted.
12. $t = 2.2$; H_0 is rejected. Food B is better than food A.

EXERCISES

1. In a sample of 8 observations, the sum of squares of deviations of the sample values from the sample mean was 94.5 and in another sample of 10 observations it was found to be 101.7. Test whether the difference of variances is significant.

2. In a sample of 10 observations, the sum of the squares of the deviations of the sample values from the sample mean was 120 and another sample of 12 observations it was 314. Test whether this difference is significant at 5% level of significance.

3. The two random samples gave he following results.

Sample	Size	Mean	$\Sigma (x - \overline{x})^2$
1	12	14	108
2	10	15	90

   Test whether the samples came from the same population.

4. The nicotine contents in two samples of tobacco are given below.

Sample I	21	24	25	26	27	
Sample II	22	27	28	30	31	36

   Can you say that the two samples came from the same populations?

5. Two independent samples of size 9 and 7 from a normal population had the following values of the variables.

Sample I:	18	13	12	15	12	14	16	14	15
Sample II:	16	19	13	16	18	13	15		

   Do the estimates of the population variance differ significantly at 5% level.

6. The means of two random samples of size 9 and 7 respectively are 196.42 and 198.82. Their respective variances are 26.94 and 18.73. Can the samples be regarded as drawn from the same normal population.

7. Two samples are composed of 7 and 9 individuals respectively and have variances $\sigma_1^2 = 9.6$ and $\sigma_2^2 = 4.8$ respectively. Is the variance $\sigma_1^2 = 9.6$ significantly greater than the variance $\sigma_2^2 = 4.8$? Test at 5% level.

8. Seven shells were fired from a 77 mm. gun and their velocities showed a variance of 150. The velocities of six shells fired from the same gun but with a different brand of power showed a variance of 120. Test whether this difference in their variability is usual.

ANSWERS

1. $F = 1.195$; Accept H_0

2. $F = 2.14$; Accept H_0 ; Difference is not significant

3. $F = 1.02$; Accept H_0 and $t = 0.742$, Accept H_0 ; Yes

4. $F = 4.07$, variances are equal; $|t| = 1.92$, not significant; came from same population.

5. $F = 1.396$; Difference is not significant. Accept H_0

6. No. the samples have not drawn from the same normal population.

7. $F = 2.075$; First variances cannot be regarded as significantly greater than the second Accept H_0

8. $F = 1.125$; Accept H_0 ; Not significant.

EXERCISES

1. The following table shows the results of tossing one dice 120 times.

Face of dice:	1	2	3	4	5	6	Total
No. of times face appeared:	13	33	14	7	36	17	120

Use χ^2-test to prove the hypothesis that the dice is fair.

2. Five dices were thrown 192 times and the number of times 4, 5 or 6 were obtained are follows.

No. of dice throwing 4,5 or 6:	5	4	3	2	1	0
Frequency:	6	46	70	48	20	2

Use χ^2 test of assess the goodness of fit.

3. In 120 throws of a single die, the following distribution was obtained.

Faces:	1	2	3	4	5	6	Total
Frequency:	30	25	18	10	22	15	120

Show that the dice is biased.

4. The following table gives the number of aircraft accidents that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week.

Days:	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total
No. of accidents:	14	16	8	12	11	9	14	84

5. Apply the χ^2 test of goodness of fit to the following data:

Observed frequency:	1	5	20	28	42	22	15	5	2
Expected frequency:	1	6	18	25	40	25	18	6	1

6. The following mistakes per page were obtained a book

No. of mistakes per page:	0	1	2	3	4
Frequency:	211	90	19	5	0

Fit a Poisson distribution to the data and test the goodness of fit.

ANSWERS

1. $\chi^2 = 34.4$; the dice is not fair, Reject H_0
2. $\chi^2 = 18.03$; the fit is poor; Reject H_0
3. $\chi^2 = 85.81$; Reject H_0. The dice is biased.
4. $\chi^2 = 4.17$; Accept H_0. Uniformly distributed.
5. $\chi^2 = 1.685$, Accept H_0; the fit is good.
6. $\chi^2 = 0.070$; Accept H_0; the fit is good.

SHORT QUESTIONS AND ANSWERS

1. Define Type I error and Type II error.

Type I Error:

Rejecting H_0, when it is true

Type II Error:

Accepting H_0, when it is false.

2. Give two applications of chi-square test.

The applications of chi-square test are

(i) Test of goodness of fit

(ii) Test of independence of attributes.

3. Explain null and alternate hypothesis.

Null hypothesis (H_0)

The hypothesis which is tested for possible rejection under the assumption that is true.

Alternate hypothesis (H_1)

Any hypothesis which is complementary to the null hypothesis is called an alternative hypothesis (H_1)

4. Give the formula for the chi-test of independence of attributes for

a	b
c	d

$$\chi^2 = \frac{(a+b+c+d)\,(ad-bc)^2}{(a+b)\,(a+c)\,(b+d)\,(c+d)}$$

5. What are the application of F-test.

Application of F test

(i) To test whether there is any significant difference between two estimates of population variance.

(ii) To test if the two sample have come from the same population.

6. What is called a critical region?

A region (corresponding to a statistic) in the sample space s which lead to the rejection of a null hypothesis H_o is called critical region (or) region of rejection.

7. Define level of significance.

The probability level below which a true null hypothesis is rejected is called level of significance. The levels of significance usually 5% and 1%

8. Calculate χ^2 for the following data.

O_i	**37**	**44**	**19**
E_i	**31**	**38**	**31**

$$\chi^2 = \Sigma\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

O_i	E_i	$\frac{(O_i - E_i)^2}{E_i}$
37	31	1.161
44	38	0.947
19	31	4.645

$$\chi^2 = 6.753$$

9. Find the value of χ^2 for the data in the contegency table.

2	**10**
8	**4**

$$\chi^2 = \frac{(a+b+c+d)\,(ad-bc)^2}{(a+b)\,(a+c)\,(b+d)\,(c+d)}$$

$$= \frac{(8-80)^2\,(24)}{10 \times 14 \times 12 \times 12}$$

$$\chi = 6.17$$

10. What is goodness of fit test?

It is chi square test

The value of $\chi^2 = \dfrac{\Sigma\,(O_i - E_i)^2}{E_i}$

Where,

O_i : Observed frequency

E_i : Expected frequency

11. A sample of 900 members has a mean 3.4 and standard deviation 2.61. Is the sample from a large population of mean 3.25.

$$Z = \frac{\overline{x} - \mu}{\left(\dfrac{S}{\sqrt{n}}\right)}$$

$$= \frac{3.4 - 3.25}{\dfrac{2.61}{\sqrt{900}}} = \frac{0.15}{\dfrac{2.61}{30}}$$

$$|Z| = \frac{0.15 \times 30}{2.61}$$

$$= 1.724$$

12. Two random samples of 11 and 9 items taken from two normal populations showed the sample *S.ds* as 0.8 and 0.5 respectively. Calculate the F-statistic.

By F-test

$$F = \frac{s_1^2}{s_2^2}$$

$$S_1^2 = \frac{n_1\, s_1^2}{n_1 - 1}$$

$$S_2^2 = \frac{n_2\, s_2^2}{n_2 - 1}$$

Given: $n_1 = 11, s_1 = 0.8$

$n_2 = 9,\ s_2 = 0.5$

$$S_1^2 = \frac{11 \times (0.8)^2}{11 - 1}$$

$$S_2^2 = \frac{9 \times (0.5)^2}{9 - 1}$$

$$F = \frac{s_1^2}{s_2^2} = 2.503$$

13. What are the applications of *t*-distribution.

1. To test the significant difference between the means of two independent samples.
2. To test the significant difference between the means of two dependent samples or paired observation.
3. To test the significance difference between population mean and mean of a random sample.
4. To test the significance of an observed correlation coefficient.

14. What is sampling distribution?

From a population a number of samples are drawn of equal size n. Find out the mean of each sample. The means of samples are not equal. The means with their respective frequencies are grouped. The frequency distribution formed is known as sampling distribution.

15. Mention the various steps involved in testing of hypothesis.

(i) Set up a null hypothesis H_0.

(ii) Set up the alternative hypothesis H_1.

(iii) Select the appropriate level of significance (α)

(iv) Compute the test statistic $z = \frac{t - E(t)}{S.E(t)}$

(v) We compare the calculate z with critical value z_α at given level of significance (α)

16. What are the expected frequency of 2×2 contingency table given below

a	***b***
c	***d***

Expected frequency table:

$\frac{(a+b)(a+c)}{N}$	$\frac{(a+b)(b+d)}{N}$
$\frac{(a+c)(c+d)}{N}$	$\frac{(c+d)(b+d)}{N}$

17. Find the standard error of sample mean from the following data $n = 14$, $\mu = 18.5$, $\bar{x} = 17.85$, $s = 1.955$

$$\text{Standard Error } = \frac{s}{\sqrt{n-1}} = \frac{1.955}{\sqrt{14-1}}$$

$$= 0.54$$

18. Define sampling distribution.

The probability distribution of a sample statistic is called sampling distribution.

UNIT - II

Design of Experiments

2.0 INTRODUCTION

By Experiment we mean collection of data for a Scientific Investigation according to some specific sampling procedures.

Design of experiment

The logical construction of the experiment in which the degree uncertainty with which the inference is drawn may be well defined.

In Engineering experimentation plays an important role in new project design, manufacturing process and process improvements.

Purpose

To obtain maximum information with minimum cost and labour.

Basic principles of experimental design

- Randomization
- Replication
- Local control

Complete block designs (Three designs)

- Completely Randomized Design (CRD)
- Randomized Block Design (RBD)
- Lain Square Design (LSD)

ANOVA: Analysis of variance

Analysis of variance of is developed by R.A. Fisher in 1980. Analysis of variance is a powerful statistical tool for test of significance.

ANOVA is a technique that will enable us to test the significance of the difference among more than two samples.

General application of ANOVA

- Pharmacy
- Biology
- Microbiology
- Agriculture
- Statistics
- Marketing
- Business research
- Finance
- Mechanical calculation

Three important designs of ANOVA:

(i) 1-way ANOVA → Completely Randomized Design (CRD)

(ii) 2-way ANOVA → Randomized Block Design (RBD)

(iii) 3-way ANOVA → Latin Square Design (LSD)

ANOVA: [Analysis of variance]

- It is a powerful statistical tool for tests of significance.
- The Assumptions are

(i) The observations are Independent.

(ii) Parent population from which observations are taken is normal.

(iii) Various treatment and environmental effects are additive in nature.

ANOVA

The separation of variance ascribable to one group of causes from the variance ascribable to other groups.

Abbreviations

SST	–	Sum of squares of Total
SSC	–	Sum of squares of Columns
SSR	–	Sum of squares of Rows
SSV	–	Sum of squares of varieties
SSE	–	Sum of squares of Error
MSC	–	Mean square Column
MSR	–	Mean Square Row
MSE	–	Mean Square Error
CF	–	Correlation factor
N	–	Number of observations

2.1 ONEWAY CLASSIFICATIONS (CRD)

One-way classification: observations are classified according to one factor. This is exhibited in column-wise.

Completely Randomized Design [CRD]
Oneway Classification
(Working procedure of one-way classification):

Step 1: Setting hypothesis

H_0 : There is no significant difference among the treatments.

H_1 : There is some significant difference among the treatments.

Step 2: Find correlation factor

$$\text{C.F} = \frac{(\text{Grand total})^2}{N} = \frac{(T)^2}{N}$$

N – The number of observations.

T – The Total value of all observations.

Step 3: Calculate the sum of squares of Total (SST)

$$SST = [\text{ Sum of squares of all the data }] - CF$$

Step 4: Calculate the sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \dots \right] - CF$$

Here n_1 is the number of elements in column 1.

n_2 is the number of elements in column 2.

Step 5: Calculate the sum of squares of error [SSE]

$$SSE = SST - SSC$$

Step 6: One way table

Sources of variation	Sum of squares	d.f	Mean square	*F*-ratio
Between columns	SSC	$C-1$	$MSC = \frac{SSC}{C-1}$	$F_c = \frac{MSC}{MSE}$ if $MSC > MSE$
Error	SSE	$N-C$	$MSE = \frac{SSE}{N-C}$	$F_c = \frac{MSE}{MSC}$ if $MSE > MSC$

Step 7: Table value

Table F_c with d.f $(N-C, C-1)$ at 5% or 1% LOS.

Step 8: Conclusion

(i) If Cal $F_c <$ Table F_c, we accept H_0

(ii) If Cal $F_c >$ Table F_c, we reject H_0.

Merits

- Design is very flexible.
- Maximum use of Experimental units.
- For experiments with small number of treatments.

Demerits

- Since the randomization is not restricted in any directions to ensure that units receiving one treatment similar to those receiving the other treatment.

WORKED EXAMPLES

Example 1: The following are the number of mistakes made in 5 successive days by 4 technicians working for a photographic laboratory. Test whether is there any significant difference among the 4 technicians mistakes.

Technicians

I	II	III	IV
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

Solution:

Step 1: Given data:

I	II	III	IV
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11
$C_1 = 49$	$C_2 = 59$	$C_3 = 55$	$C_4 = 50$

Here C_1, C_2, C_3, C_4 are respective columns total.

Null Hypothesis (H_0) : There is no significant difference among technician mistakes.

Alternative Hypothesis (H_1) : There is some significant difference among technicians mistakes.

$$N = 20 \quad \text{(Total number of data)}$$

$$C = 4 \quad \text{(Columns)}$$

$$G = \text{Grand Total} = C_1 + C_2 + C_3 + C_4$$

$$G = 49 + 59 + 55 + 50 = 213$$

Step 2: Calculate (Correlation factor):

$$CF = \frac{G^2}{N}$$

$$= \frac{(213)^2}{20}$$

$$= \frac{452369}{20}$$

$$\boxed{\mathbf{C.F = 2268.45}}$$

Step 3: Calculate SST (Sum of squares of total)

$$SST = [6^2 + 14^2 + 10^2 + 8^2 + 11^2 + 14^2 + 9^2 + 12^2 + 10^2 + 14^2$$
$$+ 10^2 + 12^2 + 7^2 + 15^2 + 11^2 + 9^2 + 12^2 + 8^2 + 10^2 + 11^2)] \text{ CF}$$

$$= 2383 - 2268.45$$

$$\boxed{\boldsymbol{SST = 114.55}}$$

Step 4: Calculate *SSC* (sum of square of columns)

$$SSC = \left[\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \frac{C_3^2}{n_3} + \frac{C_4^2}{n_4}\right] - \text{C.F}$$

$$= \left\{\left(\frac{49^2}{5}\right) + \left(\frac{59^2}{5}\right) + \left(\frac{55^2}{5}\right) + \left(\frac{50^2}{5}\right)\right\} - 2268.45$$

$$= 2281.4 - 2268.45$$

$$\boxed{\boldsymbol{SSC = 12.95}}$$

Step 5: Calculate *SSE* (sum of squares of error)

$$SSE = SST - SSC$$

$$= 114.55 - 12.95$$

$$\boxed{SSE = 101.6}$$

Step 6: One Way ANOVA Table

Sources of variation	Sum of squares	Degree of freedom	Mean square value	*F*-ratio
Between Columns	SSC = 12.95	$C-1$ = 3	$MSC = \frac{SSC}{C-1}$ $\frac{12.95}{3} = 4.317$	$F = \frac{6.35}{4.317}$ $F = 1.471$
Error	SSE = 101.6	$N-C$ = 16	$MSE = \frac{SSE}{N-C}$ $\frac{101.6}{16} = 6.35$	
	$SST =$ 114.55	$N-1 = 19$		

Step 7: Conclusion:

$$\boxed{\textbf{Calculated value} = 1.471}$$

Table value

At 5% LOS (Level of significance)

degree of freedom $= (16, 3)$ (See *F*-distribution 5% table)

Table value $= 8.66$

Calculated value < Table value

H_0 is Accepted

There is no significant difference among technicians mistakes.

Example 2: As a part of investigation of the collapse of the roof of the building, a testing laboratory is given all the available bolts that connected all the steal structure at three different positions on the roof. The force required to share each of these bolts are as follows.

Position 1:	**90**	**82**	**79**	**98**	**83**	**91**	
Position 2:	**105**	**89**	**93**	**104**	**89**	**95**	**86**
Position 3:	**83**	**89**	**80**	**94**			

Perform an Analysis of variance to test at 5% level of significance. Whether the difference among sample means at 3 positions are significant.

Solution:

Step 1: Given data

P_1	P_2	P_3
90	105	83
82	89	89
79	93	80
98	104	94
83	89	
91	95	
	86	
$C_1 = 523$	$C_2 = 661$	$C_3 = 346$

Null Hypothesis (H_0) : There is no significant difference between sample means of 3 position.

Alternative Hypothesis (H_1) : There is some significant difference between sample means of 3 position

$C_1 = 523 \quad C_2 = 661 \quad C_3 = 346$

$N = 17$

$C = 3$

$G = C_1 + C_2 + C_3$

$= 523 + 661 + 346$

$G = 1530$

Step 2: Calculate Correlation factor

$$\text{C.F} = \frac{G^2}{N}$$

$$\text{C.F} = \frac{(1530)^2}{17}$$

$$\textbf{C.F = 137700}$$

Step 3: Calculate Sum of square of total (SST)

$$SST = [90^2 + 82^2 + 79^2 + 98^2 + 83^2 + 91^2 + 105^2 + 89^2 + 93^2 + 104^2 + 89^2 + 95^2 + 86^2 + 83^2 + 89^2 + 80^2 + 94^2] - \text{CF}$$

$= 138638 - 137700$

$\boxed{\boldsymbol{SST = 938}}$

Step 4: Calculate *SSC* (sum of squares of columns)

$$SSC = \left[\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \frac{C_3^2}{n_3}\right] - \text{C.F}$$

$$= \left[\frac{(523)^2}{6} + \frac{(661)^2}{7} + \frac{(346)^2}{4}\right] - 137700$$

$= 137934.45 - 137700$

$\boxed{\boldsymbol{SSC = 234.45}}$

Step 5: Calculate Sum of square of error (SSE)

$$SSE = SST - SSC$$

$$= 938 - 234.45$$

$$\boxed{\mathbf{SSE = 703.55}}$$

Step 6: 1 - Way ANOVA Table

Source of variation	**Sum of squares**	**Degrees of freedom**	**Mean square value**	***F*-ratio**
Between Columns	SSC = 234.45	$C - 1$ = 2	$MSC = \frac{234.45}{2}$ $= 117.23$	$F = \frac{117.23}{50.253}$
Error	SSE = 703.55	$N - C$ = 14	$MSE = \frac{703.55}{14}$ $= 50.253$	$F = 2.333$

Step 7: Conclusion

Calculated value = 2.333

Table value

at 5% LoS

Degree of freedom $= (2, 14)$ (See F-distribution 5% table)

Table value = 3.74

Calculate value < Table value

H_0 is accepted

There is no significant difference between sample means of 3 position.

Example 3: The following are the prices of commodity in 3 cities.

Bombay:	**16**	**8**	**12**	**14**
Calcutta:	**14**	**10**	**10**	**6**
Delhi:	**4**	**10**	**8**	**8**

Do the data indicate that the prices of the 3 cities are significantly different.

Solution:

Step 1: Given data

Bombay	Calcutta	Delhi
16	14	4
8	10	10
12	10	8
14	6	8
$C_1 = 50$	$C_2 = 40$	$C_3 = 30$

H_0 : There is no significant difference between the prices in the 3 cities.

H_1 : There is a significant difference between the prices in the 3 cities.

$$C_1 = 50 \quad C_2 = 40 \quad C_3 = 30$$

$$N = 12$$

$$C = 3$$

$$G = C_1 + C_2 + C_3$$

$$= 50 + 40 + 30$$

$$G = 120$$

Step 2: Calculate Correlation factor (C.F)

$$\text{C.F} = \frac{G^2}{N}$$

$$= \frac{(120)^2}{12}$$

$$\boxed{\textbf{C.F} = \textbf{1200}}$$

Step 3: Calculate Sum of square of total: (SST)

$$SST = [16^2 + 8^2 + 12^2 + 14^2 + 14^2 + 10^2 + 10^2 + 6^2 + 4^2 + 10^2 + 8^2 + 8^2] - \text{C.F}$$

$$= 1336 - 1200$$

$$\boxed{\boldsymbol{SST = 136}}$$

Step 4: Calculate Sum of square of columns (*SSC*)

$$SSC = \left[\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \frac{C_3^2}{n_3}\right] - \text{C.F}$$

$$= \left[\frac{50^2}{4} + \frac{40^2}{4} + \frac{30^2}{4}\right] - 1200$$

$$= 1250 - 1200$$

$$\boxed{\boldsymbol{SSC = 50}}$$

Step 5: Sum of square of error (*SSE*)

$$SSE = SST - SSC$$

$$= 136 - 50$$

$$\boxed{\boldsymbol{SSE = 86}}$$

Step 6: One Way ANOVA Table

Source of variation	Sum of squares	Degrees of freedom	Mean square value	*F*-ratio
Between columns	$SSC = 50$	$C - 1 = 2$	$MSC = \frac{50}{2} = 25$	$F = \frac{25}{9.55}$
Error	$SSE = 86$	$N - C = 9$	$MSE = \frac{86}{9} = 9.55$	$F = 2.62$

Step 7: Conclusion

Calculate value = 2.62

Table value LOS = 5%

Degree of Freedom = (2, 9)

Table value = 4.26

Calculated value < Table value

H_0 is accepted.

There is some significant difference between the prices in the 3 cities.

Example 4: There are 3 typists working in an office the times (in minutes) they spend for the tea break in addition to the allowed much tea break are observed and noted below:

A	**25**	**18**	**30**	**32**	**35**	**37**	**19**			
B	**24**	**22**	**26**	**28**	**30**	**32**	**28**	**26**		
C	**28**	**20**	**27**	**19**	**29**	**35**	**30**	**23**	**27**	**32**

Can the difference in average that the 3 typists spend for the tea break be attributed to chance variation.

Solution: Given Data

A	*B*	*C*
25	24	28
18	22	20
30	26	27
32	28	19
35	30	29
37	32	35
19	28	30
	26	23
		27
		32
$C_1 = 196$	$C_2 = 216$	$C_3 = 270$

Step 1: Setting Hypothesis

H_0: There is no significant difference among the columns.

H_1: There is some significant difference among the columns.

Step 2: Calculate Correlation Factor (C.F)

$$\text{Grand total} = C_1 + C_2 + C_3$$

$$= 196 + 216 + 270$$

$$\text{G.T} = 682$$

$$\text{C.F} = \frac{(\text{Grand total})^2}{N} = \frac{\text{T}^2}{\text{N}}$$

$$= \frac{(682)^2}{25}$$

$$\boxed{\textbf{C.F} = \textbf{18604.96}}$$

Step 3: Calculate sum of squares of total [SST]

$$SST = \begin{bmatrix} 25^2 + 18^2 + 30^2 + 32^2 + 35^2 + 37^2 + 19^2 + 24^2 + 22^2 \\ + 26^2 + 28^2 + 30^2 + 32^2 + 28^2 + 26^2 + 28^2 + 20^2 + 27^2 \\ + 19^2 + 29^2 + 35^2 + 30^2 + 23^2 + 27^2 + 32^2 \end{bmatrix}$$

$$= 192544 - 18604.96$$

$$\boxed{SST = 649.04}$$

Step 4: Calculate sum of squares of column [SSC]

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} \right] - CF$$

$$= \left[\frac{(196)^2}{7} + \frac{(216)^2}{8} + \frac{(270)^2}{10} \right] - 18604.96$$

$$= 18610 - 18604.96$$

$$\boxed{SSC = 5.04}$$

Step 5: Calculate sum of squares of error [SSE]

$$SSE = SST - SSC$$

$$= 649.04 - 5.04$$

$$\boxed{SSE = 644}$$

Step 6: One - way classification table

Sources of variation	Sum of squares	d.f	Mean square	*F*-ratio
Between Columns	$SSC = 5.04$	$C - 1$ $= 3 - 1$ $= 2$	$MSC = \frac{SSC}{C - 1}$ $= \frac{5.04}{2} = 2.52$	$F = \frac{29.27}{2.52}$ $F_C = 11.61$
Error	$SSE = 644$	$N - C$ $= 25 - 3$ $= 22$	$MSE = \frac{SSF}{N - C}$ $= \frac{644}{22} = 29.27$	

Step 7: Conclusion

Table value At 5% LOS

d.f $= (22, 2)$

Table value $= 3.44$

Calculated value = 11.61

Since the table value is less than the calculated value

H_0 is rejected,

H_1 is accepted

Hence there is some significant difference among columns.

Example 5: A completely randomised design experiment with 10 plots and treatments gave the following results.

Plot No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Treatment:	***A***	***B***	***C***	***A***	***C***	***C***	***A***	***B***	***A***	***B***
Yield:	**5**	**4**	**3**	**7**	**5**	**1**	**3**	**4**	**1**	**7**

Solution: Given Data

A	*B*	*C*
5	4	3
7	4	5
3	7	1
1		

Step 1: Setting hypothesis

H_0: There is no significant difference among column.

H_1: There is some significant difference among column.

Step 2: Calculate Correlation Factor [C.F]

A	***B***	***C***
5	4	3
7	4	5
3	7	1
1		
$C_1 = 16$	$C_2 = 15$	$C_3 = 9$

$$\text{Grand total} = C_1 + C_2 + C_3$$
$$= 16 + 15 + 9$$
$$= 40$$
$$\text{C.F} = \frac{(\text{Grand total})^2}{N} = \frac{T^2}{N}$$
$$= \frac{(40)^2}{10}$$
$$\boxed{\textbf{C.F} = \textbf{160}}$$

Step 3: Calculate sum of squares of total [SST]

$$SST = [5^2 + 7^2 + 3^2 + 1^2 + 4^2 + 4^2 + 7^2 + 3^2 + 5^2 + 1^2] - C.F$$
$$= [200] - [160]$$
$$\boxed{\boldsymbol{SST = 40}}$$

Step 4: Calculate sum of squares of column [SSC]

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3}\right] - \text{C.F}$$
$$= \left[\frac{(16)^2}{4} + \frac{(15)^2}{3} + \frac{(9)^2}{3}\right] - 160$$
$$SSC = 166 - 160 = 6$$
$$\boxed{\boldsymbol{SSC = 6}}$$

Step 5: Calculate sum of squares of error [SSE]

$$SSE = SST - SSC$$

$$= 40 - 6$$

$$\boxed{\boldsymbol{SSE = 34}}$$

Step 6: One way classification Table

Sources of variation	Sum of squares	d.f	Mean square	*F*-ratio
Between Columns	$SSC = 6$	$C - 1$ $S - 1 = 2$	$MSC = \frac{SSC}{C-1}$ $= \frac{6}{2} = 3$	$F = \frac{4.85}{3}$ $F_c = 1.45$
Error	$SSE = 34$	$N - C$ $10 - 3 = 7$	$MSE = \frac{SSE}{N-C}$ $= \frac{34}{7}$ $= 4.85$	

Step 7: Conclusion

$$\boxed{\textbf{Calculated value} = \mathbf{1.65}}$$

Table value at 5% at LOS:

Degrees of freedom $= (7, 2)$

Table value $= 19.35$

Since the table value is greater than the calculated value

H_0 is accepted

Hence there is no significant difference among columns.

Example 6: The accompanying data resulted from are experiment comparing the degree of soiling for fabric co-polymerized with the three different mixtures of methacryclic acid analysis is the given classification.

Mixture	**1**	**0.56**	**1.12**	**0.90**	**1.07**	**0.94**
Mixture	**2**	**0.72**	**0.69**	**0.87**	**0.78**	**0.91**
Mixture	**3**	**0.62**	**1.08**	**1.07**	**0.99**	**0.93**

Solution: Given data

M_1	M_2	M_3
0.56	0.72	0.62
1.12	0.69	1.08
0.90	0.87	1.07
1.07	0.78	0.99
0.94	0.91	0.93

Step 1: Setting hypothesis

H_0: There is no significant difference among columns.

H_1: There is some significant difference among the columns.

Step 2: Calculate Correlation Factor [C.F]

M_1	M_2	M_3
0.56	0.72	0.62
1.12	0.69	1.08
0.90	0.87	1.07
1.07	0.78	0.99
0.94	0.91	0.93
$C_1 = 4.59$	$C_2 = 3.97$	$C_3 = 4.69$

$$\text{Grand total} = C_1 + C_2 + C_3$$

$$= 4.59 + 3.97 + 4.69$$

$$= 13.25$$

$$\text{C.F} = \frac{(\text{Grand total})^2}{\text{N}} = \frac{(13.25)^2}{15} = 11.70$$

Step 3: Calculate sum of squares of total [SST]

$$SST = \begin{bmatrix} 0.56^2 + 1.12^2 + 0.90^2 + 1.07^2 + 0.94^2 + 0.72^2 + \\ 0.69^2 + 0.87^2 + 0.78^2 + 0.98^2 + 0.62^2 + 1.08^2 + 1.07^2 + \\ 0.99^2 + 0.93^2 \end{bmatrix} \text{C.F}$$

$$= [12.13] - [11.70]$$

$$\boxed{\boldsymbol{SST = 0.43}}$$

Step 4: Calculate sum of squares of column [SSC]

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} \right] - \text{C.F}$$

$$= \left[\frac{(4.59)^2}{5} + \frac{(3.97)^2}{5} + \frac{(4.69)^2}{5} \right] - 11.70$$

$$= [11.76] - [11.70]$$

$$\boxed{\boldsymbol{SSC = 0.06}}$$

Step 5: Calculate sum of squares of error [SSE]

$$SSE = SST - SSC$$

$$= 0.43 - 0.06$$

$$\boxed{\boldsymbol{SSE = 0.37}}$$

Step 6: One way classification table

Sources of variation	**Sum of squares**	**d.f**	**Mean square**	***F*-ratio**
Between Column	$SSC = 0.06$	$C - 1$ $3 - 1 = 2$	$MSC = \dfrac{SSC}{C-1}$ $= \dfrac{0.06}{2} = 0.03$	$F = \dfrac{0.03}{0.03}$ $= 1$
Error	$SSE = 0.37$	$N - C$ $= 15 - 3$ $= 12$	$MSE = \dfrac{SSE}{N-C}$ $= \dfrac{0.27}{12} = 0.030$	

Step 7: Conclusion

Calculated value = 1

Table value at 5% LOS

Degrees of freedom = (12, 2)

Table value = 19.41

Since the table value is greater than the calculated value.

H_0 is accepted.

There is no significant difference among columns.

Example 7: The following table shows the lines in hours of four of brands of electric lamps.

Brand	***A***	**1610**	**1610**	**1650**	**1680**	**1700**	**1720**	**1800**	
	B	**1580**	**1640**	**1640**	**1700**	**1750**			
	C	**1460**	**1550**	**1600**	**1620**	**1640**	**1660**	**7940**	**1820**
	D	**1510**	**1520**	**1530**	**1570**	**1600**	**1680**		

Perform an analysis of variance test the homogeneity of the mean lives of the four brands of lamps.

Solution:

Step 1: Setting hypothesis

H_0: There is no significant difference among columns.

H_1: There is some significant difference among columns.

Step 2: Calculate Correction Factor [C.F]

A	*B*	*C*	*D*
1610	1580	1460	1510
1610	1640	1550	1520
1650	1640	1600	1530
1680	1700	1620	1570
1700	1750	1640	1600
1720		1660	1680
1800		1740	
		1820	
$C_1 = 11770$	$C_2 = 8310$	$C_3 = 13090$	$C_4 = 9410$

$$\text{Grand total} = C_1 + C_2 + C_3 + C_4$$

$$= 11770 + 8310 + 13090 + 9410$$

$$= 42580$$

$$\text{C.F} = \frac{(\text{Grand total})^2}{N}$$

$$= \frac{(42580)^2}{26}$$

$$\boxed{\textbf{C.F} = \mathbf{69732938.46}}$$

Step 3: Calculate sum of squares of total [SST]

$$SST = \left[\begin{array}{l} 1610^2 + 1610^2 + 1650^2 + 1680^2 + 1700^2 + 1720^2 \\ + 1800^2 + 1580^2 + 1640^2 + 1640^2 + 1700^2 + 1750^2 \\ + 1460^2 + 1550^2 + 1600^2 + 1620^2 + 1640^2 + 1660^2 \\ + 1740^2 + 1620^2 + 1510^2 + 1520^2 + 1530^2 + 510^2 \\ + 1600^2 + 1600^2 \end{array}\right] - \text{C.F}$$

$$= 69928000 - 69732938.46$$

$$\boldsymbol{SST = 195061.54}$$

Step 4: Calculate sum of squares of column [SSC]

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} + \frac{(C_4)^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{(11770)^2}{7} + \frac{(8310)^2}{5} + \frac{(13090)^2}{8} + \frac{(9410)^2}{6}\right] - 69732938.46$$

$$= 69778163.45 - 69732938.46$$

$$\boldsymbol{SSC = 45224.99}$$

Step 5: Calculate sum of squares of error [SSE]

$$SSE = SST - SSC$$

$$= 195061.54 - 45224.99$$

$$\boldsymbol{SSE = 149836.55}$$

Step 6: One way classification table

Sources of variations	**Sum of squares**	**d.f**	**Mean square**	***F*-ratio**
Between Column	SSC $= 45224.99$	$C-1$ $= 4-1$ $= 3$	$MSE = \frac{SSC}{C-1}$ $= \frac{45224.99}{3}$ $= 15074.99$	$F = \frac{15074.99}{6810.75}$ $F_C = 2.21$
Error	SSE $= 149836.55$	$N-C$ $= 26-4$ $= 22$	$MSE = \frac{SSE}{N-C}$ $= \frac{149836.55}{22}$ $= 6810.75$	

Step 7: Conclusion

Calculated value = 2.21

Table value at 5% LOS

Degrees of freedom = (3,22)

Table value = 3.05

Since the table value is greater than the calculated value

H_0 is accepted.

Hence there is no significant difference among column.

2.2 RANDOMIZED BLOCK DESIGN [RBD] [TWO WAY CLASSIFICATION]

Define RBD

In two way classification of analysis of variance, we consider one classification along column wise and the other one along row-wise.

Working procedure of RBD

Step 1: Setting a hypothesis

H_{01} : There is no significant difference between columns.

H_{11} : There is some significant difference between columns.

H_{02} : There is no significant difference between rows.

H_{12} : There is some significant difference between rows.

Step 2: Find correlation factor: (CF)

$$\text{C.F} = \frac{(\text{Grand total})^2}{N} = \frac{(T)^2}{N}$$

N – The number of observations.

T – The Total value of all observations.

Step 3: SST

$$SST = [\text{ Sum of squares of all thedata} + \ldots] - CF$$

Step 4: SSC

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \ldots\right] - CF$$

Here n_1 is the number of elements in column 1.

n_2 is the number of elements in column 2.

Step 5: SSR

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \ldots\right] - CF$$

Here n_1 is the number of elements in Row 1

n_2 is the number of elements in Row 2

Step 6: SSE

$$SSE = SST - SSC - SSR$$

Step 7: Two way classification

Sources of variation	**Sum of squares**	**d.f**	**Mean square**	***F*-ratio**
Between columns	SSC	$C - 1$	$MSC = \frac{SSC}{C - 1}$	$F_c = \frac{MSC}{MSE}$ if $MSC > MSE$
Between rows	SSR	$r - 1$	$MSR = \frac{SSR}{r - 1}$	$F_R = \frac{MSR}{MSE}$ if $MSR > MSE$
Error	SSE	$(C - 1) \times (r - 1)$	$MSE = \frac{SSE}{(r - 1) \times (C - 1)}$	

Step 8: Conclusion

Case (i) Between columns

If Cal $F_c <$ Tab F_c, we accept H_0.

If Cal $F_c >$ Tab F_c, we reject H_0.

Case (ii) Between rows

If Cal $F_R <$ Tab F_R, we accept H_0.

If Cal $F_R >$ Tab F_R, we reject H_0.

WORKED EXAMPLES

Example 1: Five doctors each test treatments for a certain disease and observe the number of days each takes to recover. The results are as follows: (Recovery time in days)

Doctor	**Treatments**				
	1	**2**	**3**	**4**	**5**
A	**10**	**14**	**23**	**19**	**20**
B	**11**	**15**	**24**	**17**	**21**
C	**9**	**12**	**20**	**16**	**19**
D	**8**	**13**	**17**	**17**	**20**
E	**12**	**15**	**19**	**15**	**22**

Discuss the difference between (a) Doctors and (b) Treatments

Solution:

Step 1: Given data:

Treatments (C)							
Doctors							**Row Total**
A	10	14	23	19	20	$R_1 = 86$	
B	11	15	24	17	21	$R_2 = 88$	
C	9	12	20	16	19	$R_3 = 76$	
D	8	13	17	17	20	$R_4 = 75$	
E	12	15	19	15	22	$R_5 = 83$	
Column Total	$C_1 = 50$	$C_2 = 69$	$C_3 = 103$	$C_4 = 84$	$C_5 = 102$	408	Grand Total

H_{01} : There is no significant difference between treatments.

H_{11} : There is some significant difference between treatments.

H_{02} : There is no significant difference between doctors.

H_{12} : There is some significant difference between doctors.

$$N = 25 \qquad r = 5$$

$$G = 408 \qquad C = 5$$

Step 2: Calculate Correlation factor (CF)

$$\text{C.F} = \frac{G^2}{N} = \frac{(408)^2}{25}$$

$$\textbf{C.F = 6658.56}$$

Step 3: Calculate Sum of squares of total (*SST*)

$$SST = \begin{bmatrix} 10^2 + 14^2 + 23^2 + 19^2 + 20^2 + 11^2 + 15^2 + 24^2 + \\ 17^2 + 21^2 + 9^2 + 12^2 + 20^2 + 16^2 + 19^2 + 8^2 + 13^2 \\ 17^2 + 20^2 + 12^2 + 15^2 + 19^2 + 15^2 + 22^2 \end{bmatrix} - \text{C.F}$$

$$SST = 7130 - 6658.56$$

$$\boxed{\boldsymbol{SST = 471.44}}$$

Step 4: Calculate Sum of squares of column (*SSC*)

$$SSC = \left[\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \frac{C_3^2}{n_3} + \frac{C_4^2}{n_3} + \frac{C_5}{n_5}\right] - \text{C.F}$$

$$= \left[\left(\frac{50^2}{5}\right) + \left(\frac{69^2}{5}\right) + \left(\frac{103^2}{5}\right) + \left(\frac{84^2}{5}\right) + \left(\frac{102^2}{5}\right)\right] - 6658.56$$

$$= 7066 - 6658.56$$

$$\boxed{\boldsymbol{SSC = 407.44}}$$

Step 5: Calculate Sum of squares of rows (*SSR*)

$$SSR = \left[\frac{R_1^2}{n_1} + \frac{R_2^2}{n_2} + \frac{R_3^2}{n_3} + \frac{R_4^2}{n_4} + \frac{R_5}{n_5}\right] - \text{C.F}$$

$$SSR = \left[\left(\frac{86^2}{5}\right) + \left(\frac{88^2}{5}\right) + \left(\frac{76^2}{5}\right) + \left(\frac{75^2}{5}\right) + \left(\frac{83^2}{5}\right)\right] - 6658.56$$

$$= 6686 - 6658.56$$

$$\boxed{\boldsymbol{SSR = 27.44}}$$

No. of items in Row 1 $= n_1$

No. of items in Row 2 $= n_2$

No. of items in Row 3 $= n_3$

Step 6: Calculate Sum of squares of error

$SSE = SST - SSC - SSR$

$= 471.44 - 407.44 - 27.44$

$$\boxed{SSE = 36.56}$$

Step 7: Two Way ANOVA Table

Source of variation	**Sum of squares**	**Degrees of freedom**	**Mean square value**	***F*-ratio**
Between columns	SSC 407.44	$C-1$ 4	$\frac{SSC}{(C-1)} = \frac{407.44}{4}$ $MSC = 101.86$	$F_C = \frac{101.86}{2.285}$
Between rows	SSR 77.44	$r-1$ 4	$\frac{SSR}{r-1} = \frac{27.44}{4}$ $MSR = 6.86$	$F_C = 44.85$
Error	SSE 36.56	$(C-1)$ $(r-1)$ 16	$\frac{SSE}{(C-1)(r-1)}$ $= \frac{86.56}{2}$ $MSE = 2.285$	$F_R = \frac{6.86}{2.285}$ $F_R = 3.002$

Step 8: Conclusion

Case 1: (Treatment)

Calculated value $= 44.85$

Table value

$$\text{LOS} = 5\%$$

$$\text{d.f} = (4, 16)$$

Table value $= 3.0$

Calculated value > Table value

H_{01} is rejected.

There is a some significant difference between treatments.

Case: 2 (Doctors)

Calculated value $= 3.002$

Table value

LOS $= 5\%$

d.f $= (4, 16)$

Table value $= 3.01$

Calculated value < Table value

H_{02} is accepted.

There is no significant difference between doctors.

Example 2: A company appoints 4 salesman A, B, C and D and observes their sales in 3 seasons - summer, winter and monsoon.

		Salesman			
		A	***B***	***C***	***D***
Seasons	**Summer**	**45**	**40**	**38**	**37**
	Winter	**43**	**41**	**45**	**38**
	Monsoon	**39**	**39**	**41**	**41**

Carry out the analysis of variance at 5% LoS.

Step 1: Given data

	Salesman				Row Total
	45	40	38	37	$R_1 = 160$
Seasons	43	41	45	38	$R_2 = 167$
	39	39	41	41	$R_3 = 160$
Column Total	$C_1 = 127$	$C_2 = 120$	$C_3 = 124$	$C_4 = 116$	Grand Total 487

H_{01} : There is no significant difference between salesman

H_{11} : There is some significant difference between salesman

H_{02} : There is no significant difference between seasons.

H_{12} : There is some significant difference between seasons

$$N = 12 \qquad r = 3$$

$$G = 487 \qquad C = 4$$

Step 2: Calculate Correlation factor

$$\text{C.F} = \frac{G^2}{N}$$

$$\text{C.F} = \frac{(487)^2}{12}$$

$$\boxed{\textbf{C.F} = \textbf{19764.08}}$$

Step 3: Calculate Sum of square of total (*SST*)

$$SST = \left[\begin{array}{l} 45^2 + 40^2 + 88^2 + 37^2 + 43^2 + 41^2 + 45^2 + \\ 38^2 + 39^2 + 39^2 + 41^2 + 41^2 \end{array}\right] - \text{C.F}$$

$$SST = 19841 - 19764.08$$

$$\boxed{SST = 76.92}$$

Step 4: Calculate Sum of square of columns (*SSC*)

$$SSC = \left[\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \frac{C_3^2}{n_3} + \frac{C_4^2}{n_4}\right] - \text{C.F}$$

$$SSC = \left[\left(\frac{(127)^2}{3}\right) + \left(\frac{(120)^2}{3}\right) + \left(\frac{(124)^2}{3}\right) + \left(\frac{(116)^2}{3}\right)\right] - 19764.08$$

$$SSC = 19787 - 19764.08$$

$$\boxed{\boldsymbol{SSC = 22.92}}$$

Step 5: Calculate Sum of square of rows (*SSR*)

$$SSR = \left[\frac{R_1^2}{n_1} + \frac{R_2^2}{n_2} + \frac{R_3^2}{n_3}\right] - \text{C.F}$$

$$SSR = \left[\left(\frac{(160)^2}{4}\right) + \left(\frac{(167)^2}{4}\right) + \left(\frac{(160)^2}{4}\right)\right] - 19764.08$$

$$SSR = 19772.25 - 19764.08$$

$$\boxed{\boldsymbol{SSR = 8.17}}$$

Step 6: Calculate Sum of squares of error (SSE)

$$SSE = SST - SSE - SSR$$

$$= 76.92 - 22.92 - 8.17$$

$$\boxed{\boldsymbol{SSE = 48.83}}$$

Step 7: Two Way ANOVA Table

Source of variation	Sum of squares	Degrees of freedom	Mean square value	*F*-ratio
Between columns	SSC 22.92	$C - 1$ 3	$= \frac{22.92}{3}$ $= 7.64$	
Between rows	SSR 8.17	$r - 1$ 2	$= \frac{8.17}{2}$ $= 4.085$	$F_C = \frac{7.64}{7.6385}$ $F_C = 0.9941$
Errors	SSE 45.63	$(c - 1)(r - 1)$ 6	$= \frac{45.83}{6}$ $= 7.6383$	$F_R = \frac{7.6383}{4.085}$ $F_R = 1.869$

Step 8: Conclusion

Case 1 (Sales man)

Calculated value = 0.9941

Table value

$$\text{LOS} = 5\%$$

$$\text{d.f} = (3, 6)$$

Table value $= 4.76$

Calculated value < Table value

H_{01} is Accepted.

There is no significant difference between salesman.

Case 2 (Season)

Calculated value = 1.869

Table value

$$\text{LOS} = 5\%$$

$$\text{d.f} = (6, 12)$$

Table value: 19.33

Calculated value < Table value

H_{02} is Accepted.

There is no significant difference between seasons.

Example 3: An experiment was designed to study the performance of 4 different detergents for cleaning fuel injectors. The following cleanliness readings were obtained with specially designed equipment for 12 tables of gas distributed over 3 different models of engines.

	Engine 1	**Engine 2**	**Engine 3**	**Total**
Detergent *A*	**45**	**43**	**51**	**139**
Detergent *B*	**47**	**46**	**52**	**145**
Detergent *C*	**48**	**50**	**55**	**153**
Detergent *D*	**42**	**37**	**49**	**128**
Total	**182**	**176**	**207**	**565**

Solution:

Step 1: Setting hypothesis

H_{01}: There is no significant difference between Engines.

H_{11}: There is some significant difference between engines.

H_{02}: There is no significant difference between detergents.

H_{12}: There is some significant difference between detergents.

Step 2: Calculate Correlation Factor (C.F)

Detergent	**Engine**			**Row Total**
	1	**2**	**3**	
A	45	43	54	$R_1 = 139$
B	47	46	52	$R_2 = 145$
C	48	50	55	$R_3 = 153$
D	42	37	49	$R_4 = 128$
Column Total	$C_1 = 182$	$C_2 = 176$	$C_3 = 207$	Grand Total 565

$$\text{C.F.} = \frac{(\text{Grand total})^2}{N}$$

$$= \frac{(565)^2}{12}$$

$$= 26602.08$$

Step 3: Calculate sum of squares of total (SST)

$$SST = \begin{bmatrix} 45^2 + 47^2 + 48^2 + 42^2 + 43^2 + 46^2 + 50^2 \\ + 31^2 + 51^2 + 52^2 + 55^2 + 49^2 \end{bmatrix} - \text{C.F}$$

$$= [268.67] - [26602.08]$$

$$\boxed{\boldsymbol{SST = 264.92}}$$

Step 4: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3}\right] - \text{C.F}$$

$$= \left[\frac{(15^2)^2}{4} + \frac{(176)^2}{4} + \frac{(207)^2}{4}\right] - 26602.08$$

$$\boxed{\boldsymbol{SSC = 135.17}}$$

Step 5: Calculate sum of squares of Rows (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \frac{(R_4)^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{(139)^2}{3} + \frac{(145)^2}{3} + \frac{(153)^2}{3} + \frac{(128)^2}{3}\right] - 2660.03$$

$$= [26713] - [26602.08]$$

$$\boxed{\boldsymbol{SSR = 110.92}}$$

Step 6: Calculate sum of squares of error (SSE)

$$SSE = SST - SSC - SSR$$

$$= 264.92 - 135.17 - 110.92$$

$$\boxed{\boldsymbol{SSE = 18.83}}$$

Step 7: Two way classification Tasle

Sources of variation	**Sum of squares**	**d.f**	**Mean square**	***F*-ratio**
Between Columns	SSC $= 135.17$	$C - 1$ $= 3 - 1$ $= 2$	$MSC = \frac{SSC}{C - 1}$ $= \frac{135.17}{2}$ $= 67.585$	$F_C = \frac{67.585}{3.13}$
Between row	SSR $= 110.92$	$R - 1$ $= 4 - 1$ $= 3$	$MSR = \frac{SSR}{r - 1}$ $= \frac{110.92}{3}$ $= 36.97$	$\boldsymbol{F_C = 21.59}$
Error	SSE $= 18.83$	$(r - 1) \times (c - 1)$ $= (2 - 1)(3 - 1)$ $= 6$	$MSE = \frac{SSE}{(r = 1)(c - 1)}$ $= \frac{18.83}{6}$ $= 3.13$	$F_R = \frac{36.97}{3.13}$ $\boldsymbol{F_R = 11.81}$

Step 8: Conclusion

Case (i) [Between Engines]

Calculated value $= 21.59$

Table value at 1% LOS

degrees of freedom $= (2, 6)$

Table value $= 10.92$

Since the table value is less than the calculated value

H_{01} is rejected.

H_{11} is accepted.

There is some significant difference between engines.

Case (ii) (Between Detergeats)

Calculated value $= 11.81$

Table value of at 1% LOS

Degrees of freedom $= (3, 6)$

Table value $= 9.78$

Since the table value is less than the calculated value

H_{02} is rejected

H_{12} is accepted

There is some significant difference between detergents.

Example 4: Three varieties A, B and C of a crop are tested in a randomised block design with four replications. The plot yield in pounds are as follows:

A	**6**	***C***	**5**	***A***	**8**	***B***	**9**
C	**8**	***A***	**4**	***B***	**6**	***C***	**9**
B	**7**	***B***	**6**	***C***	**10**	***A***	***G***

Analyse the experiment yield and state your conclusion. The given data can be arranged as follows

Solution:

Arrange the data

	Replications			
Crops	**1**	**2**	**3**	**4**
A	6	4	8	6
B	7	6	6	9
C	8	5	10	9

Step 1: Setting Hypothesis

H_{01} : There is no significant difference between replications.

H_{01} : There is some significant difference between replications.

H_{02} : There is no significant difference between crops.

H_{12} : There is some significant difference between crops.

Step 2: Calculated Correlation Factor (C.F)

	Replications				
Crops	**1**	**2**	**3**	**4**	**Row total**
A	6	4	8	6	$R_1 = 24$
B	7	6	6	9	$R_2 = 28$
C	8	5	10	9	$R_3 = 32$
Column total	$C_1 = 21$	$C_2 = 15$	$C_3 = 24$	$C_4 = 24$	Grand Total 84

$$\text{C.F} = \frac{(\text{Grand total})^2}{\text{N}} = \frac{(84)^2}{12} = 588$$

$$\boldsymbol{C.F. = 588}$$

Step 3: Calculate sum of squares of total (*SST*)

$$SST = \begin{bmatrix} 6^2 + 7^2 + 8^2 + 6^2 + 5^2 + 8^2 \\ + 6^2 + 10^2 + 6^2 + 9^2 + 9^2 \end{bmatrix} - \text{C.F}$$

$$= [624] - [588]$$

$$\boxed{\boldsymbol{SST = 36}}$$

Step 4: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} + \frac{(C_4)^2}{n_4} \right] - \text{C.F}$$

$$= \left[\frac{(21)^2}{3} + \frac{(15)^2}{3} + \frac{(24)^2}{3} + \frac{(24)^2}{3} \right] - 588$$

$$SSC = [606] - [588]$$

$$\boxed{\boldsymbol{SSC = 18}}$$

Step 5: Calculate sum of squares of rows (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \quad\right] - \text{C.F}$$

$$= \left[\frac{(24)^2}{4} + \frac{(28)^2}{4} + \frac{(32)^2}{4}\right] - 588$$

$$= 596 - 588$$

$$\boxed{SSR = 8}$$

Step 6: Calculate sum of squares of error (SSE)

$$SSE = SST - SSC - SSR$$

$$= 36 - 18 - 8$$

$$\boxed{SSE = 10}$$

Step 7: Two way classification table

Sources of variances	Sum of squares	d.f	Mean square	*F*-ratio
Between Columns	$SSC = 18$	$C - 1 = 4 - 1$ $= 3$	$MSC = \frac{SSC}{C-1}$ $= \frac{18}{3} = 6$	$F_C = \frac{6}{1.66}$ $= 3.61$
Between rows	$SSR = 8$	$R - 1 = 3 - 1$ $= 2$	$MSR = \frac{SSR}{R-1}$ $= \frac{8}{2} = 4$	$F_R = \frac{4}{1.66}$ $= 2.40$
Error	$SSE = 10$	$(r-1) \times (c-1)$ $3 \times 2 = 6$	$MSE = \frac{SSE}{(r-1)(c-1)}$ $= \frac{10}{6} = 1.66$	

Step 8: Conclusion

Case (i) (Between Replications)

Calculated value $= 3.61$

Table value at 5% LOS

degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table is greater than the calculated value

H_{01} is accepted

There is no significant difference between replications.

Case (ii) (Between Crops)

Calculated value $= 2.40$

Table value of at 5% LOS

Degrees of freedom $= (2, 6)$

Table value $= 5.14$

Since the table value is greater than the calculated value

H_{02} is accepted

There is no significant difference among crops.

Example 5: The following data represent the number of units of production per day, turned out by four randomly chosen operators using three milling machines.

Operators	**Machines**		
	M_1	M_2	M_3
1	**150**	**151**	**156**
2	**147**	**159**	**155**
3	**141**	**146**	**153**
4	**154**	**152**	**159**

Perform analysis of variance and test the hypothesis.

(a) that the machines are not significantly different

(b) then the operators are not significantly different at 5% level.

Solution:

Step 1: Set up a hypothesis

H_{01} : There is no significant difference among machines.

H_{11} : There is some significant difference among machines.

H_{02} : There is no significant difference among operators.

H_{12} : There is some significant difference among operators.

Step 2: Calculate correlation factor (CF)

Operators	**Machines**			**Row total**
	M_1	M_2	M_3	
1	150	151	156	$R_1 = 457$
2	147	159	156	$R_2 = 461$
3	141	146	153	$R_3 = 440$
4	154	152	159	$R_4 = 465$
Column total	$C_1 = 592$	$C_2 = 608$	$C_3 = 623$	1823

$$\text{C.F} = \frac{(\text{Grand total})^2}{\text{N}} = \frac{(1823)^2}{12} = 276944.08$$

$$\boxed{\boldsymbol{C.F. = 276944.08}}$$

Step 3: Calculate sum of squares of total (SST)

$$SST = \begin{bmatrix} 150^2 + 147^2 + 141^2 + 154^2 + 151^2 + 159^2 + 146^2 \\ + 152^2 + 156^2 + 155^2 + 153^2 + 159^2 \end{bmatrix} - \text{C.F}$$

$$= 277529 - 27644.08$$

$$\boxed{\boldsymbol{SST = 314.92}}$$

Step 4: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3}\right] - \text{C.F}$$

$$= \left[\frac{(592)^2}{4} + \frac{(608)^2}{4} + \frac{(623)^2}{4}\right] - 276944.08$$

$$= [277064.25] - [276944.08]$$

$$\boxed{\boldsymbol{SSC = 120.17}}$$

Step 5: Calculate sum of squares of Rows (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \frac{(R_4)^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{(457)^2}{3} + \frac{(461)^2}{3} + \frac{(440)^2}{3} + \frac{(465)^2}{3}\right] - 276944.08$$

$$= [277064.25] - [276944.08]$$

$$\boxed{\boldsymbol{SSR = 120.92}}$$

Step 6: Calculate sum of squares of error (SSE)

$$SSE = SST - SSC - SSR$$

$$= 314.92 - 120.17 - 120.92$$

$$\boxed{\boldsymbol{SSE = 73.83}}$$

Step 7: Two way classification table

Sources of variances	**Sum of squares**	**d.f**	**Mean square**	***F*-ratio**
Between Columns	$SSC = 120.17$	$C - 1 = 3 - 1 = 2$	$MSC = \frac{SSC}{C-1}$ $= \frac{120.17}{12}$ $= 10.01$	$F_C = \frac{12.30}{10.01}$ $= 1.22$

Sources of variances	Sum of squares	d.f	Mean square	F-ratio
Between rows	$SSR = 120.92$	$R - 1 = 4 - 1 = 3$	$MSR = \frac{SSR}{R-1}$ $= \frac{120.92}{3}$ $= 40.30$	$F_R = \frac{40.30}{12.30}$ $= 3.27$
Error	$SSE = 73.83$	$(C-1) \times (R-1)$ $= 3 \times 2 = 6$	$MSE = \frac{SSE}{(C-1) \times (R-1)}$ $= \frac{93.83}{6}$ $= 12.30$	

Conclusion

Case (i) (Betwenn Machines)

Calculated value $= 1.22$

Table value at 5% LOS

Degrees of freedom $= (6, 2)$

Table value $= 19.33$

Since the table is greater than the calculated value

H_{01} is accepted

There is some no significant difference between machines.

Case (ii) (Between Operators)

Calculated value $= 3.27$

Table value of at 5% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.96$

Since the table value is greater than the calculated value

H_{02} is accepted

There is no significant difference between operators.

Example 6: The following data represent the number of units production per day turned out by different workers using 4 different types of machines.

Workers	**Machine type**			
	A	***B***	***C***	***D***
1	**44**	**38**	**47**	**36**
2	**46**	**40**	**52**	**43**
3	**34**	**36**	**44**	**32**
4	**43**	**38**	**46**	**33**
5	**38**	**42**	**49**	**39**

Test whether the five mean differ with respect to mean productivity and test whether the mean productivity is the same for the four different machines types.

Solution:

Step 1: Set-up hypothesis

H_{01} : There is no significant difference between machine types.

H_{11} : There is some significant difference between machine types.

H_{02} : There is no significant difference between workers.

H_{12} : There is some significant difference between workers.

Step 2: Calculate Correlation Factor (C.F)

Machine type					
Workers	***A***	***B***	***C***	***D***	**Row total**
1	44	38	47	36	$R_1 = 165$
2	46	40	52	43	$R_2 = 181$
3	34	36	44	32	$R_3 = 146$
4	43	38	46	33	$R_4 = 160$
5	38	42	49	39	$R_5 = 168$
Column total	$C_1 = 205$	$C_2 = 194$	$C_3 = 238$	$C_4 = 183$	820

$$C.F = \frac{(\text{Grand total})^2}{N} = \frac{(820)^2}{20} = 33620$$

$$\boxed{\boldsymbol{C.F. = 33620}}$$

Step 3: Calculate sum of square of total (SST)

$$SST = \begin{bmatrix} 44^2 + 46^2 + 34^2 + 43^2 + 38^2 + 40^2 + 36^2 + 38^2 + 42^2 \\ + 47^2 + 52^2 + 44^2 + 46^2 + 49^2 + 36^2 + 43^2 + 32^2 \\ + 33^2 + 39^2 \end{bmatrix} - C.F$$

$$= 34194 - 33620$$

$$\boxed{\boldsymbol{SST = 574}}$$

Step 4: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} + \frac{(C_4)^2}{n_4} \right] - C.F$$

$$= \left[\frac{(205)^2}{5} + \frac{(194)^2}{5} + \frac{(238)^2}{5} + \frac{(183)^2}{5} \right] - 33620$$

$$= 33958.8 - 33620$$

$$\boxed{\boldsymbol{SSC = 338.8}}$$

Step 5: Calculate sum of squares of rows (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \frac{(R_4)^2}{n_4} + \frac{(R_5)^2}{n_5} \right] - C.F$$

$$= \left[\frac{(165)^2}{4} + \frac{(181)^2}{4} + \frac{(146)^2}{4} + \frac{(160)^2}{4} + \frac{(168)^2}{4} \right] - 33620$$

$$SSR = 33781.5 - 33620$$

$$\boxed{\boldsymbol{SSR = 161.5}}$$

Step 6: Calculate sum of squares of error (SSE)

$$SSE = SST - SSC - SSR$$

$$= 574 - 338.8 - 161.5$$

$$\boxed{\boldsymbol{SSE = 73.7}}$$

Step 7: Two way classification table

Sources of variances	Sum of squares	d.f	Mean square	*F*-ratio
Between Column	$SSC = 338.8$	$C - 1 = 4 - 1$ $= 3$	$MSC = \frac{SSC}{C-1}$ $= \frac{338.8}{3} = 112.93$	$F_C = \frac{112.93}{6.14}$ $F_C = 18.39$
Between row	$SSR = 161.5$	$R - 1 = 5 - 1$ $= 4$	$MSR = \frac{SSR}{R-1}$ $= \frac{161.5}{4}$ $= 40.37$	$F_R = \frac{140.37}{6.14}$
Error	$SSE = 73.7$	$(r-1) \times (c-1)$ $3 \times 4 = 12$	$MSE = \frac{SSE}{(r-1)(c-1)}$ $= \frac{73.7}{12}$ $= 6.14$	$F_R = 26.57$

Step 8: Conclusion

Case (i)

Calculated value $= 18.39$

Table value at 5% LOS

Degrees of freedom $= (3, 12)$

Table value $= 3.49$

Since the table is less than the calculated value

H_{01} is rejected

H_{11} is accepted

There is some significant difference between machine type.

Case (ii)

Calculated value $= 6.57$

Table value of at 5% LOS

Degrees of freedom $= (4, 12)$

Table value $= 3.26$

Since the table value is less than the calculated value

H_{02} is rejected

H_{12} is accepted

There is some significant difference between workers.

LATIN SQUARE DESIGN (Three Way ANOVA)

A latin square is the square arrangement of m rows and n columns such that each symbol appear only once in each row and columns.

Example

3×3 Latin		
A	B	C
B	C	A
C	A	B

4×4 Latin Square

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

Note:

Here 2×2 latin square is not possible because for $n = 2$ the degrees of freedom is zero. Hence comparison is not possible for ANOVA.

2.3 LATIN SQUARE DESIGN

2.3.1 Introduction

The Latin square takes its name from a figure of mathematical puzzle that was studied many years before, use as a plan of experiment.

Latin squares are very extensively used in agricultural trials in order to eliminate fertility trends in two directions, simultaneously. The data are classified according to the different criteria, (i.e.,) according to columns, rows and varieties and are arranged in a square known as Latin square.

2.3.2 Merits and Demerits of Latin square design

Merits

1. Latin square design controls variability in two directions of the experimental material.
2. The analysis of the design is simple and straight forward and is a three way classification of analysis of variance.

Demerits

1. The process of randomization is not as simple as in RBD.
2. The number of treatments should be equal to the number of rows and number of columns.
3. The experimental area should be in the form of a square.
4. It is suitable only in the case of smaller number of treatments (preferably less than 10).
5. A 2×2 Latin square is not possible.

2.3.3 Working Rule

The analysis is done in a way similar to two-way classification. The different sums of squares are obtained as follows:

Step 1: Find N.

5. Working rule

The analysis is one in a way similar to two-way classification. The different sums of squares are obtained as follows:

Step 1: Find N. **Step 2:** Find T.

Step 3: Find $\frac{T^2}{N}$ **Step 4:** Find SST

Step 5: Find SSC **Step 6:** Find SSR

Three Way ANOVA - Latin Square Design

Sources of variation	Sum of squares	degrees of freedom	Mean square value	F-ratios
Between Columns	SSC	$n-1$	$\frac{SSC}{n-1}$	$F_1 = \frac{\text{Greater variance}}{\text{Smaller variance}}$
Between Rows	SSR	$n-1$	$\frac{SSR}{n-1}$	
Between Treatments	SST	$n-1$	$\frac{SST}{n-1}$	$F_2 = \frac{\text{Greater variance}}{\text{Smaller variance}}$
Error	SSE	$(n-1)(n-2)$	$\frac{SSE}{(n-1)(n-2)}$	$F_3 = \frac{\text{Greater variance}}{\text{Smaller variance}}$
Total	SST			

Note:

1. Why a $\mathbf{2 \times 2}$ Latin square is not possible?

A $\mathbf{2 \times 2}$ Latin square design is not possible. The degree of freedom for error in a $\boldsymbol{m \times m}$ Latin square design is $\boldsymbol{(m-1)(m-2)}$. For $\boldsymbol{m = 2}$, the d.f is 0 and hence comparisons are not possible. Hence a $\mathbf{2 \times 2}$ Latin square design is not possible.

WORKED EXAMPLES

Example 1: Analyse the variance in the following Latin square of yields (in kgs) of crops where A, B, C and D denote different methods of cultivation. Examine whether the different methods of cultivation have given significantly different yields.

D **122**	***A*** **121**	***C*** **123**	***B*** **122**
B **124**	***C*** **123**	***A*** **122**	***D*** **125**
A **120**	***B*** **119**	***D*** **120**	***C*** **121**
C **122**	***D*** **123**	***B*** **121**	***A*** **122**

Solution:

Step 1: Given data

D 122	A 121	C 123	B 122
B 124	C 123	A 122	D 125
A 120	B 119	D 120	C 121
C 122	D 123	B 121	A 122

To find New Latin Square

(Old value – 120)

New Latin Square

					Row Total
	D 2	*A* 1	*C* 3	*B* 2	$R_1 = 8$
	B 4	*C* 3	*A* 2	*D* 5	$R_2 = 14$
	A 0	*B* – 1	*D* 0	*C* 1	$R_3 = 0$
	C 2	*D* 3	*B* 1	*A* 2	$R_4 = 8$
Column Total	$C_1 = 8$	$C_2 = 6$	$C_3 = 6$	$C_4 = 10$	30

Variety $v = 4$, Column $C = 4$, Row $R = 4$

$N = 16$

$= 30$ Grand Total

H_{01} : There is no significant difference between columns.

H_{11} : There is some significant difference between columns.

H_{02} : There is no significant difference between rows.

H_{12} : There is some significant difference between rows.

H_{03} : There is no significant difference between varieties.

H_{13} : There is some significant difference between varieties.

Step 2: Calculate C.F

$$\text{C.F} = \frac{(G)^2}{N}$$

$$= \frac{(30)^2}{16}$$

$$\boxed{\textbf{C.F} = \textbf{56.25}}$$

Step 3: Calculate The sum of squares of total: (*SST*)

$$SST = \begin{bmatrix} (2)^2 + (0)^2 + (3)^2 + (2)^2 + (4)^2 + (3)^2 + (2)^2 + (5)^2 + \\ (0)^2 + (-1)^2 + (0)^2 + (1)^2 + (0)^2 + (2)^2 + \\ (3)^2 + (1)^2 + (2)^2 \end{bmatrix} - \text{C.F}$$

$$= 92 - 56.25$$

$$\boxed{SST = 35.75}$$

Step 4: Calculate Sum of square of columns: (*SSC*)

$$SSC = \left[\frac{C_1^2}{n} + \frac{C_2^2}{n} + \frac{C_3^2}{n} + \frac{C_4^2}{n}\right] - \text{C.F}$$

$$= \left[\left(\frac{8^2}{4}\right) + \left(\frac{6^2}{4}\right) + \left(\frac{6^2}{4}\right) + \left(\frac{10^2}{4}\right)\right] - 56.25$$

$$= 59 - 56.25$$

$$\boxed{SSC = 2.75}$$

Step 5: Calculate Sum of squares of rows: (*SSR*)

$$SSR = \left[\frac{R_1^2}{n_1} + \frac{R_2^2}{n_2} + \frac{R_3^2}{n_3} + \frac{R_4^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{8^2}{4} + \frac{14^2}{4} + \frac{0^2}{4} + \frac{8^2}{4}\right] - \text{C.F}$$

$$= 81 - 56.25$$

$$\boxed{SSR = 24.75}$$

Step 6: Calculate Sum of squares of varieties (Arrange the data in same varieties)

A	*B*	*C*	*D*
1	2	3	2
2	4	3	5
0	– 1	1	0
2	1	2	3
$V_1 = 5$	$V_2 = 6$	$V_3 = 9$	$V_4 = 10$

Step 7: Calculate SSV: Sum of square of varieties

$$SSV = \left[\frac{V_1^2}{n_1} + \frac{V_2^2}{n_2} + \frac{V_3^2}{n_3} + \frac{V_4^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{5^2}{4} + \frac{6^2}{4} + \frac{9^2}{4} + \frac{10^2}{4}\right] - 56.25$$

$$= 60.5 - 56.25$$

$$\boxed{\boldsymbol{SSV = 4.25}}$$

No. of items in variety $1 = n_1$

No. of items in variety $2 = n_2$

Step 8: Calculate Sum of square of error (*SSE*)

$$SSE = SST - SSC - SSR - SSV$$

$$= 35.75 - 2.75 - 24.75 - 4.25$$

$$SSE = 35.75 - 2.75 - 24.75 - 4.25$$

$$\boxed{\boldsymbol{SSE = 4}}$$

Step 9: Conclusion

3-Way ANOVA or Latin square design table

Sources of variation	**Sum of square**	**Degrees of freedom**	**Mean square value**	***F*-ratio**
Between columns	SSC = 2.75	$C - 1$ $4 - 1 = 3$	$= \frac{2.75}{3}$ $MSC = 0.917$	$F_C = \frac{0.917}{0.667}$ $F_C = 1.375$
Between rows	SSR= 24.75	$R - 1$ $4 - 1 = 3$	$\frac{24 - 75}{3}$ $MSR = 8.25$	$F_R = \frac{8.25}{0.667}$ $F_R = 12.369$
Between varieties	SSV $= 4.25$	$v - 1$ $4 - 1 = 3$	$\frac{4.25}{3}$ $MSV = 1.417$	$F_V = \frac{1.417}{0.667}$
Error	$SSE = 4$	$(V - 1)$ $(V - 2) = 6$	$MSE = \frac{4}{6}$ 0.667	$F_V = 2.124$

Case 1: (Columns)

Calculate value $= 1.375$

Table value at 5% level of significance

$$\text{df} = (3, 6)$$

Table value $= 4.76$

Calculate value < Table value

H_0 is accepted

There is no significant difference between columns.

Case 2: (Rows)

Calculated value $= 12.369$

Table value at 5% level of significance

d.f $= (3, 6)$

Table value is 4.76

Calculated value > Table value

H_{02} is rejected

There is some significant difference between rows

Case 3: (Varieties)

Calculated value $= 2.124$

Table value at 5% level of significance

d.f $= (3, 6)$

Table value $= 4.76$

Calculated value < Table value

H_{03} is accepted.

There is no significant difference between varieties.

Example 2: Analyse the following results of Latin square design. The letters A, B, C, D denotes the treatment and the figures denotes the observation.

A	*D*	*C*	*B*
12	20	16	10
D	*A*	*B*	*C*
18	14	11	14
B	*C*	*D*	*A*
12	15	19	13
C	*B*	*A*	*D*
16	11	15	20

Solution:

Here 4 letters are given and no. of rows is equal to no. of columns. Therefore it is Latin square design method (or) 3-way ANOVA.

Step 1: Given data

Given Latin Square

A 12	*D* 20	*C* 16	*B* 10
D 18	*A* 14	*B* 11	*C* 14
B 12	*C* 15	*D* 19	*A* 13
C 16	*B* 11	*A* 15	*D* 20

To find new latin square (old value – 12)

New Latin square

A 0	*D* 8	*C* 4	*B* – 2	$R_1 = 10$
D 6	*A* 2	*B* – 1	*C* 2	$R_2 = 9$
B 0	*C* 3	*D* 7	*A* 1	$R_3 = 11$
C 4	*B* – 1	*A* 3	*D* 8	$R_4 = 4$
$C_1 = 10$	$C_2 = 12$	$C_3 = 13$	$C_4 = 9$	Grand Total 44

$$v = 4$$

$$N = 16$$

$$G = 44$$

H_{01} : There is no significant difference between treatments.

H_{11} : There is no significant difference between treatments.

H_{12} : There is no significant difference between treatments.

H_{13} : There is no significant difference between treatments.

Step 2: CF: (Correlation Factor)

$$\text{C.F} = \frac{(G)^2}{16}$$

$$\mathbf{C.F = 121}$$

Step 3: Calculate Sum of square of total (SST)

$$SST = \begin{bmatrix} (0)^2 + (8)^2 + (4)^2 + (-2)^2 + (6)^2 + (2)^2 + (-1)^2 + (2)^2 + \\ (0)^2 + (3)^2 + (7)^2 + (1)^2 + (4)^2 + (-1)^2 + (3)^2 + (8)^2 \end{bmatrix} - \text{C.F}$$

$$= 278 - 121$$

$$\boxed{\boldsymbol{SST = 157}}$$

Step 4: Calculate Sum of squares of column (SSC)

$$SSC = \left[\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \frac{C_3^2}{n_3} + \frac{C_4^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{10^2}{4} + \frac{12^2}{4} + \frac{13^2}{4} + \frac{9^2}{4}\right] - 121$$

$$= 123.5 - 121$$

$$\boxed{\boldsymbol{SSC = 2.5}}$$

Step 5: Calculate Sum of squares of rows (SSR)

$$SSR = \left[\frac{R_1^2}{n_1} + \frac{R_2^2}{n_2} + \frac{R_3^2}{n_3} + \frac{R_4^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{10^2}{4} + \frac{9^2}{4} + \frac{11^2}{4} + \frac{14^2}{4}\right] - 121$$

$$= 124.5 - 121$$

$$\boxed{\boldsymbol{SSR = 3.5}}$$

Step 6: Calculate Sum of squares of varieties (SSV)

A	B	C	D
0	− 2	4	8
2	− 1	2	6
1	0	3	− 1
3	− 1	4	8
$V_1 = 6$	$V_2 = -4$	$V_3 = 13$	$V_4 = 29$

$$SSV = \left[\frac{V_1^2}{n_1} + \frac{V_2^2}{n_2} + \frac{V_3^2}{n_3} + \frac{V_4^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{(6)^2}{4} + \frac{(-4)^2}{4} + \frac{(13)^2}{4} + \frac{(29)^2}{4}\right] - 121$$

$$= 265.5 - 121$$

$$\boxed{\mathbf{SSV = 144.5}}$$

Step 7: Calculate Sum of squares of Errors (*SSE*)

$$SSE = SST - SSC - SSR - SSV$$

$$= 157 - 2.5 - 3.5 - 144.5$$

$$\boxed{\mathbf{SSE = 6.5}}$$

Step 8: Three way Classification Table

Sources of variations	Sum of squares	Degree of freedom	Mean square value	*F*-ratio
Between columns	$SSC = 2.5$	$C - 1 = 3$	$= \frac{2.5}{3}$ $MSC = 0.833$	$F_C = \frac{1.086}{0.833}$ $F_C = 1.300$
Between rows	$SSR = 3.5$	$R - 1$ 3	$= \frac{3.5}{3}$ $MSR = 1.167$	$F_R = \frac{1.167}{1.083}$
Between varieties	$SSV = 144.5$	$v - 1$ 3	$= \frac{144.5}{3}$ $MSV = 48.167$	$F_R = 1.078$
Error	$SSE = 6.5$	$(v - 1)$ $(v - 2)$ $= 6$	$= \frac{6.5}{6}$ $MSE = 1.083$	$F_V = 44.476$

Case 1: Columns

Calculated value $= 1.300$

Table value at 5% level of significance

$\text{d.f} = (6, 3)$

Table value $= 8.94$

Calculated value < Table value

H_{01} is accepted.

There is no significant difference between columns

Case 2: Rows

Calculated value $= 1.078$

Table value:

$$\text{LOS} = 5\%$$

$$\text{d.f} = (3, 6)$$

Table value $= 4.76$

Calculated value < Table value

H_{02} is accepted.

There is no significant difference between rows.

Case 3: Varieties

Calculated value $= 44.476$

Table value:

$$\text{LOS} = 5\%$$

$$\text{d.f} = (3, 6)$$

Table value $= 4.76$

Calculated value > Table value

H_{03} is rejected

There is some significant difference between varieties.

Example 3: The following data resulted from an experiment to compare 3 burners $B1, B2, B3$, A thin square. Design was used as the tests were mode on 3 engines 2nd were spread over 3 days perform an analysis of variance at 5% level of significance on that data.

		Engine		
		1	**2**	**3**
	1	**B1-16**	**B2-17**	**B3-20**
Day	**2**	**B2-16**	**B3-21**	**B1-15**
	3	**B3-15**	**B1-12**	**B2-13**

Solution:

Step 1: Setup hypothesis

H_{01} : There is no significant difference between columns

H_{11} : There is some significant difference between columns

H_{02} : There is no significant difference between rows

H_{12} : There is some significant difference between rows

H_{03} : There is no significant difference between varieties

H_{13} : There is some significant difference between varieties

Step 2: Given latin square

		Engine		
		1	2	3
	1	B1-16	B2-17	B3-20
Day	2	B2-16	B3-21	B1-15
	3	B3-15	B1-12	B2-13

New Latin Square [Given value − 12]

				Row Total
	B1-4	B2-5	B3-8	$R_1 = 17$
	B2-4	B3-9	B1-3	$R_2 = 16$
	B3-3	B1-0	B2-1	$R_3 = 4$
Column Total	$C_1 = 11$	$C_2 = 14$	$C_3 = 12$	Grand Total 37

Step 3: Calculate correlation factor (C.F)

$$\text{C.F} = \frac{(\text{Grand total})^2}{\text{N}} = \frac{(37)^2}{9}$$

$$\boxed{\textbf{C.F} = \textbf{152.11}}$$

Step 4: Calculate sum of squares total (SST)

$$SST = [4^2 + 4^2 + 3^2 + 5^2 + 9^2 + 0^2 + 8^2 + 3^2 + 1^2] - \text{C.F}$$

$$= 221 - 152.11$$

$$\boxed{\mathbf{SST = 68.89}}$$

Step 5: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3}\right] - \text{C.F}$$

$$= \left[\frac{(11)^2}{3} + \frac{(14)^2}{3} + \frac{(12)^2}{3}\right] - 152.11$$

$$SSC = 153.66 - 152.11$$

$$\boxed{\mathbf{SSC = 1.55}}$$

Step 6: Calculate sum of square of Row (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \right] - \text{C.F}$$

$$= \left[\frac{(17)^2}{3} + \frac{(16)^2}{3} + \frac{(4)^2}{3}\right] - 152.11$$

$$= 187 - 152.11$$

$$\boxed{\mathbf{SSR = 34.89}}$$

Step 7: Calculate sum of squares of varieties (SSV)

B_1	B_2	B_3
4	5	8
3	4	9
0	1	3
$V_1 = 7$	$V_2 = 10$	$V_3 = 20$

$$SSR = \left[\frac{(V_1)^2}{n_1} + \frac{(V_2)^2}{n_2} + \frac{(V_3)^2}{n_3}\right] - \text{C.F}$$

$$= \left[\frac{(7)^2}{3} + \frac{(10)^2}{3} + \frac{(20)^3}{3}\right] - 152.11$$

$$= 183 - 152.11$$

$$\boxed{\boldsymbol{SSV = 30.89}}$$

Step 8: Calculate sum of squares of error (SSE)

$$SSE = SST - SSC - SSR - SSV$$

$$= 68.89 - 1.55 - 34.89 - 30.89$$

$$\boxed{\boldsymbol{SSE = 1.56}}$$

Step 9: Three way classification table

Sources of variances	**Sum of squares**	**d.f**	**Mean square**	***F*-ratio**
Between Column	$SSC = 1.55$	$C - 1 =$ $3 - 1 = 2$	$MSC = \frac{SSC}{C-1}$ $= \frac{1.55}{2} = 0.77$	$F_C = \frac{0.78}{0.77}$ $F_C = 1.01$
Between rows	$SSR = 34.89$	$R - 1 =$ $3 - R = 2$	$MSR = \frac{SSR}{R-1}$ $= \frac{34.89}{2} = 17.44$	$F_R = \frac{17.44}{0.78}$ $F_R = 22.35$
Between varieties	$SSV = 30.89$	$(V - 1)$ $= 3 - 1 = 2$	$MSV = \frac{SSV}{V-1}$ $= \frac{30.89}{2}$ $MSV = 15.44$	
Error	$SSE = 1.56$	$(V-1)(V-2)$ $= (3-1) \times (3-2)$ $= 2 \times 1 = 2$	$MSE =$ $\frac{SSE}{(V-1)(V-2)}$ $= \frac{1.56}{2} = 0.78$	$F_v = \frac{15.44}{0.78}$ $F_v = 19.79$

Step 10: Conclusion

Case (i) (Columns)

Calculated value $= 1.01$

Table value at 5% LOS

Degrees of freedom $= (2, 2)$

Table value $= 19.00$

Since the table value is greater than the calculated value

H_{01} is accepted

There is no significant difference between columns.

Case (ii) (Rows)

Calculated value $= 22.35$

Table value of at 5% LOS

Degrees of freedom $= (2, 2)$

Table value $= 19.00$

Since the table value is less than the calculated value

H_{02} is rejected

H_{12} is accepted

There is some significant difference between rows.

Case (iii) (Varieties)

Calculated value $= 19.79$

Table value of at 5% LOS

Degrees of freedom $= (2, 2)$

Table value $= 19.00$

Since the table value is less than the calculated value

H_{03} is rejected

H_{13} is accepted

There is some significant difference between varieties.

Example 4: Carry out analysis of variance at 0.01% LOS

A_{12}	D_{20}	C_{16}	B_{10}
D_{18}	A_{14}	B_{11}	C_{14}
B_{12}	C_{15}	D_{19}	A_{13}
C_{16}	B_{11}	A_{15}	D_{20}

Solution:

Step 1: Set up a hypothesis

H_{01} : There is no significance difference between column

H_{11} : There is some significance difference between column

H_{02} : There is no significance difference between rows

H_{12} : There is some significance difference between rows

H_{03} : There is no significant difference between varieties

H_{13} : There is some significant difference between varieties

Given Latin Square Design

A_{12}	D_{20}	C_{16}	B_{10}
D_{18}	A_{14}	B_{11}	C_{14}
B_{12}	C_{15}	D_{19}	A_{13}
C_{16}	B_{11}	A_{15}	D_{20}

New Latin Square Design (old Latin Square $-$ 10)

					Row Total
	A_2	D_{10}	C_6	B_0	$R_1 = 18$
	D_8	A_4	B_1	C_4	$R_2 = 17$
	B_2	C_5	D_9	A_3	$R_3 = 19$
	C_6	B_1	A_5	D_{10}	$R_4 = 22$
Column Total	$C_1 = 18$	$C_2 = 20$	$C_3 = 2$	$C_4 = 17$	76

Step 2: Calculate Correction Factor

$$\text{C.F} = \frac{(\text{Grand total})^2}{\text{N}} = \frac{(76)^2}{16} = 361$$

Step 3: Calculate sum of squares of total (SST)

$$SST = \left[\begin{array}{l} 2^2 + 8^2 + 2^2 + 6^2 + 10^2 + 4^2 + 5^2 + 1^2 + 6^2 + 1^2 + 9^2 \\ + 5^2 + 0^2 + 4^2 + 3^2 + 10^2 \end{array}\right] - \text{C.F}$$

$$= 518 - 361$$

***SST* = 157**

Step 4: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} + \frac{(C_4)^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{(18)^2}{4} + \frac{(20)^2}{4} + \frac{(21)^2}{4} + \frac{(17)^2}{4}\right] - 361$$

$$= 363.5 - 361$$

***SSC* = 2.5**

Step 5: Calculate sum of squares of Rows (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \frac{(R_4)^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{16)^2}{4} + \frac{(17)^2}{4} + \frac{(19)^2}{4} + \frac{(22)^2}{4} \right] - 361$$

$$= 364.5 - 361$$

$$\boldsymbol{SSR = 3.5}$$

Step 6: Calculate sum of squares of varieties SSV

A	***B***	***C***	***D***
2	0	6	10
4	1	4	6
3	2	5	9
5	1	6	10
$v_1 = 14$	$v_2 = 4$	$v_3 = 21$	$v_4 = 37$

$$SSV = \left[\frac{(V_1)^2}{n_1} + \frac{(V_2)^2}{n_2} + \frac{(V_3)^2}{n_3} + \frac{(V_4)^2}{n_4} \right] - \text{C.F}$$

$$= \left[\frac{(14)^2}{4} + \frac{(4)^2}{4} + \frac{(21)^2}{4} + \frac{(37)^2}{4} \right] - 361$$

$$= 505.5 - 361$$

$$\boldsymbol{SSV = 144.5}$$

Step 7: Calculate sum of squares of error (SSE)

$$SSE = SS7 - SSC - SSR - SSV$$

$$= 157 - 2.5 - 3.5 - 144.5$$

$$\boldsymbol{SSE = 6.5}$$

Step 8: Three way classification table

Sources of variances	Sum of squares	d.f	Mean square	F-ratio
Between Column	$SSC = 2.5$	$C-1 = 4-1$ $= 3$	$MSC = \frac{2.5}{3}$ $= 0.833$	$F_c = \frac{1.883}{0.033}$ $F_c = 1.300$
Between rows	$SSR = 3.5$	$R-1 = 4-1$ $= 3$	$MSR = \frac{3.5}{3}$ $= 1.166$	$F_R = \frac{1.166}{1.083}$ $F_R = 1.076$
Between varieties	$SSV = 144.5$	$V-1 = 4-1$ $= 3$	$MSV = \frac{144.5}{3}$ $= 48.169$	
Error	$SSE = 6.5$	$(V-1)(V-2)$ $= (4-1)(4-2)$ $= (3)(2) = 6$	$MSE = \frac{6.5}{6}$ $= 1.083$	$F_V = \frac{48.166}{1.083}$ $F_V = 44.474$

Conclusion

Case (i) Columns

Calculated value $= 1.300$

Table value at 1% LOS

Degrees of freedom $= (6, 3)$

Table value $= 8.94$

Since the table value is greater than calculated value

H_{01} is accepted

There is no significant difference between column

Case (ii) Rows

Calculated value $= 1.076$

Table value of at 1% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is greater than the calculated value

H_{02} is accepted

There is no significant difference between row.

Case (iii) Varieties

Calculated value $= 44.474$

Table value of at 1% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is less than the calculated value

H_{03} is rejected

H_{13} is accepted

There is some significant difference between varieties.

Example 5: Analysis the variance in the following latin square.

A_8	C_{18}	B_9
C_9	B_{18}	A_{16}
B_{11}	A_{10}	C_{20}

Solution:

Step 1: Set up a hypothesis

H_{01}: There is no significant difference between columns

H_{11}: There is some significant difference between columns

H_{02}: There is no significant difference between rows

H_{12}: There is some significant difference between rows

H_{03} : There is no significant difference between varieties

H_{13} : There is some significant difference between varieties

Step 2: Given Latin Square Design

A_8	C_{18}	B_9
C_9	B_{18}	A_{16}
B_{11}	A_{10}	C_{20}

New Latin Square Design = [Old value − 8]

				Row total
	A_0	C_{10}	B_1	$R_1 = 11$
	C_1	B_{10}	A_8	$R_2 = 19$
	B_3	A_2	C_{12}	$R_3 = 17$
Column Total	$C_1 = 4$	$C_2 = 22$	$C_3 = 21$	47 − GrandTotal

Step 3: Calculate correlation factor (CF)

$$\text{C.F} = \frac{(\text{Grand total})^2}{N} = \frac{(47)^2}{9} = 245.44$$

$$\boxed{\boldsymbol{C.F. = 245.44}}$$

Step 4: Calculate sum of squares of total (SST)

$$SST = [0^2 + 1^2 + 3^2 + 10^2 + 2^2 + 1^2 + 8^2 + 12^2] - \text{C.F}$$

$$= 423 - 245.44$$

$$\boxed{\boldsymbol{SST = 177.56}}$$

Step 5: Calculate sum of square of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3}\right] - \text{C.F}$$

$$= \left[\frac{(4)^2}{3} + \frac{(22)^2}{3} + \frac{(21)^2}{3} \right] - 245.44$$

$$= 313.66 - 245.44$$

$$\boxed{SSC = 68.22}$$

Step 6: Calculate sum of squares of rows (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} \right] - \text{C.F}$$

$$= \left[\frac{(11)^2}{3} + \frac{(19)^2}{3} + \frac{(17)^2}{3} \right] - 245.44$$

$$= 257 - 245.44$$

$$\boxed{SSR = 11.56}$$

Step 6: Calculate sum of squares of varieties (SSV)

A	*B*	*C*
0	1	10
8	10	1
2	3	12
$V_1 = 10$	$V_2 = 14$	$V_3 = 23$

$$SSV = \left[\frac{(V_1)^2}{n_1} + \frac{(V_2)^2}{n_2} + \frac{(V_3)^2}{n_3} \right] - \text{C.F}$$

$$= \left[\frac{(10)^2}{3} + \frac{(14)^2}{3} + \frac{(23)^2}{3} \right] - 245.44$$

$$= 275 - 245.44$$

$$\boxed{SSV = 29.56}$$

Step 7: Calculate sum of square of error (SSE)

$$SSE = SST - SSC - SSR - SSV$$

$$= 177.56 - 68.22 - 11.56 - 29.56$$

$$\boxed{SSE = 68.22}$$

Step 8: Three way classification table

Sources of variances	Sum of squares	d.f	Mean square	*F*-ratio
Between Column	$SSC = 68.22$	$C - 1 = 3 - 1$ $= 2$	$MSC = \frac{SSC}{C-1}$ $= \frac{68.22}{2}$ $= 34.11$	$F_C = \frac{34.11}{34.11}$ $F_C = 1$
Between Row	$SSR = 11.56$	$R - 1 = 3 - 1$ $= 2$	$MSR = \frac{SSR}{R-1}$ $\frac{11.56}{2} = 5.78$	$F_R = \frac{34.11}{5.78}$ $F_R = 5.90$
Between Varieties	$SSV = 29.56$	$V - 1 = 3 - 1$ $= 2$	$MSV = \frac{29.56}{2}$ $= 14.78$	
Error	$SSE = 68.22$	$(V-1)(V-2)$ $= (3-1)(3-2)$ $= 2 \times 1$ $= 2$	$MSE = \frac{SSE}{(r-1)(r-2)}$ $= \frac{68.22}{2}$ $= 34.11$	$F_v = \frac{34.11}{14.78}$ $F_v = 2.30$

Step 10: Conclusion

Case (i) Columns

Calculated value $= 1$

Table value at 5% at LOS

Degrees of freedom $= (2, 3)$

Table value $= 19.00$

Since the table value is greater than calculated value

H_{01} is accepted

There is no significant difference between columns

Case (ii) Rows

Calculated value $= 5.90$

Table value of at 5% LOS

Degrees of freedom $= (2, 2)$

Table value $= 19.00$

Since the table value is greater than the calculated value

H_{02} is accepted

There is no significant difference between rows.

Case (iii) Varieties

Calculated value $= 2.30$

Table value of at 5% LOS

Degrees of freedom $= (2, 2)$

Table value $= 19.00$

Since the table value is less greater than the calculated value

H_{03} is accepted.

There is no significant difference between varities.

Example 6: Analyse the variance in the following Latin square.

20 ***B***	**17** ***C***	**25** ***D***	**34** ***A***
23 ***A***	**21** ***D***	**15** ***C***	**24** ***B***
24 ***D***	**26** ***A***	**21** ***B***	**19** ***C***
26 ***C***	**23** ***B***	**27** ***A***	**22** ***D***

Solution:

Step 1: Set up a hypothesis

H_{01} : There is no significant difference between columns

H_{11} : There is some significant difference between columns

H_{02} : There is no significant difference between rows

H_{12} : There is some significant difference between varieties rows

H_{03} : There is no significant difference between varieties

H_{13} : There is some significant difference between varieties

Step 2: Given Latin Square Design

20 *B*	17 *C*	25 *D*	34 *A*
23 *A*	21 *D*	15 *C*	24 *B*
24 *D*	26 *A*	21 *B*	19 *C*
26 *C*	23 *B*	27 *A*	22 ***D***

New Latin Square Design = [old value -15]

				Row Total
B_5	C_2	D_{30}	A_9	$R_1 = 36$
A_8	D_4	C_0	B_9	$R_2 = 23$
D_9	A_{11}	B_6	C_4	$R_3 = 30$
C_{11}	B_8	A_{12}	D_7	$R_4 = 38$
$C_1 = 33$	$C_2 = 27$	$C_3 = 28$	$C_4 = 39$	127 Grand Total

Step 3: Calculate correlation factor (C.F)

$$\text{C.F} = \frac{(\text{Grand total})}{N} = \frac{(127)^2}{16} = 1008.06$$

$$\boxed{\boldsymbol{CF = 1008.06}}$$

Step 4: Calculate sum of squares of total (SST)

$$SST = \begin{bmatrix} 5^2 + 8^2 + 9^2 + 11^2 + 2^2 + 6^2 + 11^2 + 8^2 + 10^2 + 0^2 \\ + 6^2 + 12^2 + 19^2 + 9^2 + 4^2 + 7^2 \end{bmatrix} - \text{C.F}$$

$$= 1303 - 1008.06$$

$$\boxed{\boldsymbol{SST = 294.94}}$$

Step 5: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} + \frac{(C_4)^2}{n_4} \right] - \text{C.F}$$

$$SSC = \left[\frac{(33)^2}{4} + \frac{(27)^2}{4} + \frac{(18)^2}{4} + \frac{(39)^2}{4} \right] - 1008.06$$

$$= 1030.75 - 1008.06$$

$$\boxed{\boldsymbol{SSC = 22.69}}$$

Step 6: Calculate sum of squares of Rows (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \frac{(R_4)^2}{n_4} \right] - \text{C.F}$$

$$= \left[\frac{(36)^2}{4} + \frac{(23)^2}{4} + \frac{(30)^2}{4} + \frac{(38)^2}{4} \right] - 1008.06$$

$$= 1042.25 - 1008.06$$

$$\boxed{\boldsymbol{SSR = 34.19}}$$

Step 7: Calculate sum of squares of varieties (SSV)

A	*B*	*C*	*D*
19	5	2	10
8	9	0	6
11	6	4	9
12	8	11	7
$V_1 = 50$	$V_2 = 28$	$V_3 = 17$	$V_4 = 32$

$$SSV = \left[\frac{(V_1)^2}{n_1} + \frac{(V_2)^2}{n_2} + \frac{(V_3)^2}{n_3} + \frac{(V_4)^2}{n_4} \right] - \text{C.F}$$

$$= \left[\frac{(50)^2}{4} + \frac{(28)^2}{4} + \frac{(17)^2}{4} + \frac{(32)^2}{4} \right] - 1008.06$$

$$= 1149.25 - 1008.06$$

$$\boxed{\boldsymbol{SSV = 141.19}}$$

Step 8: Calculate sum of squares of error (SSE)

$$SSE = SST - SSC - SSR - SSV$$

$$= 294.94 - 22.69 - 34.19 - 141.1$$

$$\boxed{\boldsymbol{SSE = 96.87}}$$

Step 9: Three way classification Table

Sources of variance	**Sum of squares**	**d.f**	**Mean square**	***F*-ratio**
Between Columns	$SSC = 22.69$	$C - 1 = 4 - 1$ $= 3$	$MSC = \frac{SSC}{C-1}$ $\frac{22.69}{3} = 7.56$	$F_C = \frac{16.14}{7.56}$ $= 2.13$
Between Rows	$SSR = 34.19$	$R - 1 = 4 - 1$ $= 3$	$MSR = \frac{SSR}{R-1}$ $= \frac{34.19}{3} = 11.39$	$F_R = \frac{16.14}{11.39}$ $= 1.41$
Between Varieties	$SSV = 141.19$	$V - 1 = 4 - 1$ $= 3$	$MSV = \frac{SSV}{V-1}$ $= \frac{141.19}{3}$ $= 144.56$	
Error	$SSE = 96.87$	$(V-1)(V-2)$ $= (4-1)(4-2)$ $= 3 \times 2 = 6$	$MSE = \frac{SSE}{(V-1)(V-2)}$ $= \frac{96.87}{6} = 16.14$	$F_V = \frac{16.14}{47.56}$ $= \frac{47.56}{16.14}$ $= 2.91$

Step 10: Conclusion

Case (i) Columns

Calculated value $= 2.06$

Table value at 5% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is greater than the calculated value

H_{01} is accepted

There is no significant difference between columns.

Case (ii) Rows

Calculated value $= 2.541$

Table value of at 5% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is greater than the calculated value

H_{02} is accepted

There is no significant difference between rows.

Case (iii) Varieties

Calculated value $= 2.91$

Table value of at 5% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is greater than the calculated value

H_{03} is rejected

H_{13} is accepted

There is some significant difference between varieties.

Example 4: A variable trial was conducted on wheat with 4 varieties in a latin square design. The plan of the experiment and the per plot yield are given below.

Solution:

C	**25**	***B***	**23**	***A***	**20**	***D***	**20**
A	**19**	***D***	**19**	***C***	**21**	***B***	**10**
B	**19**	***A***	**14**	***D***	**17**	***C***	**20**
D	**17**	***C***	**20**	***B***	**21**	***A***	**15**

Analyse data and intersect the result

Step 1: Set up a Hypothesis

H_{01} : There is no significant difference between columns

H_{11} : There is some significant difference between columns

H_{02} : There is no significant difference between rows

H_{12} : There is some significant difference between varieties rows

H_{03} : There is no significant difference between varieties

H_{13} : There is some significant difference between varieties

Step 2: Given Latin Square Design

C	25	*B*	23	*A*	20	*D*	20
A	19	*D*	19	*C*	21	*B*	10
B	19	*A*	14	*D*	17	*C*	20
D	17	*C*	20	*B*	21	*A*	15

New Latin Square [Old Value – 14]

					Row total
	C11	B9	A6	D6	$R_1 = 32$
	A5	D5	C7	B4	$R_2 = 21$
	B5	A0	D3	C6	$R_3 = 14$
	D3	C6	B7	A1	$R_4 = 17$
Column total	$C_1 = 24$	$C_2 = 20$	$C_3 = 23$	$C_4 = 17$	Grand Total 84

Step 3: Calculate Correlation Factor (CF)

$$\text{C.F} = \frac{(\text{Grand total})^2}{N} = \frac{(84)^2}{16} = 441$$

$$\boxed{\boldsymbol{C.F = 441}}$$

Step 4: Calculate the sum of squares of total (SST)

$$SST = \begin{bmatrix} 11^2 + 5^2 + 5^2 + 3^2 + 9^2 + 5^2 + 0^2 + 6^2 \\ 6^2 + 7^2 + 3^2 + 7^2 + 6^2 + 4^2 + 6^2 + 1^2 \end{bmatrix} - \text{C.F}$$

$$= 554 - 441$$

$$\boxed{\boldsymbol{SST = 113}}$$

Step 5: Calculate sum of squares of column (SSC)

$$SSC = \left[\frac{(C_1)^2}{n_1} + \frac{(C_2)^2}{n_2} + \frac{(C_3)^2}{n_3} + \frac{(C_4)^2}{n_4}\right] - \text{C.F}$$

$$= \left[\frac{(24)^2}{4} + \frac{(20)^2}{4} + \frac{(23)^2}{4} + \frac{(17)^2}{4}\right] - 441$$

$$= 448.5 - 44.1$$

$$\boxed{\boldsymbol{SSC = 7.5}}$$

Step 6: Calculate sum of squares of Row (SSR)

$$SSR = \left[\frac{(R_1)^2}{n_1} + \frac{(R_2)^2}{n_2} + \frac{(R_3)^2}{n_3} + \frac{(R_4)^2}{n_4} \right] - \text{C.F}$$

$$= \left[\frac{(32)^2}{4} + \frac{(21)^2}{4} + \frac{(24)^2}{4} + \frac{(17)^2}{4} \right] - 441$$

$$= 487.5 - 441$$

$$\boxed{\mathbf{SSR = 46.5}}$$

Step 7: Calculate sum of squares of varieties (SSV)

A	*B*	*C*	*D*
6	9	11	6
5	4	7	5
0	5	6	3
1	7	6	3
$V_1 = 12$	$V_2 = 25$	$V_3 = 30$	$V_4 = 17$

$$SSV = \left[\frac{(V_1)^2}{n_1} + \frac{(V_2)^2}{n_2} + \frac{(V_3)^2}{n_3} \frac{(V_4)^2}{n_4} \right] - \text{C.F}$$

$$= \left[\frac{(12)^2}{4} + \frac{(25)^2}{4} + \frac{(30)^2}{4} + \frac{(17)^2}{4} \right] - 441$$

$$= 489.5 - 441$$

$$\boxed{\mathbf{SSV = 48.5}}$$

Step 8: Calculate sum of squares of error (SSE)

$$SSE = SS7 - SSC - SSR - SSV$$

$$= 113 - 7.5 - 46.5 - 48.5 = 10.5$$

$$\boxed{\mathbf{SSV = 10.5}}$$

Step 9: Three way classification table

Sources of variance	Sum of squares	d.f	Mean square	*F*-ratio
Between Columns	$SSC = 7.5$	$C - 1 = 4 - 1$ $= 3$	$MSC = \dfrac{SSC}{C-1}$ $= \dfrac{7.5}{3}$ $= 2.5$	$F_C = \dfrac{2.5}{1.75}$ $= 1.42$
Between Rows	$SSR = 46.5$	$R - 1 = 4 - 1$ $= 3$	$MSR = \dfrac{SSR}{R-1}$ $= \dfrac{46.5}{3}$ $= 15.5$	$F_R = \dfrac{15.5}{1.75}$ $= 3.85$
Between Varieties	$SSV = 48.5$	$V - 1 = 45.1$ $= 3$	$MSV = \dfrac{SSV}{V-1}$ $= \dfrac{48.5}{3}$ $= 16.16$	$F_v = \dfrac{16.16}{1.75}$ $= 9.23$
Error	$SSE = 10.5$	$(V-1)(V-2)$ $= (4-1)(4-2)$ 3×2 $= 6$	$MSE = \dfrac{10.5}{0.6}$ $= 1.75$	

Step 10: Conclusion

Case (i) Between Columns

Calculated value $= 1.42$

Table value at 5% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is greater than the calculated value

H_{01} is accepted

There is no significant difference between columns.

Case (ii) Between rows

Calculated value $= 3.85$

Table value of at 5% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is less than the calculated value

H_{02} is rejected

H_{12} is accepted

There is some significant difference between rows.

Case (iii) Between varieties

Calculated value $= 9.23$

Table value of at 5% LOS

Degrees of freedom $= (3, 6)$

Table value $= 4.76$

Since the table value is less than the calculated value

H_{03} is rejected

H_{13} is accepted

There is some significant difference between varieties.

Since the tasle value is less than the calculated value.

H_{03} is rejected.

H_{13} is accepted.

There is some significant difference between varieties.

EXERCISES

1. The following are the numbers of mistakes made in 5 successive days of 4 technicians working for a photographic laboratory:

Technician I (X_1)	Technician II (X_2)	Technician III (X_3)	Technician IV (X_4)
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

Test at the level of significance $\alpha = 0.01$, whether the differences among the 4 sample means, can be attributed to chance. *[A.U. A/M 2004, A.U A/M 2011]*

2. The accompanying data resulted from an experiment comparing the degree of soiling for fabric co-polymerized with the three different mixtures of methacryclic acid. Analysis is the given classification.

Mixture 1	.56	1.12	.90	1.07	.94
Mixture 2	.72	.69	.87	.78	.91
Mixture 3	.62	1.08	1.07	.99	.93

[A.U A/M 2017 R-13] [A.U N/D 2019 R-13]

3. In order to determine whether there is significant difference in the durability of 3 makes of computers, samples of size 5 are selected from each make and the frequency of repair during the first year of purchase is observed. The results are as follows:

Makes		
A	*B*	*C*
5	8	7
6	10	3
8	11	5
9	12	4
7	4	1

In view of the above data. What conclusion you draw?

4. Table show the milk production of arbitrary units in four days with three different cattle feed A, B and C. Analyse the results of cattle-feed brands.

A	10	8	11	12
B	9	10	9	10
C	12	14	10	11

5. A completely randomised design experiment with 10 plots and 3 treatments have the following results: *[A.U N/D 2007]*

Plot No:	1	2	3	4	5	6	7	8	9	10
Treatment:	A	B	C	A	C	C	A	B	A	B
Yield:	5	4	3	7	5	1	3	4	1	7

Analyse the results for treatment effects.

6. The following table shows the lives in hours of four brands of electric lamps.

Brand *A* :	1610	1610	1650	1680	1700	1720	1800	
B :	1580	1640	1700	1750				
C :	1460	1550	1600	1620	1640	1660	1740	1820
D :	1510	1520	1530	1570	1600	1680		

Perform an analysis of variance test the homogeneity of the mean lives of the four brands of Lamps.

[A.U. A/M. 2008] [A.U N/D 2011]
[A.U Tvli M/J 2011] [A.U A/M 2015]

7. There are typists working in an office. The times (in minutes) they spend for the tea-break in addition to the allowed lunch tea break are observed and noted below:

A :	25	18	30	32	35	37	19			
B :	24	22	26	28	30	32	28	26		
C :	28	20	27	19	29	35	30	23	27	32

Can the difference in average times that the 3 typists spend for the tea break be attributed to chance variation.

8. A random sample is selected from each of 3 makes of ropes and their breaking strength (in certain units) are measured with the following results.

I : 70, 72, 75, 80, 83

II : 60, 65, 57, 84, 87, 73

III : 100, 110, 108, 112, 113, 120, 107

Test whether the breaking strength of the ropes differ significantly.

9. An experiment was designed to study the performance of 4 different detergents for cleaning fuel injectors. The following "cleanliness" readings were obtained with specially designed equipment for 12 tanks of gas distributed over 3 different models of engines:

	Engine 1	Engine 2	Engine 3	Total
Detergent A	45	43	51	139
Detergent B	47	46	52	145
Detergent C	48	50	55	153
Detergent D	42	37	49	128
Total	182	176	207	565

Perform the ANOVA and test at 0.01 level of significance, whether there are differences in the detergents or in the engines. *[A.U. Model] [A.U. N/D. 2004] [A.U CBT A/M 2011] [A.U M/J 2007, N/D 2008] [A.U N/D 2015 R13]*

10. Analyse the following RBD and find your conclusion.

Treatments					
		T_1	T_2	T_3	T_4
	B_1	12	14	20	22
	B_2	17	27	19	15
Blocks	B_3	15	14	17	12
	B_4	18	16	22	12
	B_5	19	15	20	14

[A.U N/D 2013]

11. A tea company appoints four salesmen A, B, C and D and observes their sales in three seasons - summer, winter and monsoon. The figures (in lakhs) are given in the following table.

Seasons	Salesmen				Season's Total
Summer	36	36	21	35	128
Winter	28	29	31	32	120
Monsoon	26	28	29	29	112
Salesmen's Total	90	93	81	96	360

(i) Do the salesmen significantly differ in performance?
(ii) Is there significant difference between the seasons?

[A.U N/D 2012] [A.U M/J 2016 R13]

12. Three varieties A, B and C of a crop are tested in a randomised block design with four replications. The plot yield in pounds are as follows:

A	6	C	5	A	8	B	9
C	8	A	4	B	6	C	9
B	7	B	6	C	10	A	6

Analyse the experimental yield and state your conclusion.

[A.U N/D 2011] [A.U A/M 2019 R-13]

13. The following data represent the number of units production per day turned out by different workers, using 4 different types of machines.

		Machine type			
		A	B	C	D
	1	44	38	47	36
	2	46	40	52	43
Workers	3	34	36	44	32
	4	43	38	46	33
	5	38	42	49	39

Test whether the five men differ with respect to mean productivity and test whether the mean productivity is the same for the four different machine types.

[A.U. M/J 2006, N/D 2007, M/J 2013]
[N/D 2010, A/M 2011]

14. The table shows the yield of paddy in arbitrary units obtained from four different varieties planted in five blocks where each block is divided into four plots. Test at 5% level whether the yields vary significantly with (i) soil differences (ii) differences in the type of paddy.

Blocks	Types of Paddy			
	I	II	III	IV
A	12	15	10	14
B	15	19	12	11
C	14	18	15	12
D	11	16	12	16
E	16	17	11	14

[A.U. N/D 2017 (R17)] [A.U N/D 2019 (R17)]

15. Table below shows the seeds of 4 different types of corns planted in 3 blocks. Test at 0.05 level of significance whether the yields in kilograms per unit area vary significantly with different types of corns.

		Types of Corns		
		I	II	III
Blocks	*A*	4.5	6.4	7.2
	B	8.8	7.8	9.6
	C	5.9	6.8	5.7

[A.U. A/M 2019 R-17] [A.U N/D 2020 R-17]

16. Five doctors each test treatments for a certain disease and observe the number of days each takes to recover. The results are as follows: (Recovery time in days)

Doctor	Treatments				
	1	2	3	4	5
A	10	14	23	19	20
B	11	15	24	17	21
C	9	12	20	16	19
D	8	13	17	17	20
E	12	15	19	15	22

Discuss the difference between (a) doctors and (b) treatments.

17. The following is a Latin square of a design, when 4 varieties of seeds are being tested. Set up the analysis of variance table and state your conclusion. You may carry out suitable change of origin and scale.

A	105	*B*	95	*C*	125	*D*	115
C	115	*D*	125	*A*	105	*B*	105
D	115	*C*	95	*B*	105	*A*	115
B	95	*A*	135	*D*	95	*C*	115

[A.U. M/J 2013, N/D 2013] [A.U A/M 2017 R-8]

18. A variable trial was conducted on wheat with 4 varieties in a Latin square design. The plan of the experiment and the per plot yield are given below:

C	25	*B*	23	*A*	20	*D*	20
A	19	*D*	19	*C*	21	*B*	18
B	19	*A*	14	*D*	17	*C*	20
D	17	*C*	20	*B*	21	*A*	15

Analyse data and interpret the result.

[A.U. M/J 2012] [A.U N/D 2016 R-13] [A.U A/M 2017 R-13]
[A.U N/D 2019 R-13] [A.U N/D 2022 (R-21)]

19. A farmer wishes to test the effects of four different fertilizers A, B, C, D on the yield of wheat. In order to eliminate sources of error due to variability in soil fertility, he uses the fertilizers, in a Lain square arrangement as indicated in the following table, where the numbers indicate yields in bushels per unit area.

A 18	*C* 21	*D* 25	*B* 11
D 22	*B* 12	*A* 15	*C* 19
B 15	*A* 20	*C* 23	*D* 24
C 22	*D* 21	*B* 10	*A* 17

Perform an analysis of variance to determine, if there is a significant difference between the fertilizers at $\alpha = 0.05$ level of significance.

[A.U. M/J 2007] [Tvli M/J 2009] [A.U N/D 2012]
[A.U A/M 2019 R-17] [A.U N.D 2020 R-17]

20. Analyze the variance in the latin square of yields (in kgs) paddy where P, Q, R, S denote the different methods of cultivation.

S122	P121	R123	Q122
Q124	R123	P122	S125
P120	Q119	S120	R121
R122	S123	Q121	P122

21. The following data resulted from an experiment to compare 3 burners B1, B2, B3, A Latin square design was used as the tests were made on 3 engines and were spread over 3 days. Perform an analysis of variance at 5% level of significance on the data.

		Engine		
		1	2	3
	1	B1-16	B2-17	B3-20
Day	2	B2-16	B3-21	B1-15
	3	B3-15	B1-12	B2-13

[A.U. N/D 2021 (R-17)] [A.U A/M 2022 (R-21)]

22. Analyse the variance in the following Latin square.

A8	C18	B9
C9	B18	A16
B11	A10	C20

23. Analyse the variance in the following Latin square.

20 *B*	17 *C*	25 *D*	34 *A*
23 *A*	21 *D*	15 *C*	24 *B*
24 *D*	26 *A*	21 *B*	19 *C*
26 *C*	23 *B*	27 *A*	22 *D*

SHORT QUESTIONS AND ANSWERS

1. State the assumptions For ANOVA?

1. The samples are independently drawn from the populations.
2. The populations are normally distributed.
3. The variance of the populations are equal.

2. What are types of ANOVA?

1. One way classification $\rightarrow$ classified according to one factor.
2. Two way classification $\rightarrow$ classified according to two factor.

3. What are the basic principles of experimental design?

(i) Replications

(ii) Randomizations

(iii) Local control.

4. What is meant by completely randomized design?

(i) The statistical analysis of the completely randomized design.

(ii) One way classification is to test the hypothesis whether the means of populations are all equal.

5. What is a latin square design?

A latin square is a square arrangement of m-rows and m-columns such that each symbol appears once and only in each row and column.

6. Show that 2×2 latin square is not possible.

The degrees of freedom for error in $m \times m$ latin square design $(m-1)(m-2)$. For $m = 2$, the d.f. is zero and hence comparisons are not possible. Hence a 2×2 latin square design is not possible

7. Write the advantages of the Latin square design over the other design.

(i) LSD controls more of the variation than RBD.

(ii) The analysis is simple.

(iii) It is only slightly more complicated than for RBD.

(iv) The analysis remains relatively simple even with missing data.

8. Write down the format of the ANOVA table for two factors of classification.

Source of variation	**Sum of square**	**Degrees of freedom**	**Mean sum of squares**	***F*-ratio**
Between columns	SSC	C-1	$MSC = \frac{SSC}{C-1}$	$F_C = \frac{\text{Greater Variance}}{\text{Smaller Variance}}$
Between rows	SSR	R-1	$MSR = \frac{SSR}{R-1}$	$F_R = \frac{\text{Greater Variance}}{\text{Smaller Variance}}$
Error	SSE	$(C-1) \times (R-1)$	$\frac{SSE}{(C-1) \times (R-1)}$	

9. Why a 2×2 latin square is not possible? Explain.

Since degrees of freedom for error is $(r-1)(r-2)$ the value of F will not exist if $r = 2$

10. Write any two difference between RBD and LSD

	RBD	**LSD**
(i)	Number of treatments = Number of replications	Number of treatments ≠ Number of replications
(ii)	Statistical analysis remains simple	Statistical analysis remains slightly complicated
(iii)	3 way layout performed on a square field	I way layout performed on square (or) rectangular field

11. Mention the various steps involved is testing of hypothesis.

(i) Formulate H_0 and H_1

(ii) Choose the level of significance α.

(iii) Compute the test statistic.

(iv) Draw the conclusion.

12. Compare RBD, CRD.

RBD		**CRD**	
(i)	To influence two factors	(i)	To influence one factor
(ii)	No restriction further treatments	(ii)	No restriction on treatment & replication
(iii)	Use only rectangular or square field	–	

UNIT - III

Solution of Equations and Eigen Value Problems

3.0 INTRODUCTION

Given a quadratic equation $ax^2 + bx + c = 0$, the closed form solution namely, a formula giving the solution as a function of the coefficients a, b, c is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Similar formulas are available for cubic and quartic polynomial equations but we rarely remember them. For higher order polynomial and non-polynomial equations, it is difficult and in many cases, impossible to get closed form solutions. For example to formula exists for equation such as $a_x + blog_x = c$, $ae^{-x} + b + anx = 5$. These are called transcendental equations. However, when the coefficients of such equations are pure numbers, it is always possible to compute the roots to any desired degree of accuracy.

Another approach to solve these equations is by **trial and error**. As a technique of solving equations, trial and error is very general. It is repugnant to a human problem solver. On the other hand, for a computer, it is the most natural technique to use.

Some of the problems will be of the type of determinining roots of a polynomial equation of the form $f(x) = 0$. The equation $f(x) = 0$ is classified into two types.

$f(x) = 0$

Algebraic Equations	Transcendental Equations
Ex. $f(x) = 0$	Ex. $e^x + \log x = 0$
$x^2 + 4x = 0$	$\sin x - x = 0$
$x^{10} + 4x^3 - x + 1 = 0$	$f(x)$ is given implicitly
$f(x)$ may be given explicitly.	

There are two types solutions to the polynomial equations (i) Direct (ii) Indirect solution

Solution to the polynomial

Direction Solution	Iterative Solution
(or)	(or)
Analytical Solution	Numerical Solution
(or)	(or)
Exact Solution	approximate Solution

Introduction

Consider the equation of the form $f(x)=0$. If $f(x)$ is a quadratic, cubic or biquadratic expression then algebraic formula are available for finding the roots.

The problem of solving the equation $f(x)=0$ is of great importance in Science and Engineering.

Algebraic and Transcendental Equations

If $f(x)$ is purely a polynomial in x then it is called algebraic equation.

Example: $x^4-3x^3-2x^2+18x-9=0$

If $f(x)$ contains trigonometric, logarithmic, exponential function then the equation $f(x)=0$ is called transcendental equations.

Example: $x \tan x - \sinh x$, $e^x - \sin x$, $xe^x - \sin x = 0$

Any value of 't' for which $f(t)=0$ is called a root or solution of the equation $f(x)=0$.

Note

1. Every equation has root, real or imaginary.
2. Every polynomial of degree n has exactly n roots.

When $f(x)$ is a polynomial of higher degree or an expression involving transcendental function, algebraic methods are not available to find the exact solution. Hence we are looking for numerical methods to solve such expressions.

Some of the numerical methods available to solve algebraic and transcendental equations are listed below.

- Bisection method
- Regular falsi-method
- Iteration method
- Newton-Raphson method

In this unit we are going to study only Newton Raphson and Iteration method.

3.1 NEWTON RAPHSON METHOD

[Newton's Iteration method]

- It is also known as method of tangents.

This method starts with an initial approximation to the solution of an equation, a better and closer approximation to the solution can be found by using an iterative procedure.

Derivation of N-R Method Formula

Let t be a root of $f(x) = 0$ and x_0 be an approximation to t. If $h = t - x_0$, then by Taylor series.

$$f(x_0 + h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots \quad \text{... (1)}$$

Since $t = x_0 + h$ is the solution,

$$f(x_0 + h) = 0$$

Omitting the higher powers of h in (1) we get,

$$0 = f(x_0) + hf'(x_0)$$

$$hf'(x_0) = -f(x_0)$$

$$h = \frac{-f(x_0)}{f'(x_0)}$$

Since some terms were omitted in (1), $x_1 = x_0 + h$ is exactly but it is better approximation than x_0.

$$\therefore\ x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximations are given by $x_2, x_3, \ldots, x_{n+1}$, where

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad \ldots (2)$$

$(n = 0, 1, 2, 3 \ldots)$

which is the Newton-Raphson formula.

3.1.1 Convergence criterion of Newton-Raphson Method

Comparing (2) with $x_{n+1} = \phi(x_n)$ (say)

where $\phi(x_n) = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(or) $\phi(x) = x - \dfrac{f(x)}{f'(x)}$

Since Iteration method converges if $|\phi'(x)| < 1$

Now

$$\phi'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}$$

$$\phi(x) = \frac{[f'(x)]^2 - [f'(x)]^2 + f(x)f''(x)]}{[f'(x)]^2}$$

$$\phi(x) = \frac{f(x) f''(x)}{[f'(x)]^2}$$

Newton method converges if

$$\left| \frac{f(x) f''(x)}{[f'(x)]^2} \right| < 1$$

i.e., $|f(x) f''(x)| < [f'(x)]^2$ in the interval.

Note

The Error at any stage is proportional to the square of the error in the previous stage. So, the order of convergence of Newton – Raphson method is two i.e, the convergence is Quadratic.

3.1.2 Newton – Raphson method

WORKED EXAMPLES

Example 1: Find the real positive root of $3x - \cos x - 1 = 0$ by Newton – Raphson method correct to six decimal places.

Solution:

[**Note:** Use Radian mode in the calculator]

Given

$$3x - \cos x - 1 = 0$$

Step 1

Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

By trial & Error method,

$$\left.\begin{aligned} f(0) &= -2 \ (-\text{ ve}) \\ f(1) &= 1.459698 + (+\text{ ve}) \end{aligned}\right] \text{sign changes}$$

$\therefore$ There is a root of the equation which lies between 0 and 1.

Step 2

The formula for Newton – Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{[3x_n - \cos x_n - 1]}{[3 + \sin x_n]}$$

Step 3

Since $|f(0)| > |f(1)|$

$\because$ choose $x_0 = 1$

Iteration	$\boldsymbol{x_n}$	$\boldsymbol{x_{n+1} = x_n - \frac{[3x_n - \cos x_n - 1]}{[3 + \sin x_n]}}$
$n = 0$	$x_0 = 1$	$x_1 = x_0 - \frac{[3x_0 - \cos x_0 - 1]}{[3 + \sin x_0]}$ $x_1 = 0.620016$
$n = 1$	$x_1 = 0.620016$	$x_2 = x_1 - \frac{[3x_1 - \cos x_1 - 1]}{[3 + \sin x_1]}$ $x_2 = 0.607121$
$n = 2$	$x_2 = 0.607121$	$x_3 = x_2 - \frac{[3x_2 - \cos x_2 - 1]}{[3 + \sin x_2]}$ $x_3 = 0.607102$
$n = 3$	$x_3 = 0.67102$	$x_4 = x_3 - \frac{[3x_3 - \cos x_3 - 1]}{[3 + \sin x_3]}$ $x_4 = 0.607102$

Since x_3 and x_4 values are same

Hence the root of the Equation is **0.607102**.

Example 2: By using Newton – Raphson method, find the root of $x^4 - x - 10 = 0$, correct to 5 decimal places.

Solution:

Given

$$x^4 - x - 10 = 0,$$

Step 1

Let $f(x) = x^4 - x - 10$

$$f'(x) = 4x^3 - 1$$

By trail & Error method

$$f(0) = -10 \ (-\text{ ve})$$

$$f(1) = 10 \ (-\text{ ve})$$

$$f(2) = 4 \ (+\text{ ve})$$

$\therefore$ There is a root of the equation which lies between 1 and 2.

Step 2

The formula for Newton – Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{[x_n^4 - x_n - 10]}{[4x_n^3 - 1]}$$

Step 3

Since $|f(1)| > |f(2)|$

$\therefore$ Choose $x_0 = 2$

Iteration	x_n	$x_{n+1} = x_n - \frac{[x_n^4 - x_n - 10]}{[4x_n^3 - 1]}$
$n = 0$	$x_0 = 2$	$x_1 = x_0 - \frac{[x_0^4 - x_0 - 10]}{[4x_0^3 - 1]}$ $x_1 = 1.87097$
$n = 1$	$x_1 = 1.87097$	$x_2 = x_1 - \frac{[x_1^4 - x_1 - 10]}{[4x_1^3 - 1]}$ $x_2 = 1.85578$
$n = 2$	$x_2 = 1.85578$	$x_3 = x_2 - \frac{[x_2^4 - x_2 - 10]}{[4x_2^3 - 1]}$ $x_3 = 1.85558$
$n = 3$	$x_3 = 1.85558$	$x_4 = x_3 - \frac{[x_3^4 - x_3 - 10]}{[4x_3^3 - 1]}$ $x_4 = 1.85558$

Since x_3 and x_4 values are same

Hence the root of the equation is **1.85558**.

Example 3: Find a root of $x \log_{10} x - 1.2 = 0$ by Newton's method correct to three decimal places?

Solution:

Given

$$x \log_{10} x - 1.2 = 0$$

Step 1

Let $f(x) = x \log_{10} x - 1.2$

$$f'(x) = x\frac{1}{x}\log e + \log_{10} x\,(1) - 0$$

$$\therefore f'(x) = \log e + \log_{10} x$$

By trail & Error method

$f(1) = -1.200$ (– ve)

$\left.\begin{array}{l} f(2) = -0.598\ (-\text{ ve}) \\ f(3) = 0.231\ (+\text{ ve}) \end{array}\right]$ sign changes

$\therefore$ There is a root of the equation which lies between 2 and 3.

Step 2

The Newton – Raphson method formula:

$$x_{n+1} = x_n - \frac{f[x_n]}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{[x_n \log_{10} x_n - 1.2]}{[\log e + \log_{10} x_n]}$$

Step 3

Since $|f(2)| > |f(3)|$

Choose $x_0 = 3$

Iteration	x_n	$x_{n+1} = x_n - \frac{[x_n \log_{10} x_n - 1.2]}{[\log e + \log_{10} x_n]}$
$n = 0$	$x_0 = 3$	$x_1 = 2.746$
$n = 1$	$x_1 = 2.746$	$x_2 = 2.741$
$n = 2$	$x_2 = 2.741$	$x_3 = 2.741$

Since x_2 and x_3 values are same.

Hence the root of the equation is **2.741**

Example 4: Find the root of $4x - e^x = 0$ that lies between 2 and 3 by Newton's method.

Solution:

Given

$$4x - e^x = 0$$

Let $f(x) = 4x - e^x$

$$f'(x) = 4 - e^x$$

$$\left.\begin{aligned} f(2) &= -0.61094 (+\text{ ve}) \\ f(3) &= -8.08554 (+\text{ ve}) \end{aligned}\right] \text{sign changes}$$

$\therefore$ There is a root of the equation which lies between 2 and 3.

Step 2

The formula for Newton – Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{[4x_n - e^{x_n}]}{[4 - e^{x_n}]}$$

Step 3

Since $|f(2)| < |f(3)|$

Choose $x_0 = 2$

Iteration	x_n	$x_{n+1} = x_n - \frac{[4x_n - e^{x_n}]}{[4 - e^{x_n}]}$
$n = 0$	$x_0 = 2$	$x_1 = 2.18027$
$n = 1$	$x_1 = 2.18027$	$x_2 = 2.15395$
$n = 2$	$x_2 = 2.15395$	$x_3 = 2.15329$
$n = 3$	$x_3 = 2.15329$	$x_4 = 2.15329$

Since x_3 and x_4 values are same.

Hence the root is **2.15329.**

Example 5: Using Newton – Raphson method find the positive root of $x^3 = 6x - 4$ correct to four decimal places.

Solution:

Given

$$x^3 = 6x - 4$$

Step 1

Let $f(x) = x^3 - 6x + 4$

$$f'(x) = 3x^2 - 6$$

By trail & Error method

$$\left.\begin{array}{l} f(0) = 4\ (+\text{ ve}) \\ f(1) = -1 - \text{ve} \end{array}\right] \text{sign changes}$$

$\therefore$ There is a root of the equation which lies between 0 and 1.

Step 2

The formula for Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{[x_n^3 - 6x_n + 4]}{[3x_n^2 - 6]}$$

Step 3

Since $|f(0)| > |f(1)|$

Choose $x_0 = 1$

Iteration	x_n	$x_{n+1} = \dfrac{[x_n^3 - 6x_n + 4]}{[3x_n^2 - 6]}$
$n = 0$	$x_0 = 1$	$x_1 = 0.6667$
$n = 1$	$x_1 = 0.6669$	$x_2 = 0.7302$
$n = 2$	$x_2 = 0.7302$	$x_3 = 0.7320$
$n = 3$	$x_3 = 0.7320$	$x_4 = 0.7321$
$n = 4$	$x_4 = 0.7321$	$x_5 = 0.7321$

Since x_4 and x_5 values are same

Hence the root is **0.7321**.

Example 6: Find the positive root of $f(x) = 2x^3 - 3x - 6 = 0$. By Newton Raphson method correct to 5 decimal places.

Solution:

Given

$$f(x) = 2x^3 - 3x - 6$$

$$f(x) = 2x^3 - 3x - 6$$

$$f'(x) = 6x^2 - 3$$

By trail & error method

$$f(0) = -6 \ (-\text{ve})$$

$$\left.\begin{aligned} f(1) &= -7 \ (+\text{ve}) \\ f(2) &= 4 \ (+\text{ve}) \end{aligned}\right] \text{sign changes}$$

$\therefore$ There is a root of the equation which lies between 1 and 2.

Step 2

Formula for Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{[2x_n^3 - 3x_n - 6]}{[6x_n^2 - 3]}$$

Step 3

Since $|f(1)| > |f(2)|$

Choose $x_0 = 2$

Iteration	x_n	$x_{n+1} = x_n - \frac{[2x_n^3 - 3x_n - 6]}{[6x_n^2 - 3]}$
$n = 0$	$x_0 = 2$	$x_1 = 3.33333$
$n = 1$	$x_1 = 3.33333$	$x_2 = 2.42118$
$n = 3$	$x_2 = 2.42118$	$x_3 = 1.95112$
$n = 4$	$x_3 = 1.95112$	$x_4 = 1.79982$
$n = 5$	$x_4 = 1.79982$	$x_5 = 1.78394$
$n = 6$	$x_5 = 1.78394$	$x_6 = 1.78377$
$n = 7$	$x_6 = 1.78377$	$x_7 = 1.78377$

Since x_6 and x_7 values are same

Hence the root is **1.78377.**

Example 7: Using Newton - Raphson method, establish the formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ to calculate the square root of *N*. And find the square root of 5 correct to four places of decimals.

[A.U May/June 2006, MA 038]

Solution:

Let $x = \sqrt{N}$

$\therefore\ x^2 = N$

$x^2 - N = 0$

The solution of this equation is $\sqrt{N}$

Let $f(x) = x^2 - N\ ;\ f(x_n) = x_n^2 - 2$

$f'(x) = 2x\ ;\ f'(x_n) = 2x_n$

By Newton - Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{x_n^2 - N}{2x_n}\right)$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

i.e, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ is the iterative formula to calculate $\sqrt{N}$.

To find the square root of 5, take $N = 5$. Now $\sqrt{5}$ lies between 2 and 3.

Choose $x_0 = 2$

Iteration	x_n	$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$
$n = 0$	$x_0 = 2$	$x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = 2.25$
$n = 1$	$x_1 = 2.25$	$x_2 = 2.23611$
$n = 2$	$x_2 = 2.23611$	$x_3 = 2.23607$
$n = 3$	$x_3 = 2.23607$	$x_4 = 2.23607$

Since x_3 and x_4 values are same.

Hence $\sqrt{5}$ is approximately equal to 2.2361.

Example 8: Use Newton - Raphson method to determine a root of the equation $\cos x - xe^x = 0$, correct to five decimal places. Start with $x = 1$.

[AU Nov/Dec 2006 MA 038
Apr/May 2004 MA 038]

Solution:

Let $f(x) = \cos x - xe^x$

$f'(x) = -\sin x - xe^x - e^x$ [uv rule]

$$x_{n+1} = x_n - \left(\frac{\cos x_n - x_n e^{x_n}}{-\sin x_n - x_n e^{x_n} - e^{x_n}}\right)$$

$$\boldsymbol{x_{n+1} = \frac{-x_n \sin x_n - x_n^2 e^{x_n} - \cos x_n}{-\sin x_n - x_n e^{x_n} - e^{x_n}}}$$

Choose $x_0 = 1$

Iteration	x_0	x_{n+1}
$n = 0$	$x_0 = 1$	$x_1 = 0.65308$
$n = 1$	$x_1 = 0.65308$	$x_2 = 0.53134$
$n = 2$	$x_2 = 0.53134$	$x_3 = 0.51791$
$n = 3$	$x_3 = 0.51791$	$x_4 = 0.51776$
$n = 4$	$x_4 = 0.51776$	$x_5 = 0.51776$

Since x_4 and x_5 values are same.

$\therefore$ The required root is **0.51776.**

Example 9: Using Newton - Raphson's method, solve $x \log_{10} x = 12.34$. Start with $x_0 = 10$.

[AU, Apr/May 2004, MA 038]

Solution:

Let

$$f(x) = x \log_{10} x - 12.34$$

$$= 0.4343x \log_e x - 12.34$$

$$f'(x) = 0.4343\,(1 + \log_e x)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{0.4343x_n \log_e x_n - 12.34}{0.4343\,(1 + \log_e x_n)}\right)$$

$$x_{n+1} = \frac{0.4343x_n + 12.34}{0.4343\,(1 + \log_e x_n)}$$

Choose $x_0 = 10$

Iteration	x_n	x_{n+1}
$n = 0$	$x_0 = 10$	$x_1 = 11.63135$
$n = 1$	$x_1 = 11.63135$	$x_2 = 11.59477$
$n = 2$	$x_2 = 11.59477$	$x_3 = 11.59475$
$n = 3$	$x_3 = 11.59475$	$x_4 = 11.59475$

Since x_3 and x_4 values are same.

$\therefore$ The required is 11.5948.

Example 10: Find the root of the equation $3x + \sin x = e^x$ by Newton's method. *[AU CBE, June 2009, 070030010]*

Solution:

Let $f(x) = 3x + \sin x - e^x$

$f'(x) = 3 + \cos x - e^x$

$$\left.\begin{array}{l} f(0) = -1 \ (-\text{ve}) \\ f(1) = 1.1232 \ (+\text{ve}) \end{array}\right] \text{sign changes}$$

Hence there is a root between 0 and 1

By Newton - Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{3x_n + \sin x_n - e^{x_n}}{3 + \cos x_n - e^{x_n}} \right)$$

$$= \left(\frac{3x_n + x_n \cos x_n - x_n e^{x_n} - 3x_n - \sin x_n + e^{x_n}}{3 + \cos x_n - e^{x_n}} \right)$$

$$x_{n+1} = \frac{x_n \cos x_n - x_n e^{x_n} - \sin x_n + e^{x_n}}{3 + \cos x_n - e^{x_n}}$$

Choose $x_0 = 0$

Iteration	x_n	x_{n+1}
$n = 0$	$x_0 = 0$	$x_1 = 0.33333$
$n = 1$	$x_1 = 0.33333$	$x_2 = 0.36017$
$n = 2$	$x_2 = 0.36017$	$x_3 = 0.36042$
$n = 3$	$x_3 = 0.36042$	$x_4 = 0.36042$

Since x_3 and x_4 value are same.

$\therefore$ The required root is **0.36042.**

Example 11: Using Newton - Raphson method find the positive root of $x^3 - 5x + 3 = 0$

Solution:

Step 1

Given

$$x^3 - 5x + 3 = 0$$

Let $f(x) = x^3 - 5x + 3$

$f'(x) = 3x^2 - 5$

By trial and Error

$$\left.\begin{array}{l} f(0) = 3 \ (+\text{ve}) \\ f(1) = -1 \ (-\text{ve}) \end{array}\right] \text{Sign changes}$$

$\therefore$ The root lies between 0 and 1.

Step 2

The formula for $N-R$ method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - 5x_n + 3)}{(3x_n^2 - 5)}$$

Step 3

Since $|f(0)| > |f(1)|$ choose $x_0 = 1$.

Iteration	(x_n)	$x_{n+1} = x_n - \dfrac{(x_n^3 - 5x_2 + 3)}{(3x_n^2 - 5)}$
$n = 0$	$x_0 = 2$	$x_1 = 0.5000$
$n = 1$	$x_1 = 0.5000$	$x_2 = 0.6471$
$n = 2$	$x_2 = 0.6471$	$x_3 = 0.6566$
$n = 3$	$x_3 = 0.6566$	$x_4 = 0.6566$

Since x_3 and x_4 values are same.

The positive root of the equation is **0.6566**.

Example 12: Using Newton - Raphson method find the positive root of $x^3 - 2x - 5 = 0$.

Solution:

Step 1: Given

$$x^3 - 2x - 5 = 0$$

$$\text{Let} \quad f(x) = x^3 - 2x - 5$$

$$f'(x) = 3x^2 - 2$$

By trial and Error method

$$f(0) = -5 \ (-\text{ve})$$

$$f(1) = -6 \ (-\text{ve})$$

$$\left.\begin{array}{l} f(2) = -1 \ (-\text{ve}) \\ f(3) = 16 \ (+\text{ve}) \end{array}\right\} \text{sign changes}$$

$\therefore$ The root lies between 2 and 3.

Since $|f(2)| < |f(3)|$

Choose $x_0 = 2$.

Step 2

The formula for Newton - Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^2 - 2x_n - 5)}{(3x_n^2 - 2)}$$

Iteration	(x_n)	$x_{n+1} = x_n - \dfrac{(x_n^3 - 2x_n - 5)}{(3x_n^2 - 2)}$
$n = 0$	$x_0 = 2$	$x_1 = 7.0000$
$n = 1$	$x_1 = 7.0000$	$x_2 = 4.7655$
$n = 2$	$x_2 = 4.7655$	$x_3 = 3.3487$
$n = 3$	$x_3 = 3.3487$	$x_4 = 2.5316$
$n = 4$	$x_4 = 2.5316$	$x_5 = 2.1739$
$n = 5$	$x_5 = 2.1739$	$x_6 = 2.0979$
$n = 6$	$x_6 = 2.0979$	$x_7 = 2.0946$
$n = 7$	$x_7 = 2.0946$	$x_8 = 2.0946$

Since x_7 and x_8 values are same.

The positive root of the equation is **2.0946.**

Example 13: Use the Newton - Raphson method find the root of the equation $x^3 - x - 1 = 0$.

Solution:

Given

$$x^3 - x - 1 = 0$$

Let $f(x) = x^3 - x - 1$

$$f'(x) = 3x^2 - 1$$

By trial and Error method

$$f(0) = -1 \ (-\text{ve})$$

$$\left.\begin{array}{l} f(1) = -1 \ (-\text{ve}) \\ f(2) = 5 \ (+\text{ve}) \end{array}\right\} \text{sign changes}$$

$\therefore$ The root lies between 1 and 2.

Since $|f(1)| < |f(2)|$, choose $x_0 = 1$.

The formula of Newton - Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - x_n - 1)}{(3x_n^2 - 1)}$$

Iteration	(x_n)	$x_{n+1} = x_n - \frac{(x_n^3 - x_n - 1)}{(3x_n^2 - 1)}$
$n = 0$	$x_0 = 1$	$x_1 = 1.5000$
$n = 1$	$x_1 = 1.5000$	$x_2 = 1.3478$
$n = 2$	$x_2 = 1.3478$	$x_3 = 1.3252$
$n = 3$	$x_3 = 1.3252$	$x_4 = 1.3247$
$n = 4$	$x_4 = 1.3247$	$x_5 = 1.3247$

Since x_4 and x_5 values are same.

The positive root of equation is **1.3247.**

Example 14: Find by Newton's method, the real positive root of $x = \cos x$, correct to 3 decimal places.

Solution:

Let $f(x) = x - \cos x \Rightarrow f'(x) = 1 + \sin x$

$f(0) = -1$ (– ve), $f(1) = 0.4597$ (+v), the root lies between 0 and 1.

We choose the initial approximation $x_0 = 0.5$

By Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \ldots$$

Iteration	x_n	$x_{n+1} = x_n - \frac{[x_n - \cos x_n]}{[1 + \sin x_n]}$
$n = 0$	$x_0 = 0.5$	$x_1 = 0.7553$
$n = 1$	$x_1 = 0.7553$	$x_2 = 0.7392$
$n = 2$	$x_2 = 0.7392$	$x_3 = 0.7391$
$n = 3$	$x_3 = 0.7391$	$x_4 = 0.7391$

First approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{0.5 - \cos(0.5)}{1 + \sin(0.5)} = 0.5 - \left[\frac{-0.3773}{1.479}\right]$$

(Calculations in radians)

$\therefore \quad x_1 = 0.7553$

Second approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7553 - \frac{0.0272}{1.6855}$$

$\therefore$ $$x_2 = 0.7392$$

Third approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.7392 - \frac{0.00019}{1.6737}$$

$\therefore$ $$x_3 = 0.7391$$

Fourth approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.7391 - \frac{0.000025}{1.6736}$$

$\therefore$ $$x_4 = 0.7391$$

$\therefore$ The root correct to three decimals is 0.739.

Since x_3 and x_4 values are same.

Hence the root of the equation is **0.7391.**

Example 15: Find the positive root of $2x - \log_{10} x = 7$ by Newton-Raphson method.

Solution:

$$f(x) = 2x - \log_{10} x = 7$$

$$f'(x) = 2 - \frac{1}{x}$$

$$f(1) = -5 \text{ (–ve)}$$

$$f(2) = -3.30103 \text{ (–ve)}$$

$$\left.\begin{array}{l} f(3) = -1.47712 \text{ (–ve)} \\ f(4) = 0.39794 \text{ (+ve)} \end{array}\right] \text{Sign changes}$$

The root lies between 3 and 4.

Initial approximation $x_0 = 3.7$

Formula : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3.7 - \frac{(-0.16820)}{1.72973}$$

$$\mathbf{x_1 = 3.79724}$$

$$x_2 = 3.79724 - \frac{f(3.79724)}{f'(3.79724)}$$

$$= 3.79724 - \frac{(0.01501)}{1.73665}$$

$$\mathbf{x_2 = 3.78860}$$

$$x_3 = 3.78860 - \frac{f(3.78860)}{f'(3.7886)}$$

$$= 3.78860 - \frac{-(0.00128)}{1.73605}$$

$$\mathbf{x_3 = 3.78934}$$

$$x_4 = 3.78934 - \frac{f(3.78934)}{f'(3.78934)}$$

$$\mathbf{x_4 = 3.78927}$$

∴ The root of the equation is 3.78927

Example 16: For the following function $x^3 - 2x + 0.5 = 0$ find the smallest positive root.

Solution

$$f(x) = x^3 - 2x + 0.5$$

$$f'(x) = 3x^2 - 2$$

$$f(0) = 0.5 \text{ (+ve)}$$

$$f(1) = -0.5 \text{ (−ve) sign changes}$$

Solution interval [0, 1]

Formula : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Initial approximation $x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{(-0.375)}{-(125)}$$

$$x_1 = 0.2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.2 - \frac{f(0.2)}{f'(0.2)}$$

$$= 0.2 - \frac{(0.108)}{(-1.88)}$$

$$\mathbf{x_2 = 0.25745}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.25742 - \frac{f(0.25742)}{f'\,0.25742)}$$

$$= 0.25742 - \frac{0.00222}{-1.80120}$$

$$\mathbf{x_3 = 0.25865}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.25865 - \frac{f(0.25865)}{f'(0.25865)}$$

$$= 0.25865 - \frac{(0.000003639}{-1.79930}$$

$$\mathbf{x_4 = 0.25865}$$

$\therefore$ The root of the equation is 0.25865

Example 17: Evaluate the value of 1/7 using Newton-Raphson method.

Solution:

$x = 1/7$ (i.e.,) $7x - 1 = 0$

$f(x) = 7x - 1$

$f(0) = 1$ (+ve)

$f(1) = -6$ (–ve)

$\therefore$ Solution interval [0, 1]

Formula: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Choose Initial approximation $x_0 = 0.2$

$\therefore \qquad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$= 0.2 - \frac{f(0.2)}{f'(0.2)} = 0.2 - \frac{(7(0.2) - 1)}{7}$$

$$x_1 = 0.14286$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.14286 - \frac{f(0.14286}{f'(0.14286)}$$

$$= 0.14286 - \frac{(7(0.14286) - 1)}{7} = 0.4286$$

$$x_2 = 0.14286$$

∴ The root of the equation is **0.14286**.

EXERCISES

1. Determine the smallest positive root of the equation $f(x) = x \sin x + \cos x = 0$ by Newton - Raphson method.
 [**Ans:** 2.7983]

2. Determine the roots by Newton Raphson method $f(x) = e^{-x} - x = 0$

3. Solve by Newton's method, the following equation correct to six decimal places $x \log_{10} x = 4.7772393$ [**Ans:** 6.089114]

4. Find the smallest root of the equation by Newton - Raphson method $e^x = \sin x$ [**Ans:** 0.5885]

5. Solve by Newton Raphson method $f(x) = x\, e^x - \cos x = 0$
 [**Ans:** 0.517757]

6. Solve the following equation by Newton's method $f(x) = \log x - \cos x = 0$ [**Ans:** 1.3029]

7. Find a zero of a polynomial $x^2 - 5x + 2 = 0$ by Newton - Raphson method. [**Ans:** 4.5616]

3.2 ITERATION METHOD (FIXED POINT ITERATION METHOD)

To find the real root of an equation

$$f(x) = 0 \quad \ldots (1)$$

We rewrite it as

$$x = \phi(x) \quad \ldots (2)$$

Equation (1) and (2) are equivalent, so their roots are the same. The root of equation (2) is got as the abscission of the point of intersection of the curves $y = x$ and $y = \phi(x)$ and it is called the fixed point.

3.2.1 Procedure

We first an initial approximate value x_0 of the required root. Now using equation (2), we find a better approximation x_1 as $x_1 = \phi(x_0)$. We continue the iteration by finding

$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2)$$

If the sequence $x_0, x_1, x_2, \ldots$ converges to a limit α then α, is the required root of the equation $f(x) = 0$.

Note

The process converges only when the absolute value of the slope of $y = \phi(x)$ curve is the less than the slope of $y = x$ curve.

i.e., when $|\phi'(x)| < 1$

Also if the slope of $y = \phi(x)$ is closer to zero, the convergence will be faster.

Fixed point theorem

If $\phi(x)$ and $\phi'(x)$ are continuous in the interval I in which the root α of the equation $x = \phi(x)$ lies, then the sequence of approximations $x_0, x_1, x_2, \ldots$ converges to the root α if

(i) $|\phi'(x)| < 1|$ for all x in I

(ii) the initial approximation x_0 is chosen in I

Note

The sequence $x_0, x_1, x_2, \ldots$ may not converge always. Convergence of the iteration process depends on $\phi(x)$. The conditions for convergence are given in fixed point theorem. Hence we should rewrite the given equation $f(x) = 0$ in the form $x = \phi(x)$, in such a way that $|\phi'(x)| < 1$ for all $x \in I$.

3.2.2 sufficient Condition for the Iteration Function $g(x)$ to Converge to the Actual Root

Theorem

If the modulus of he iteration function is less than 1 (i.e.,) $|g(x)| < 1$ in the solution interval, $[c, d]$ of the given equation and $g(x)$ is continuous in the solution interval, then for any initial value x_0 taken from the solution interval $[c, d]$ the sequence of iteration function converges to the actual root.

Proof: $f(x) = 0$ is the given equation to be solved

$f(x)$ is the rewritten in the form $x = g(x)$ where $g(x)$ is the iteration function.

Let x be the exact solution of (1)

$$\therefore \quad f(\eta) = 0$$

Let $x_1, x_2, x_3 \ldots. x_k, x_{k+1}, \ldots$ are the approximations of the actual root.

The $(k+1)^{th}$ — approximation is got by substituting k^{th} approximation is (2)

$$x_{k+1} = \phi\,(x_k) \qquad \text{... (3)}$$

Since η is the exact root it should satisfy (2)

$$\therefore\ \eta = \phi\,(\eta) \qquad \text{... (4)}$$

$(3) - (4) \Rightarrow$

$$\eta - x_{k+1} = \phi\,(n) - \phi\,(x_k) \qquad \text{... (5)}$$

$$\eta - x_{k+1} = (\eta - x_k)\,\phi\,'(t_k)\ t_k \in\ (x_k, \eta)$$

$$\in_{k+1} = \in_k \phi\,'\,(t_k)$$

$$= \varepsilon_{k-1}\,\phi\,'\,(t_{k-1})\,f\,'(t_k)\ \ \text{[by induction]}$$

$$= \varepsilon_{k-2}\,\phi\,'\,(t_{k-2})\,\phi\,'(t_{k-1})\,\phi\,'\,(t_1)$$

$$\varepsilon_{k-1} = \varepsilon_f\,'(t_1)\,\phi\,'(t_1)\,\phi\,'(t_2) \ldots \phi\,'(t_k)$$

$$\varepsilon_{k+1} \le |\,\varepsilon_0\,|\,c^{k+1} \qquad \text{if } |\,g'(t_1)\,| \le c$$

$$|\,\varepsilon_{k+1}\,| \le \varepsilon_0\,|\,c^{k+1} \qquad i = 0, 1, 2, \ldots k.$$

If right hand side becomes zero then the error $|\in_{k+1}| \to 0$.

R.H.S. $\to 0$ only if $c^{k+1} \to 0$

$c^{k+1} \to 0$ if $c < 1$ which implies that

$|\,\phi\,'(t_1)\,| < 1$

$\therefore$ The iteration converges when $|\,g'(t)\,| < 1$ in the solution interval.

Order of Convergence

Order of convergence of Newton - Raphson method: 2 (Quadratic)

Let x_n, x_{n+1} are the approximate roots at the n^{th} and $(n+1)^{\text{th}}$ stage. α: exact root (may be positive or negative).

$e_{n+1} = (\alpha - x_n)$: error at the $(n+1)^{\text{th}}$ stage

$e_n = (\alpha - x_n)$: error at the n^{th} stage

$$f(\alpha) = f(x_n + e_n) = f(x_n) + \frac{e_n}{11} f'(x_n) + \frac{e_n^2}{21} f''(x_n) + ...$$

$$0 = f(x_n) + \frac{e_n}{1!} f'(x_n) + \frac{e_n^2}{21} f''(x_n) + \ldots$$

$$0 = f(x_n) + (\alpha - x_n) - f'(x_n) + \frac{(a - x_n)^2}{2!} f''(x_n) \; ..... \qquad \ldots (1)$$

$$0 \sim f(x_n) + (x_{n+1} - x_n) f'(x_n$$

$$\begin{bmatrix} x_{n+1} \sim \alpha \\ \text{where } \alpha \text{ is the actual root. } x_{n+1} \text{ is the} \\ \text{approximate root at the } (n+1)^{\text{th}} \text{ iteration} \end{bmatrix}$$

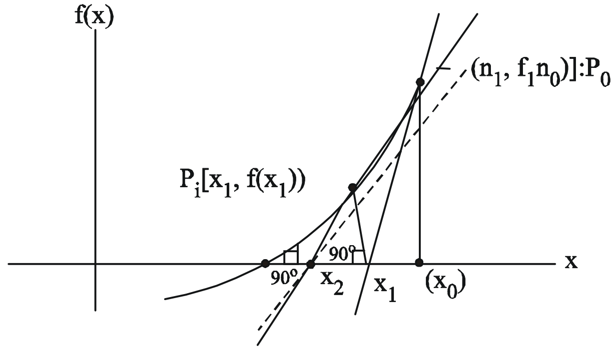
$$(1) - (2) \Rightarrow 0 = (K - k_{n+1})(f'(x_n) + \frac{(K - k_n)^2}{2!} f''(x_n)$$

$$\Rightarrow 0 = e_{n+1} f'(x_n) + \frac{e_n^2}{2!} f''(x_n)$$

$$e_{n+1} = \frac{-e_n^2}{2!} \frac{f''(x_n)}{f'(x_n)}$$

∴ Error at the $(n+1)^{\text{th}}$ stage is approximately the square of the error at the previous stage (ie) The error is of quadratic convergence.

Geometrical Interpretation



Let x_0 be the initial approximation. The ordinate at x_0 touch the curve at the point P_0. $(x_0, f(x_0))$. The tangent at $P_0(x_0, f_1(x_0))$ crosses the x-axis at x_1 which is nearer to the exact root α than x_0. The ordinate of $f(x_1)$ touches the curve $f(x)$ at P_1 $(x_1, f(x_1))$. The tangent at $P_1(x_1, f(x_1))$ crosses the x-axis at x_2 which still nearer to the exact root α, than x_1. Continuing like this we can make the root almost equal to the exact root α.

WORKED EXAMPLES

Example 1: Find the real root of the equation $\cos x - 3x + 1 = 0$, correct to 4 decimal places using fixed point iteration method.

[A.U N/D 2022] [A.U M/J 2013]

Solution:

Note

Use "Radian" mode in the calculator (fx – 991 ms) for problems involving trigonometric functions.

Given

$$\cos x - 3x + 1 = 0 \qquad \dots (1)$$

Let $\quad f(x) = \cos x - 3x + 1$

By trial and Error method,

$$\left.\begin{aligned} f(0) &= 2 \ (+\text{ ve}) \\ f(1) &= -1.4597 \ (-\text{ ve}) \end{aligned}\right] \text{sign changes}$$

Therefore there is a root of the equation which lies between 0 and 1.

Rewrite equation (1) as

$$\cos x + 1 = 3x$$

$$x = \frac{\cos x + 1}{3} = \phi(x) \qquad \text{... (2)}$$

Differentiate (2), we get

$$\phi'(x) = \frac{1}{3}(-\sin x)$$

$$|\phi'(x)| = \left| \frac{-\sin x}{3} \right| = \frac{\sin x}{3}$$

$|\phi'(x)| < 1 \ \forall \ x$ in (0, 1).

Now we use iteration method.

Choose $x = 0.5$, (2) will becomes,

$$x_1 = \frac{1 + \cos x_0}{3} = 0.62586$$

$$x_2 = \frac{1 + \cos x_1}{3} = 0.60349$$

$$x_3 = \frac{1 + \cos x_2}{3} = 0.60779$$

$$x_4 = \frac{1 + \cos x_3}{3} = 0.60697$$

$$x_5 = \frac{1 + \cos x_4}{3} = 0.60713$$

$$x_6 = \frac{1 + \cos x_5}{3} = 0.60710$$

$$x_7 = \frac{1 + \cos x_6}{3} = 0.60710$$

Since x_6 and x_7 are same.

Hence the required root is **0.60710.**

Example 2: Solve by Iteration method: $x^3 + x^2 = 100$ for its real root.

Solution:

Given

$$x^3 + x^2 = 100 \qquad \text{... (1)}$$

Let $f(x) = x^3 + x^2 - 100$

$f(0) = -100$ (– ve)

$f(1) = -98$ (– ve)

$f(2) = -88$ (– ve)

$f(3) = -64$ (– ve)

$$\begin{bmatrix} f(4) = -20 \ (-\text{ ve}) \\ f(5) = 50 \ (+\text{ ve}) \end{bmatrix} \text{sign changed}$$

Therefore there is a root for the equation which lies between 4 and 5.

Rewrite equation (1) as

$$x^3 + x^2 = 100$$

$$x^2 (x + 1) = 100$$

$$x^2 = \frac{100}{x + 1}$$

$$x = \frac{10}{\sqrt{x + 1}} = \phi(x) \qquad \text{...(2)}$$

Differentiate (2), we get

$$\therefore\ \phi'(x) = 10 \times \left(\frac{-1}{2}\right)(x+1)^{-3/2} = \frac{-5}{(x+1)^{3/2}}$$

$$|\phi'(x)| = \left|\frac{-5}{(x+1)^{3/2}}\right| = \frac{5}{(x+1)^{3/2}} < 1\ \forall\ x \in (4, 5)$$

Clearly

$|\phi'(x)| < 1\ \forall\ x$ in (4, 5).

Now we use iteration method

Choose $x_0 = 4.2$, (2) will becomes

$$x_1 = \frac{10}{\sqrt{(x_0+1)}} = \frac{10}{\sqrt{(4.2+1)}} = 4.38529$$

$$x_2 = \frac{10}{\sqrt{(x_1+1)}} = \frac{10}{\sqrt{(4.38529+1)}} = 4.30919$$

$$x_3 = \frac{10}{\sqrt{(x_2+1)}} = \frac{10}{\sqrt{(4.30919+1)}} = 4.33996$$

$$x_4 = \frac{10}{\sqrt{(x_3+1)}} = \frac{10}{\sqrt{(4.33996+1)}} = 4.32744$$

$$x_5 = \frac{10}{\sqrt{(x_4+1)}} = \frac{10}{\sqrt{(4.32744+1)}} = 4.33252$$

$$x_6 = \frac{10}{\sqrt{(x_5+1)}} = \frac{10}{\sqrt{(4.333252+1)}} = 4.33046$$

$$x_7 = \frac{10}{\sqrt{(x_6+1)}} = \frac{10}{\sqrt{(4.33046+1)}} = 4.33129$$

$$x_8 = \frac{10}{\sqrt{(x_7+1)}} = \frac{10}{\sqrt{(4.33129+1)}} = 4.33096$$

$$x_9 = \frac{10}{\sqrt{(x_8+1)}} = \frac{10}{\sqrt{(4.33096+1)}} = 4.33109$$

$$x_{10} = \frac{10}{\sqrt{(x_9 + 1)}} = \frac{10}{\sqrt{(4.33109 + 1)}} = 4.33104$$

$$x_{11} = \frac{10}{\sqrt{(x_{10} + 1)}} = \frac{10}{\sqrt{(4.33104 + 1)}} = 4.33106$$

$$x_{12} = \frac{10}{\sqrt{(x_{11} + 1)}} = \frac{10}{\sqrt{(4.33106 + 1)}} = 4.33105$$

$$x_{13} = \frac{10}{\sqrt{(x_{12} + 1)}} = \frac{10}{\sqrt{(4.33105 + 1)}} = 4.33105$$

Since the values of x_{12} and x_{13} are equal, the root is **4.33105.**

Example 3: Find a real root of the equation $x^3 + x^2 + 1 = 0$ by iteration method. *[A.U M/J 2012] [A.U N/D 2019]*

Solution:

Let $f(x) = x^3 + x^2 - 1$

$$\left[\begin{array}{l} Now f(0) = -1 = -\text{ ve} \\ \text{and } f(1) = 1 = +\text{ ve} \end{array}\right] \text{sign changes}$$

Hence a real root lies between 0 and 1.

Now $x^3 + x^2 - 1 = 0$ can be written as

$$x^2(x + 1) - 1 = 0$$

i.e., $$x^2 = \frac{1}{x + 1}$$

i.e., $$x = \frac{1}{\sqrt{x + 1}}$$

Let $$x = \phi(x) = \frac{1}{\sqrt{x + 1}} \quad ...(2)$$

Differentiate (2), we get

Therefore, $$\phi'(x) = \frac{-\frac{1}{2\sqrt{x + 1}}}{(x + 1)} = -\frac{1}{2(x + 1)^{3/2}}$$

Clearly $|\phi'(x)| = \left|\dfrac{1}{2(x+1)^{3/2}}\right| < 1$ in (0, 1)

Let the initial approximation be $x_0 = 0.5$, (2) will becomes

$$x_1 = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.5+1}} = 0.81649$$

$$x_2 = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{0.81649+1}} = 0.74196$$

$$x_3 = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{0.74196+1}} = 0.75767$$

$$x_4 = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.75767+1}} = 0.75427$$

$$x_5 = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{0.75427+1}} = 0.75500$$

$$x_6 = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{0.75500+1}} = 0.75485$$

$$x_7 = \frac{1}{\sqrt{x_6+1}} = \frac{1}{\sqrt{0.75485+1}} = 0.75488$$

Since the difference x_6 and x_7 is very small, the root of the given equation is **0.75488.**

Example 4: Solve by iteration method $2x - \log_{10} x = 7$.

[A.U A/M 2022]

Solution:

Let $\quad f(x) = 2x - \log_{10} x - 7 \quad$ ...(1)

$$f(3) = 6 - \log_{10} 3 - 7$$

$$= 6 - 0.4771 - 7 = -\text{ ve}$$

and $\quad f(4) = 8 - \log_{10} 4 - 7$

$$= 8 - 0.602 - 7 = +\text{ ve}$$

Hence a real root lies between 3 and 4. The given equation (1) can be written as

$$x = \frac{1}{2} (\log_{10} x + 7)$$

Let $$\phi (x) = \frac{1}{2} (\log_{10} x + 7) \quad ...(2)$$

Differentiate (2) we get

$$\phi (x) = \frac{1}{2} \left[\frac{1}{x} \log_{10} e \right]$$

Since $|\phi' (x)| < 1$ for x lies in (3, 4).

Let $x_0 = 3.6$, (2) will be becomes,

$$x_1 = \frac{1}{2} (\log_{10} x_0 + 7)$$

$$= \frac{1}{2} (\log_{10} 3.6 + 7) = 3.77815$$

$$x_2 = \frac{1}{2} (\log_{10} x_1 + 7)$$

$$= \frac{1}{2} (\log_{10} 3.77815 + 7) = 3.78863$$

$$x_3 = \frac{1}{2} (\log_{10} x_2 + 7)$$

$$x_3 = \frac{1}{2} (\log_{10} 3.78863 + 7) = 3.78924$$

$$x_4 = \frac{1}{2} (\log_{10} x_3 + 7)$$

$$x_4 = \frac{1}{2} (\log_{10} 3.78924 + 7) = 3.78927$$

Since x_3 and x_4 values are very close. Hence the root is **3.78927.**

Example 5: Find a positive root of $3x - \log_{10} x = 6$, using fixed point iteration method.

Solution:

Let $f(x) = 3x - \log_{10} x - 6$ ...(1)

$f(1) = 3 - \log_{10} 1 - 6 = -3$ (– ve)

$f(2) = 3(2) - \log_{10} 2 - 6$

$= -0.3010$ (– ve)

$f(3) = 3(3) - \log_{10} 3 - 6 = 3 - \log_{10} 3$

$= 3 - 0.4771 = 2.5229$ (+ ve)

Sign changes

∴ The root lies between 2 and 3.

The given equation (1) can be written as

$$x = \frac{1}{3}[6 + \log_{10} x]$$

Let $$\phi(x) = \frac{1}{3}[6 + \log_{10} x]$$ ...(2)

Differentiate (2), we get

$$\phi'(x) = \frac{1}{3}\left[\frac{1}{x}\log_{10} x\right]$$

Clearly $|\phi'(x)| < 1$ in the interval [2,3].

Take $x_0 = 2$, (2) will becomes

$$x_1 = \frac{1}{3}[6 + \log_{10} x_0] = \frac{1}{3}[6 + \log_{10} 2]$$

$$x_1 = 2.1003$$

$$x_2 = \frac{1}{3}[6 + \log_{10} x_1] = \frac{1}{3}[6 + \log_{10} 2.1003]$$

$$x_2 = 2.1074$$

$$x_3 = \frac{1}{3}[6 + \log_{10} x_2] = \frac{1}{3}[6 + \log_{10} 2.1074]$$

$$x_3 = 2.1079$$

$$x_4 = \frac{1}{3}[6 + \log_{10} x_3] = \frac{1}{3}[6 + \log_{10} 2.1079]$$

$$x_4 = 2.10795$$

$$x_5 = \frac{1}{3}[6 + \log_{10} x_4] = \frac{1}{3}[6 + \log_{10} 2.10795]$$

$$x_5 = 2.10795$$

Since x_4 and x_5 are equal the required root is **2.10795.**

Example 6: Use the iteration method to find a root of the equation $x = \frac{1}{2} + \sin x$.

Solution:

Let $f(x) = \sin x - x + \frac{1}{2}$ ...(1)

Now $f(0) = \sin 0 - 0 + \frac{1}{2} = \frac{1}{2}$ (+ ve)

$f(1) = \sin 1 - 1 + \frac{1}{2} = 0.84 - 0.5 =$ (+ ve)

and $f(2) = \sin 2 - 2 + \frac{1}{2} = 0.909 - 1.5 =$ (– ve)

Hence a root lies between 1 and 2. The given equation (1) can be written as

$$x = \sin x + \frac{1}{2}$$

Let $x = \phi(x) = \sin x + \frac{1}{2}$ ...(2)

Differentiate (2), we get

$$|\phi'(x)| = |\cos x| < 1 \text{ in } (1, 2)$$

Hence iteration method can be applied. Let the initial approximation be $x_0 = 1$. (2) will be becomes

$$x_1 = \sin(x_0) + \frac{1}{2} = \sin 1 + \frac{1}{2}$$

$$x_1 = 0.8414 + 0.5 = 1.3414$$

$$x_2 = \sin(x_1) + \frac{1}{2} = \sin(1.3414) + \frac{1}{2}$$

$$x_2 = 0.9738 + 0.5 = 1.4738$$

$$x_3 = \sin(x_2) + \frac{1}{2} = \sin(1.4738) + \frac{1}{2}$$

$$x_3 = 0.9952 + 0.5 = 1.4953$$

$$x_4 = \sin(x_3) + \frac{1}{2} = \sin(1.4953) + \frac{1}{2}$$

$$x_4 = 0.9971 + 0.5 = 1.4972$$

$$x_5 = \sin(x_4) + \frac{1}{2} = \sin(1.4.972) + \frac{1}{2}$$

$$x_5 = 0.9972 + 0.5 = 1.4972$$

Since $\boldsymbol{x_4}$ and $\boldsymbol{x_5}$ are equal, the required root is **1.4972.**

Example 7: Solve $e^x - 3x = 0$ using fixed point iteration method correct to 4 decimal places. *[A.U M/J 2012]*

Solution:

Let $f(x) = e^x - 3x$ ...(1)

$f(0) = e^0 - 0 - 0 = 1$ (+ ve)

$f(1) = e^1 - 3(1)$

$= -0.2817 = (-\text{ ve})$

Sign changes

$\therefore$ The root lies between 0 and 1.

The given equation can (1) be written as

$$e^x = 3x$$

(or) $$x = \frac{e^x}{3}$$

Let $$x = \phi(x) = \frac{e^x}{3} \qquad \text{...(2)}$$

Differentiate (2), we get

$$\phi'(x) = \frac{e^x}{3}$$

Since $|\phi'(x)| < 1$ in the interval [0, 1].

Let, $\boldsymbol{x_0 = 0.5}$, (2) will becomes

$$x_1 = \frac{e^{0.5}}{3} = 0.5496$$

$$x_2 = \frac{e^{0.5496}}{3} = 0.5775$$

$$x_3 = \frac{e^{0.5775}}{3} = 0.5939$$

$$x_4 = \frac{e^{0.5939}}{3} = 0.6037$$

$$x_5 = \frac{e^{0.6037}}{3} = 0.6096$$

$$x_6 = \frac{e^{0.6096}}{3} = 0.6132$$

$$x_7 = \frac{e^{0.6132}}{3} = 0.6155$$

$$x_8 = \frac{e^{0.6155}}{3} = 0.6168$$

$$x_9 = \frac{e^{0.6168}}{3} = 0.6177$$

$$x_{10} = \frac{e^{0.6177}}{3} = 0.6182$$

$$x_{11} = \frac{e^{0.6182}}{3} = 0.6185$$

$$x_{12} = \frac{e^{0.6185}}{3} = 0.6187$$

$$x_{13} = \frac{e^{0.6187}}{3} = 0.6188$$

$$x_{14} = \frac{e^{0.6188}}{3} = 0.6189$$

$$x_{15} = \frac{e^{0.6189}}{3} = 0.6189$$

Since x_{14} and x_{15} are equal the required root is **0.6189.**

Example 8: Solve for x from $\cos x - xe^x = 0$ by iteration method.

Solution:

Let $f(x) = \cos x - xe^x = 0$ ...(1)

$f(0) = 1$
$f(1) = -2.1780$ sign changes

Hence there is a root between 0 and 1.

Rewrite the given equation (1) as

$$xe^x = \cos x$$

$$x = \frac{\cos x}{e^x} = \phi\,(x) \qquad \text{...(2)}$$

Differentiate (2), we get

$$\phi'\,(x) = \frac{-\,e^x \sin x - e^x \cos x}{e^{2x}}$$

Since $|\phi'\,(x)| < 1$ for $x \in (0, 1)$,

Choose $x_0 = 0.5$, (2) will becomes

$$x_1 = 0.53228$$

$$x_2 = 0.50602$$

$$x_3 = 0.52734$$

$$x_4 = 0.51000$$

$$x_5 = 0.52408$$

$$x_6 = 0.51263$$

$$x_7 = 0.52193$$

$$x_8 = 0.51437$$

$$x_9 = 0.52051$$

$$x_{10} = 0.51552$$

Proceeding this we get the root **0.51776** after 36 iterations.

3.3 SYSTEMS OF LINEAR EQUATIONS

Introduction

Many problems in engineering and science needs the solution of a system of simultaneous linear equations. The solution of a system of simultaneous linear equations is obtained by the following two types of methods.

(a) Direct methods (Gauss elimination method and Gauss-Jordan method).

(b) Indirect methods Iterative methods (Gauss-Jacobi and Gauss-Seidel method).

(a) Direct methods are those in which

1. the computation can be completed in a finite number of steps resulting in the exact solution.
2. the amount of computation involved can be specified in advance.
3. the methods is independent of the accuracy desired.

(b) Iterative methods (self correcting methods) are those which

1. begin with an approximate solution and
2. obtain an improved solution with each step of iteration
3. but would require an infinite number of steps to obtain an exact solution with out round-off errors.
4. the accuracy of the solution depends on the number of iterations performed.

Simultaneous linear equations

The system of equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ is given by

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2$$

$$a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n = b_n$$

This can be written as $AX = B$ where

$A = (a_{ij})_{n \times n}$; $X = (x_1\ x_2\ \ldots\ldots\ x_n)^T$ and $B = (b_1\ b_2\ \ldots\ldots\ b_n)^T$

This system of equations can be solved by using determinants (Cramer's rule) or by means of matrices. These methods involves tedious calculations. There are other methods too to solve such equations. In this chapter we will discuss four methods viz.

(i) Gauss-Elimination method

(ii) Pivoting

(iii) Gauss-Jordan method

(iv) Gauss-Jacobi method

(v) Gauss-Seidel method.

NUMERICAL SOLUTION OF SET OF A LINEAR ALGEBRAIC EQUATIONS

In this chapter we shall discuss some direct and iterative numerical method suitable for computer or solving simultaneous linear algebraic equations with many unknows.

Numerical Methods for solving simultaneous equations

Direct Method	**Iterative methods**
1. Gauss Elimination	1. Gauss Jacobi
2. Gauss Jordan	2. Gauss seidel

3.3.1 Gauss - Elimination method

This is an elimination method and it reduces the given system of equation to an equivalent upper triangular system which can be solved by *back substitution*.

Consider the system of equations

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_1 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

Gauss-algorithm is explained below.

Step 1

Elimination of x_1 from the second and third equations. If $a_{11} \neq 0$, the first equation is used to eliminate x_1 from the second and third equation. After elimination, the reduced system is

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{22}' x_2 + a_{23}' x_3 = b_2'$$

$$a_{32}' x_2 + a_{33}' x_3 = b_3'$$

Step 2

Elimination of x_2 from the third equation. If $a_{22}' \neq 0$, we eliminate x_2 from the third equation and the reduced upper triangular system is

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{22}' x_2 + a_{23}' x_3 = b_2'$$

$$a_{33}'' x_3 = b_3''$$

Step 3

From third equation x_3 is known. Using x_3 in the second equation, x_2 is obtained. Using both x_2 and x_3 in the first equation, the value of x_1 is obtained.

Thus in Gauss-elimination method, we start with the augmented matrix (A/B) of the given system and transform it to (U/K) by elementary row operations. Finally the solution is obtained by back substitution process.

Principle

$$(A/B) \xrightarrow{\text{Gauss–Elimination}} (U/K)$$

3.3.2 Gauss - Jordan Method

Aim: To solve the following system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots a_{2n}x_n = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots a_{nn}x_n = b_n$$

Illustration with three equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots\ldots = b_3$$

Step 1:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Step 2: Augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & b_1 \\ a_{21} & a_{22} & a_{23} & | & b_2 \\ a_{31} & a_{32} & a_{33} & | & b_3 \end{bmatrix} \quad \ldots (1)$$

Reduce the augmented system (1) into diagonal system using the procedure explained for Gauss-elimination method.

$$\begin{bmatrix} c_{11} & 0 & 0 & | & d_1 \\ 0 & c_{22} & 0 & | & d_2 \\ 0 & 0 & c_{33} & | & d_3 \end{bmatrix}$$

$$c_{11}x_1 = d_1 \Rightarrow x_1 = \frac{d_1}{c_{11}}$$

$$c_{22}x_2 = d_2 \Rightarrow x_2 = \frac{d_2}{c_{22}}$$

$$c_{33}x_3 = d_3 \Rightarrow x_3 = \frac{d_3}{c_{33}}$$

Gauss-Jordan method

This method is a modification of Gauss-elimination method. Here the elimination of unknowns is performed not only in the equations below but also in the equations above. The co-efficient matrix A of the system $AX = B$ is reduced into a diagonal or a unit matrix and the solution is obtained directly without back substitution process.

Principle

$$(A/B) \xrightarrow{\text{Gauss–Jordan}} (D/K) \text{ or } (I/K)$$

WORKED EXAMPLES

Example 1: Solve the system of equations by Gauss-elimination method.

$$\boldsymbol{x + y + z = 9 \;;\; 2x - 3y + 4z = 13 \;;\; 3x + 4y + 5z = 40}$$

Note: Solve by using calculator and cross check year answer $x = 1,\ y = 3,\ z = 5.$

Solution:

Given system of equations

$$x + y + z = 9$$

$$2x - 3y + 4z = 13$$

$$3x + 4y + 5z = 40$$

The Augmented matrix is given by

$$(A, B) \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -15 \\ 0 & 1 & 2 & 3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to 5R_3 + 1R_2 \end{array}$$

The above is an upper triangular matrix

The corresponding equations are

$$x + y + z = 9 \quad \text{... (1)}$$

$$-5y + 2z = -5 \quad \text{... (2)}$$

$$12z = 60 \quad \text{... (3)}$$

We solve these equations by back substitution method.

From equation (3) we get

$$z = \frac{60}{12}$$

$$\boxed{z = 5}$$

Substituting in equation (2), we get

$$-5y + 2(5) = -5$$

$$-5y + 10 = -5$$

$$-5y = -15$$

$$\boxed{y = 3}$$

From equation (1)

$$\Rightarrow x + 3 + 5 = 9$$

$$\Rightarrow x = 9 - 8$$

$$\Rightarrow \mathbf{x = 1}$$

$\therefore$ **The solution is $x = 1, y = 3, z = 5$**

Example 2: Solve the equations by Gauss-elimination method

$2x + y + 4z = 12$; $8x - 3y + 2z = 20$; $4x + 11y - z = 33$

[AU A/M 2019]

Solution:

Given system of equations

$$2x + y + 4z = 12$$

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

The Augmented matrix is given by

$$(A, B) \sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -189 & -189 \end{array}\right] R_3 \to 7R_3 + 9R_2$$

The above is an upper triangular matrix,

The corresponding equations are,

$$2x + 1y + 4z = 12 \quad \text{...(1)}$$

$$-7y - 14z = -28 \quad \text{...(2)}$$

$$-189z = -189 \quad \text{... (3)}$$

We solve these equation times by back susstitution method.

From equation (3), we get

$$-189z = -189$$

$$\boxed{z = 1}$$

Substituting $z = 1$ in equation (2) we get,

$$-7y - 14\,(1) = -28$$

$$-7y = -28 + 14$$

$$\boxed{y = 2}$$

From equation (1), we get

$$2x + 2 + 4\,(1) = 12$$

$$2x = 12 - 6$$

$$\boxed{x = 3}$$

$\therefore$ **The solution is $x = 3, y = z,\ z = 1$**

Example 3: Solve the following equations by Gauss-elimination method.

$$\boldsymbol{x_1 + x_2 + 2x_3 = 4\ ;\ 3x_1 + x_2 - 3x_3 = -4\ ;\ 2x_1 - 3x_2 - 5x_3 = -5}$$

Solution:

The given system of equation is

$$x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + x_2 - 3x_3 = -4$$

$$2x_1 - 3x_2 - 5x_3 = -5$$

The Augmented matrix is given by

$$(A, B) \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & 1 & -3 & -4 \\ 2 & -3 & -5 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -2 & -9 & -16 \\ 0 & -5 & -9 & -13 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -2 & -9 & -16 \\ 0 & 0 & 27 & 54 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to 2R_3 - 5R_2 \end{matrix}$$

The above is an upper triangular matrix,

The corresponding equations are,

$$x_1 + x_2 + 2x_3 = 4 \quad \text{... (1)}$$

$$-2x_2 - 9x_3 = -16 \quad \text{... (2)}$$

$$27x_3 = 54 \quad \text{... (3)}$$

We solve these equations by back substitution method.

From equation (3), we get

$$27x_3 = 54$$

$$x_3 = \frac{54}{27}$$

$$\boxed{x_3 = 2}$$

Substituting $x_3 = 2$ in (2), we get

$$-2x_2 - 9\,(2) = -16$$

$$-2x_2 = -16 + 18$$

$$\boxed{x_2 = -1}$$

From equation (1)

$$x_1 + (-1) + 2\,(2) = 4$$

$$x_1 = 4 - 3$$

$$\boxed{x_1 = 1}$$

∴ The solution is $x_1 = 1,\ x_2 = -1,\ x_3 = 2$

Example 4: Solve the system of equations by Gauss-elimination method $x + 2y + z = 3,\ 2x + 3y + 3z = 10,\ 3x - y + 2z = 13$

[AU N/D 2019]

Solution:

The given system can be written as,

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

The Augmented matrix is given by

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right] \begin{array}{l} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_3 \leftrightarrow R_3 - 7R_2$$

The above is an upper triangular matrix. The correspoinding equations are

$$x + 2y + z = 3 \qquad ...(1)$$

$$-y + z = 4 \qquad ...(2)$$

$$-8z = -24 \qquad ...(3)$$

We solve the equation using back substitution method from (3).

$-8z = -24$	$-y + z = 4$	$x + 2y + z = 3$
$\boxed{z = 3}$	$-y + 3 = 4$	$x - 2 + 3 = 3$
	$\boxed{y = -1}$	$\boxed{x = 2}$
Substitute $z = 3$ in (2)	Substitute z and y in (1)	

Hence, the solution is $x = 2, y = -1, z = 3$

Example 5: Solve the system of equations by Gauss-elimination method.

$$\mathbf{5x_1 + x_2 + x_3 + x_4 = 4\ ; \qquad x_1 + 7x_2 + x_3 + x_4 = 12}$$

$$\mathbf{x_1 + x_2 + 6x_3 + x_4 = -5\ ; \qquad x_1 + x_2 + x_3 + 4x_4 = -6}$$

Solution:

The given system can be written as $AX = B$

$$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ -5 \\ -6 \end{bmatrix}$$

The Augmented matrix is given by

$$[A, B] = \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 1 & 1 & 1 & 4 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 34 & 4 & 4 & 56 \\ 0 & 4 & 29 & 4 & -29 \\ 0 & 4 & 4 & 9 & -34 \end{array}\right] \begin{array}{l} R_2 \leftrightarrow 5R_2 - R_1 \\ R_3 \leftrightarrow 5R_3 - R_1 \\ R_2 \leftrightarrow 5R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 12 & 63 & -138 \end{array}\right] \begin{array}{l} R_2 \leftrightarrow \dfrac{R_2}{2} \\ R_3 \leftrightarrow \dfrac{R_3}{10} \\ R_4 \leftrightarrow \dfrac{R_4}{10} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 5 & 1 & 1 & 1 & 4 \\ 0 & 17 & 2 & 2 & 28 \\ 0 & 0 & 97 & 12 & -121 \\ 0 & 0 & 0 & 5967 & -11934 \end{array}\right] R_4 \leftrightarrow 97R_4 - 12R_3$$

The corresponding equation are

$$x_1 + x_2 + x_3 + x_4 = 4 \quad \text{...(1)}$$

$$17x_2 + 2x_3 + 2x_4 = 28 \quad \text{...(2)}$$

$$97x_3 + 12x_4 = -121 \quad \text{...(3)}$$

$$5967x_4 = -11934 \quad \text{...(4)}$$

The also is an upper triangular matrix. We solve these equations using back substitution method.

$4 \Rightarrow$	$(3) \Rightarrow$	$(2) \Rightarrow$
$5967x_4 = -11934$	$97x_3 + 12x_4 = -121$	$17x_2 + 2x_3 + 2x_4 = 28$
$x_4 = \dfrac{-11934}{5967}$	$97x_3 + 12(-2) = -121$	$17x_2 + 2(-1) + 2(-2) = 28$
$\boxed{\mathbf{x_4 = -2}}$	$\boxed{\mathbf{97x_3 - 24 = -121}}$	$17x_2 - 2 - 4 = 28$
	$97x_3 = -121 + 24$	$17x_2 = 28 + 6$
	$97x_3 = -97$	$17x_2 = 34$
	$\boxed{\mathbf{x_3 = -1}}$	$\boxed{\mathbf{x_2 = 2}}$

$$(1) \Rightarrow x_1 + x_2 + x_3 + x_4 = 4$$

$$5x_1 + 2 + (-1) + (-2) = 4$$

$$5x_1 - 1 = 4$$

$$5x_1 = 5$$

$$\boxed{\mathbf{x_1 = 1}}$$

Hence, the solution is $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$.

Example 6: Using Gauss-elimination method, solve the system.

$$\mathbf{3.15x - 1.96y + 3.85z = 12.95}$$

$$\mathbf{2.13x + 5.12y - 2.89z = -8.61}$$

$$\mathbf{5.92x + 3.05y + 2.15z = 6.88}$$

[M.K.U. 1981] [A.U N/D 2011]

Solution:

The given system can be written as $AX = B$.

$$\begin{bmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 21.2107 & -16.0195 & -54.992 \end{bmatrix} \begin{matrix} R_2 \leftrightarrow 3.15R_2 - 2.13R_1 \\ R_3 \leftrightarrow 3.15R_3 - 5.92R_1 \\ \end{matrix}$$

$$\sim \begin{bmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 0 & 41.7892 & -43.8398 \end{bmatrix} R_3 \leftrightarrow 20.3028R_3 - 21.2107R_2$$

The above is an upper triangular matrix.

The corresponding equations are

$$3.15x - 1.96y + 3.85z = 12.95 \quad ...(1)$$

$$20.3028y - 17.304z = -54.705 \quad ...(2)$$

$$41.7892z = 43.8398 \quad ...(3)$$

We solve these equations using back substitution method.

$$(3) \Rightarrow 41.7892z = 43.8398$$

$$\boxed{\mathbf{z = 1.049}} \quad \text{[correct to 3 decimal places]}$$

$$20.3028y - 17.304z = -54.705$$

$$(2) \Rightarrow 20.3028y - (17.304)(1.049) = -54.705$$

$$20.3028y - 18.1519 = -54.705$$

$$20.3028y = -54.705 + 18.1519$$

$$= -36.5531$$

$$\boxed{\mathbf{y = -1.800}}$$

$$(1) \Rightarrow 3.15x - 1.96y + 3.85z = 12.95$$

$$3.15x - 1.96(-1.8) + 3.85(1.049) = 12.95$$

$$3.15x + 3.528 + 4.03865 = 12.95$$

$$3.15x + 7.5665 = 12.95$$

$$3.15x = 12.95 - 7.56665$$

$$= 5.38335$$

$$\boxed{x = 1.709}$$

Hence, the solution is $x = 1.709$, $y = -1.800$, $z = 1.049$

Example 7: Solve the system by Gauss-elimination method: $x - 3y - z = -30$; $2x - y - 3z = 5$, $5x - y - 2z = 142$.

Solution:

The augmented matrix of the given system is

$$(A, B) \sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & -30 \\ 2 & -1 & -3 & 5 \\ 5 & -1 & -2 & 142 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & -30 \\ 0 & 5 & -1 & 65 \\ 0 & 14 & 3 & 292 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 5 & -1 \\ 0 & 0 & 29/5 \end{bmatrix} \left(R_3 \to R_3 - \left(\frac{14}{5}\right) R_2 \right)$$

The above is an upper triangular matrix

The equivalent system is

$$x - 3y - z = -30 \quad ...(1)$$

$$5y - z = 65 \quad ...(2)$$

$$\frac{29}{5} z = 110 \quad ...(3)$$

By back substitution, we solve these equations

$$\boxed{z = \mathbf{18.966}}$$

$$y = \frac{65 + 18.966}{5} = 16.793 \text{ and}$$

$$\boxed{y = \mathbf{16.793}}$$

$$x = -30 + 3 \times 16.793 + 18.966 = 39.345$$

$$\boxed{x = \mathbf{39.345}}$$

$\therefore$ **The solution is $x = 39.345$, $y = 16.793$, $z = 18.966$**

Example 8: Solve $3x - y = 2$

$x + 3y = 4$

using Gauss-elimination method.

[AU May/June 2009 MA 038]

Solution:

The given system of equations can be written in matrix form as

$$AX = B$$

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The augmented matrix is

$$(A, B) \sim \begin{bmatrix} 3 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & \frac{10}{3} & \frac{10}{3} \end{bmatrix} \text{ by } R_2 \to \left(\frac{-1}{3}\right) R_1 + R_2$$

The equivalent system is

The above is an upper triangular matrix $3x - y = 2$ ... (1)

$$\frac{10}{3} y = \frac{10}{3} \quad \text{... (2)}$$

We solve these equations by back substitution method

From Eqn.(2), we get $\boxed{y = 1}$

Substituting in Eqn.(1), we get

$$3x - 1 = 2$$

$$3x = 3$$

$$\boxed{x = 1}$$

$\therefore$ **The solution is $x = 1, y = 1$**

Example 9: Solve the system of equations $x + 2y + z = 8$, $2x + 3y + 4z = 20$, $4x + y + 2z = 12$ by Gauss-Jordan method.

Solution:

The given system

$$x + 2y + z = 8$$

$$2x + 3y + 4z = 20$$

$$4x + y\ 2z = 12$$

The augmented matrix is

$$(A, B) \sim \left[\begin{array}{ccc} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 1 & 2 \end{array}\right|\left|\begin{array}{c} 8 \\ 20 \\ 12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -1 & 2 & 4 \\ 0 & -7 & -2 & -20 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & -16 & -48 \end{array}\right] \quad R_3 \leftrightarrow R_3 - 7R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_3 \to R_3 + (-1/16)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 5R_3 \\ R_2 \to R_2 - 2R_3 \end{array}$$

The above is an diagonal matrix

$\therefore$ The solution is $x = 1,\ y = 2,\ z = 3.$

Example 10: Solve by Gauss-Jordan method:
$3x + 4y + 5z = 18$; $2x - 2y + 8z = 13$; $5x - 2y + 7z = 20$

Solution:

The given system

$$3x + 4y + 5z = 18$$

$$2x - 2y + 8z = 13$$

$$5x - 2y + 7z = 20$$

The augmented matrix of the given system is

$$(A, B) \sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 5 \\ 0 & 1 & -14/11 & -3/11 \\ 0 & -27 & 22 & -5 \end{array}\right] \quad R_2 \to R_2\,(-1/11)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 37/11 & 70/11 \\ 0 & 1 & -14/11 & -3/11 \\ 0 & 0 & -136/11 & -136/11 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 5R_2 \\ \\ R_3 \to R_3 + 27R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 37/11 & 70/11 \\ 0 & 1 & -14/11 & -3/11 \\ 0 & 0 & 1 & 1 \end{array}\right] R_3 \to R_3\,(-11/136)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - (37/11)\,R_3 \\ R_2 \to R_2 \to R_2 + (14/11)\,R_3 \end{array}$$

The above is a diagonal matrix

$\therefore$ **The solution is $x = 3, y = 1, z = 1$.**

Example 11: Apply Gauss Jordan method to solve the system of equations

$$\begin{aligned} x + y + z &= 9 \\ 2x - 3y + 4z &= 13 \\ 3x + 4y + 5z &= 40 \end{aligned}$$

Solution:

The above system of equations can be written in matrix form as

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

The augmented matrix is

$$(A, B) \sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{bmatrix} \text{by} \quad \begin{matrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -3R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{5} & 8 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \frac{12}{5} & 12 \end{bmatrix} \text{by} \quad \begin{matrix} R_1 \to \left(\frac{1}{5}\right) R_2 + R_1 \\ \\ R_3 \to \left(\frac{1}{5}\right) R_2 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & \frac{12}{5} & 12 \end{bmatrix} \text{by} \quad \begin{matrix} R_1 \to \left(-\frac{7}{12}\right) R_3 + R_1 \\ \\ R_2 \to \left(-\frac{5}{6}\right) R_3 + R_2 \end{matrix}$$

The above is a diagonal matrix

$$\therefore \; x = 1$$

$$-5y = -15$$

$$\frac{12}{5} z = 12$$

Hence the solution is $x = 1, y = 3, \; z = 5$

Example 12: Solve by Gauss Jordan method.

$$x + y + z + w = 2$$

$$2x - y + 2z - w = -5$$

$$3x + 2y + 3z + 4w = 7$$

$$x - 2y - 3z + 2w = 5$$

[AU N/D 2021]

Solution:

The Augmented matrix is given by

$$(A, B) \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -4 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to (-1)\, R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{bmatrix} R_2 \to R_2 \div (-3)$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{bmatrix} \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 + 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & -4 & 4 & 12 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} R_3 \to R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} R_3 \to R_4 \div -4 \\ R_4 \to R_4 \div 2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} R_1 \to R_1 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_4 \\ R_2 \to R_2 - R_4 \\ R_3 \to R_3 + R_4 \end{matrix}$$

The above is a diagonal matrix. Hence the solution is

$$\boxed{\begin{aligned} \mathbf{x} &= \mathbf{0} \\ \mathbf{y} &= \mathbf{1} \\ \mathbf{z} &= \mathbf{-1} \\ \mathbf{w} &= \mathbf{2} \end{aligned}}$$

Example 13: Using the Gauss-Jordan method solve the following system of equations

$$\mathbf{10x + y + z = 12}$$

$$\mathbf{2x + 10y + z = 13}$$

$$\mathbf{x + y + 5z = 7}$$

[AU M/J 2016]

Solution:

$$10x + y + z = 12$$

$$2x + 10y + z = 13$$

$$x + y + 5z = 7$$

Interchanging the first and the last equation then

The Augmental matrix is

$$[A, B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right]$$

$$[A, B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right] \begin{matrix} R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 10R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad R_2 \leftrightarrow \frac{R_2}{8}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{array}\right] \quad \begin{array}{l} R_1 \leftrightarrow R_1 - R_2 \\ R_3 \leftrightarrow R_3 + 9R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3 \leftrightarrow \frac{R_3}{-59.125}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_1 - 6.125R_3 \\ R_1 \leftrightarrow R_2 + 1.125R_3 \end{array}$$

The above is a Diagonal Matrix.

Hence, the solution is $x = 1, y = 1, z = 1$

Example 14: Solve the following system by Gauss elimination method

$$\mathbf{2x + 3y + 5z = 23}$$

$$\mathbf{3x + 4y + z = 14}$$

$$\mathbf{6x + 7y + 27 = 26}$$

Solution

Step 1: Writing the system in matrix form

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 1 \\ 6 & 7 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ 14 \\ 26 \end{pmatrix}$$

Step 2: Reducing the augmented matrix into upper-triangular matrix form

$$\begin{pmatrix} 2 & 3 & 5 & 23 \\ 3 & 4 & 1 & 14 \\ 6 & 7 & 2 & 26 \end{pmatrix}$$

$$\left[\begin{array}{cccc} 2 & 3 & 5 & | & 23 \\ 0 & 4+\left(\frac{-3}{2}\right)(3)+ & 1+\left(\frac{-3}{2}\right)(5) & | & 14+\left(\frac{-3}{2}\right)(23) \\ 0 & 7+(3)(3) & 2+(-3)(5) & | & 26+(13)(23) \end{array}\right] \begin{array}{l} R_2 = R_2 + \left(\frac{-3}{2}\right)R_1 \\ R_3 = R_3 + \left(\frac{-6}{2}\right)R_1 \end{array}$$

$$= \begin{bmatrix} 2 & 3 & 5 & 23 \\ 0 & \frac{-1}{2} & \frac{-13}{2} & \frac{-41}{2} \\ 0 & -2 & 13 & -43 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 & 23 \\ 0 & \frac{-1}{2} & \frac{-13}{2} & \frac{-41}{2} \\ 0 & 0 & 13 & 39 \end{bmatrix} R_3 = R_2(-4) + R_3$$

$$= \left[\begin{array}{cccc} 2 & 3 & 5 & | & 23 \\ 0 & \frac{-1}{2} & \frac{-13}{2} & | & \frac{-41}{2} \\ 0 & 0 & 13+(-4)\left(\frac{-13}{2}\right) & | & -43+(-4)\left(\frac{-41}{2}\right) \end{array}\right]$$

$$= \left[\begin{array}{cccc} 2 & 3 & 5 & | & 23 \\ 0 & \frac{-1}{2} & \frac{-13}{2} & | & \frac{-41}{2} \\ 0 & 0 & 13 & | & 39 \end{array}\right]$$

$$\Rightarrow 2x + 3y + 57 = 23$$

$$-\frac{y}{2} - \frac{13}{2}z = -\frac{41}{2}$$

$$13z = 39$$

$$\Rightarrow z = \frac{39}{13} = 3$$

$$\boxed{\begin{array}{c} \mathbf{z = 3} \\ \mathbf{y = 2} \\ \mathbf{x = 1} \end{array}}$$

Example 15: Solve the following equations by Gauss elimination method.

$$\mathbf{2x + 6y - z = -12}$$

$$\mathbf{5x - y - z = 11}$$

$$\mathbf{4x - y - 3z = 10}$$

Solution: Step 1

The system of equations are rewritten

$$\begin{bmatrix} 2 & 6 & -1 \\ 5 & -1 & 1 \\ 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{vmatrix} \\ \\ \\ \end{vmatrix} \begin{bmatrix} -12 \\ 11 \\ 10 \end{bmatrix}$$

Step 2: Reducing the augmented matrix into upper triangle matrix

$$\begin{bmatrix} 2 & 6 & -1 & -12 \\ 5 & -1 & 1 & 11 \\ 4 & -1 & 3 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & -1 & | & -12 \\ 0 & -16 & +\frac{7}{2} & | & 41 \\ 0 & 0 & \frac{69}{32} & | & \frac{11}{16} \end{bmatrix} \begin{array}{l} R_2 = R_2 + \left(\frac{-5}{2}\right)R_1 \\ R_3 = R_3 + \left(\frac{-4}{2}\right)R_1 \end{array}$$

$$\begin{bmatrix} 2 & 6 & -1 & -12 \\ 0 & -16 & +\frac{7}{2} & 41 \\ 0 & 0 & \frac{69}{32} & \frac{11}{16} \end{bmatrix}$$

$$2x + 6y - z = -12$$

$$-16y + 7/2\, z = 41$$

$$\frac{69}{32} z = \frac{11}{16}$$

$$\mathbf{z = 0.31884}$$

$$\mathbf{y = -2.4928}$$

$$\mathbf{x = 1.4784}$$

Example 16: Solve the following system by Gauss Jorden method.

$$\mathbf{2x + 3y + 5z = 23}$$

$$\mathbf{3x + 4y + z = 14}$$

$$\mathbf{6x + 7y + 2z = 26}$$

Solution

Step 1: Write the system in matrix form

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 1 \\ 6 & 7 & 2 \end{pmatrix} \begin{pmatrix} x \\ x \\ x \end{pmatrix} = \begin{pmatrix} 23 \\ 14 \\ 26 \end{pmatrix}$$

Step 2: Reducing the augmented matrix into diogonal form

$$\begin{bmatrix} 2 & 3 & 5 & 23 \\ 3 & 4 & 1 & 14 \\ 6 & 7 & 2 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} (2) & 3 & 5 & 23 \\ 0 & \frac{-1}{2} & \frac{-13}{2} & -\frac{41}{2} \\ 0 & -2 & -13 & -43 \end{bmatrix} \begin{matrix} R_2 = R_2 + \left(\frac{-3}{2}\right) R_1 \\ R_3 = R_3 + \left(\frac{-6}{2}\right) R_1 \\ R_1 = R_1 + \left(\frac{-3}{-112}\right) R_2 \end{matrix}$$

$$= \begin{bmatrix} 2 & 0 & -34 & -100 \\ 0 & \left(\frac{-1}{2}\right) & \frac{-13}{2} & \frac{-41}{2} \\ 0 & -2 & -13 & -43 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -34 & -100 \\ 0 & \left(\frac{-1}{2}\right) & \frac{-13}{2} & \frac{-41}{2} \\ 0 & 0 & (13) & 39 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -34 & -100 \\ 0 & \left(\frac{-1}{2}\right) & 0 & -1 \\ 0 & 0 & 13 & 39 \end{bmatrix}$$

$$\Rightarrow 2x = 2 \quad \Rightarrow \boxed{\boldsymbol{x = 1}}$$

$$-1/2y = -1 \quad \Rightarrow \boxed{\boldsymbol{y = 2}}$$

$$13z = 39 \quad \Rightarrow \boxed{\boldsymbol{z = 3}}$$

EXERCISES

1. $x + 2y + z = 8$
$2x + 3y + 4z = 20$
$4x + 3y + 2z = 16$ [**Ans:** $x = 1,\ y = 2,\ z = 3$]

2. $x + \frac{y}{2} + \frac{z}{3} = 1$
$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 0$
$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 0$ [**Ans:** $x = 9,\ y = -36,\ z = 30$]

3. $3x + y + 2z = 3$
$2x - 3y - z = -3$
$x + 2y + z = 4$ [**Ans:** $x = 1,\ y = 2,\ z = -1$]

4. $2x_1 + 3x_2 + x_3 = 9$
$x_1 + 2x_2 + 3x_3 = 6$
$3x_1 + x_2 + 2x_3 = 8$ [**Ans:** $x_1 = 35/18,\ x_2 = 29/18,\ x_3 = 5/18$]

5. $A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} \frac{7}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{-3}{2} & 0 & \frac{+1}{2} \\ \frac{11}{10} & \frac{-1}{5} & \frac{-1}{10} \end{pmatrix}$

6. $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 6 & 1 \\ 3 & 3 & 2 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{-7}{12} & \frac{2}{3} \\ \frac{-1}{4} & \frac{5}{12} & \frac{-1}{3} \\ \frac{-3}{4} & \frac{+1}{4} & 0 \end{pmatrix}$

7. $A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} \frac{5}{14} & \frac{-1}{14} & \frac{-1}{2} \\ \frac{-1}{14} & \frac{3}{14} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

8. $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ $\qquad A^{-1} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} & \frac{2}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{6}{5} & \frac{-4}{5} & \frac{2}{5} \\ \frac{-2}{5} & \frac{-4}{5} & \frac{6}{5} & \frac{-3}{5} \\ \frac{-1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$

3.4 INVERSE OF A MATRIX USING GAUSS-JORDAN METHOD

We can use Gauss-Jordan method to find the inverse of a non - singular matrix A.

Step 1: Form the augmented matrix (A, I) where I is the identity matrix of the same order as A.

Step 2: Transform (A, I) into the form (I, B) using elementary row operations

Step 3: Then $A^{-1} = B$.

WORKED EXAMPLES

Example 1: Find the inverse of the given matrix by Gauss-Jordan method $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} \end{bmatrix}$

[AU, May/June 2006, MA 1251]

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The augmented matrix is

$$(A, I) \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -4 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{5}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & -4 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{4} R_3 \\ R_2 \to R_2 + \frac{1}{4} R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{-5}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{array}\right] \text{by } R_3 \to \left(-\frac{1}{4}\right) R_3$$

$$\sim (I, A^{-1})$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{7}{4} & \frac{-5}{4} & \frac{1}{4} \\ -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 & -5 & 1 \\ -5 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Example 2: Using Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} \mathbf{8} & \mathbf{-4} & \mathbf{0} \\ \mathbf{-4} & \mathbf{8} & \mathbf{-4} \\ \mathbf{0} & \mathbf{-4} & \mathbf{8} \end{bmatrix}$$

[AU, Apr/May 2008, MA 1251]

Solution:

Let $A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The augmented matrix is

$$[A, I] = \left[\begin{array}{ccc|ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ 0 & 6 & -4 & \frac{1}{2} & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 + \frac{1}{2} R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 8 & 0 & -\frac{8}{3} & \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 6 & -4 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{16}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{2}{3} R_2 + R_1 \\ R_3 \to \frac{2}{3} R_2 + R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 6 & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{16}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{2} R_3 + R_1 \\ R_2 \to \frac{3}{4} R_3 + R_2 \end{array}$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array}\right] \begin{array}{l} R_1 \to \frac{1}{8} R_1 \\ R_2 \to \frac{1}{6} R_2 \\ R_3 \to \frac{3}{6} R_3 \end{array}$$

$$\sim [I, A^{-1}]$$

$$\therefore \; A^{-1} = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{bmatrix}$$

Example 3: Using Gauss-Jordan method, find the inverse of $\mathbf{A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}}$ *[AU, Nov/Dec 2004, MA 038]*

Solution:

The augmented matrix is

$$[A, I] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 12 & 3 & -1 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \to 2R_1 - R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 6 & 2 & 3 \\ 0 & 4 & 0 & -5 & -1 & -3 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to 2R_2 - 3R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \begin{array}{l} R_1 \to \left(\frac{1}{2}\right) R_1 \\ R_2 \to \left(\frac{1}{4}\right) R_2 \\ R_3 \to \left(-\frac{1}{4}\right) R_3 \end{array}$$

$$\sim [I, A^{-1}]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Example 4: Find the inverse of $A = \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{pmatrix}$ by Gauss-Jordan method.

Solution:

Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The Augmented matrix

$$(A, I) = \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$(A, I) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_1 \to R_1 - 3R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_1 \to R_1 - 3R_3$$

Thus $(A, I) \to (I, A^{-1})$

Hence

$$A^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Example 5: Find the inverse of $A = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \end{pmatrix}$ by Gauss-Jordan method.

Solution:

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The Augmented matrix

$$(A, I) = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right) R_3 \to R_3 - 3R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right) \begin{array}{l} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 + 5R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right) R_3 \to R_3 \times (1/2)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right) \begin{array}{l} R_1 \to R_1 + R_2 \\ R_2 \to R_2 - 2R_3 \end{array}$$

Thus $(A, I) \to (I, A^{-1})$

Hence $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

3.5 ITERATIVE METHODS [INDIRECT METHODS]

These methods are used to solve a special system of linear equations in which each equation must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation. Further in each equation, the absolute value of the large coefficient of the unknown is greater than the sum of the absolute values of the other coefficients of the other unknowns. Such type of simultaneous linear equations can be solved by the following iterative methods.

(i) Gauss-Jacobi method and

(ii) Gauss-Seidel method

The system of equations:

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

can be solved by iterative method if

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

i.e. the absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX = B$ are greater than the sum of the absolute values of the other coefficients of that row i.e., the system is diagonally dominant.

3.5.1 Gauss-Jacobi method

Consider $a_1 x + b_1 y + c_1 z = d_1$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

Let $|a_1| > |b_1| + |c_1|$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

Then $x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

Use initial values $x^{(0)}, y^{(0)}, z^{(0)}$ and get $x^{(1)}, y^{(1)}, z^{(1)}$. Use these values and get $x^{(2)}, y^{(2)}, z^{(2)}$. Proceed till we get the desired accuracy.

If initial values are not known, we use 0, 0, 0

3.5.2 Gauss-Seidel method

Here we use initial values $y^{(0)}$ and $z^{(0)}$ and get $x^{(1)}$. Then use $x^{(1)}$ and $z^{(0)}$ to get $y^{(1)}$. Use $x^{(1)}$ and $y^{(1)}$ to get $z^{(1)}$

Note

1. Since the current values of the unknowns at each state of iteration are used to find the values of the unknowns, the convergence in Gauss-Seidel method is faster than that in Gauss-Jacobi method. The rate of convergence in Gauss-Seidel method is nearly two - times that of Gauss-Jacobi method.
2. Iteration method is a self - correcting method i.e, any error made in computation is corrected in the subsequent iterations.

WORKED EXAMPLES

Example 1: Solve the following system of equation by Gauss-Jacobi method.

$$\mathbf{10x + 2y + z = 9;\ x + 10y - z = -22;\ -2x + 3y + 10z = 9.}$$

Solution

Given system of equations

$$10x + 2y + z = 9$$

$$x + 10y - z = -22$$

$$-2x + 3y + 10z = 22$$

The co-efficient matrix

$$A = \begin{bmatrix} 10 & 2 & 1 \\ 1 & 10 & -1 \\ -2 & 3 & 10 \end{bmatrix}$$

As the co-efficient matrix is diagonally dominant. So we rewrite the equations as

$$x = \frac{1}{10}(9 - 2y - z) \qquad ...(1)$$

$$y = \frac{1}{10}(-22 - x + z) \qquad ...(2)$$

$$z = \frac{1}{10}(22 + 2x - 3y) \qquad ...(3)$$

We start with initial values $x = 0, y = 0, z = 0$ and iterate, using the values obtained in the previous step.

First iteration

$$x^{(1)} = \frac{1}{10}(9 - 2(0) - 0) = 0.9$$

$$y^{(1)} = \frac{1}{10}(-22 - 0 + 0) = -2.2$$

$$z^{(1)} = \frac{1}{10}(22 + 2(0) - 3(0)) = 2.22$$

We use these values in the next iteration.

Second iteration

$$x^{(2)} = \frac{1}{10}(9 - 2(-2.2) - 2.2) = 1.12$$

$$y^{(2)} = \frac{1}{10}(-22 - 0.9 - 2.2) = -2.07$$

$$z^{(2)} = \frac{1}{10}(22 + 2(0.9) - 3(0\ 2.2)) = 3.04$$

Third iteration

$$x^{(3)} = \frac{1}{10}(9 - 2(-2.07) - 3.04) = 1.01$$

$$y^{(3)} = \frac{1}{10}(-22 - 1.12 + 3.04) = -2.008$$

$$z^{(3)} = \frac{1}{10}(22 + 2(1.12) - 3(-2.07)) = 3.045$$

Fourth iteration

$$x^{(5)} = \frac{1}{10}(9 - 2(-2.008) - 3.045) = 0.9972$$

$$y^{(4)} = \frac{1}{10}(-22 - 1.01 + 3.045) = -2.9965$$

$$z^{(4)} = \frac{1}{10}(22 + 2(1.01) - 3(-2.008)) = 3.0044$$

Similarly we do further iterations till we get the desired accuracy. The values are tabulated in the following table.

Iteration	***x***	***y***	***z***
0	0	0	0
1	0.9	– 2.2	2.2
2	1.12	– 2.07	3.04
3	1.01	– 2.008	3.045
4	0.9971	– 1.9965	3.0044
5	0.9989	– 1.9993	2.9984
6	1.0000	– 2.0000	2.9996
7	1.0000	– 2.0000	3.0000

Since 6^{th} and 7^{th} Iteration values are same.

$\therefore$ The required solution is $\boxed{x = 1,\ y = -2,\ z = 3}$

Example 2: Apply Gauss - Seidel iterative method to solve the system of equations:

$$\mathbf{20x + y - 2z = 17}$$

$$\mathbf{3x + 20y - z = -18}$$

$$\mathbf{2x - 3y + 20z = 25}$$

[AU A/M 2023]

Solution:

The given system of equation is

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

The co-efficient matrix

$$A = \begin{bmatrix} 20 & 1 & -2 \\ 3 & 20 & -1 \\ 2 & -3 & 20 \end{bmatrix}$$

The given co-efficient matrix is diagonally dominant. So we write x, y, z as:

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{...(1)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{...(2)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{...(3)}$$

We take initial values as $y = 0, z = 0$

First iteration

$$x^{(1)} = \frac{1}{20}(17 - 0 + 2(0)) = 0.85$$

$$y^{(1)} = \frac{1}{20}(-18 - 3(0.85) + 0) = -1.02750$$

$$z^{(1)} = \frac{1}{20}(25 - 2(0.85) + 3(-1.02750) = 1.01088$$

Second iteration

$$x^{(2)} = \frac{1}{20}(17 - (-1.02750) + 2(1.01088)) = 1.00246$$

$$y^{(2)} = \frac{1}{20}(-18 - 3(1.00246) + 1.01088) = -0.99983$$

$$z^{(2)} = \frac{1}{20}(25 - 2(1.00246) + 3(-0.99983) = 0.99978$$

We proceed in the same manner always using the latest available values. The results are tabulated below:

Iteration	x	y	z
0	0	0	0
1	0.85	– 1.02750	1.01088
2	1.00246	– 0.99983	0.99978
3	0.99997	– 1.00001	1.00000
4	1	–1	1

Since 3^{rd} and 4^{th} iteration values are same.

Hence, the solution is $x = 1,\ y = -1, z = 1$

Example 3: Solve the system of equations by Gauss-Jacobi method and Gauss Seidel method.

$$\mathbf{27x + 6y - z = 85,\ x + y + 54z = 110,\ 6x + 15y + 2z = 72}$$

[AU N/D 2023]

Solution:

Given system of equations

$$27x + 6y - z = 85$$

$$x + y + 54z = 110$$

$$6x + 15y + 2z = 72$$

The coefficient matrix is

$$A = \begin{bmatrix} 27 & 6 & -1 \\ 1 & 1 & 54 \\ 6 & 15 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

As the coefficient matrix is not diagonally dominant. Rewrite the given system as

$$27x + 6y - z = 85 \quad ...(1)$$

$$6x + 15y + 2z = 72 \quad ...(2)$$

$$x + y + 54z = 110 \quad ...(3)$$

The co-efficient matrix is

$$A = \begin{bmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{bmatrix}$$

Now, the coefficient matrix is diagonally dominant

Solve for x, y, z

$$(1) \Rightarrow 27x + 6y - z = 85$$

$$27x = 85 - 6y + z$$

$$x = \frac{1}{27}(85 - 6y + z) \quad ...(A)$$

$$(2) \Rightarrow 6x + 15y + 2z = 72$$

$$y = \frac{1}{15}[72 - 6x - 2z] \quad ...(B)$$

$$(3) \Rightarrow x + y + 54z = 110$$

$$z = \frac{1}{54}[110 - x - y] \quad ...(C)$$

We take initial values $x = 0, \ y = 0, \ z = 0$.

Gauss Jacobi Interation Table

Iteration	$x = \frac{1}{27}(85 - 6y + z)$	$y = \frac{1}{15}[72 - 6x - 27]$	$z = \frac{1}{54}[110 - x - y]$
0	0	0	0
1	3.148	4.800	2.037
2	2.157	3.269	1.890
3	2.492	3.685	1.937
4	2.401	3.545	1.923
5	2.432	3.583	1.927
6	2.423	3.570	1.926
7	2.426	3.574	1.926
8	2.425	3.573	1.926
9	2.425	3.573	1.926

Hence the solution since 8^{th} and 9^{th} iteration values are same.

$$\mathbf{x = 2.425, \quad y = 3.573, \quad z = 1.926}$$

Note: Using the latest values.

Gauss Seidel Method Table

Iteration	$x = \frac{1}{27}[85 - 6y + z]$	$y = \frac{1}{15}[72 - 6x - 2z]$	$z = \frac{1}{54}[110 - x - y]$
0	0	0	0
1	3.148	3.541	1.913
2	2.432	3.572	1.926
3	2.426	3.573	1.926
4	2.425	3.573	1.926
5	2.425	3.573	1.926

Since 4^{th} and 5^{th} interations values are same.

Hence the solution of the equation is

$$\mathbf{x = 2.425 \quad y = 3.573 \quad z = 1.926}$$

Example 4: Solve the system of equation by Gauss Seidel Iteration Method

$$\mathbf{20x + 4y - z = 32,\ 2x + 17y + 4z = 35,\ x + 3y + 10z = 24}$$

[AU N/D 2021]

Solution:

Given system of equation

$$20x + 4y - z = 32 \quad ...(1)$$

$$2x + 17y + 4z = 35 \quad ...(2)$$

$$x + 3y + 10z = 24 \quad ...(3)$$

The coefficient matrix is

$$A = \begin{bmatrix} 20 & 4 & -1 \\ 2 & 17 & 4 \\ 1 & 3 & 10 \end{bmatrix}$$

The coefficient matrix is diagonally dominant

Solve for x, y, z

$$(1) \Rightarrow 20x + 4y - z = 32$$

$$20x = 32 - 4y + z$$

$$\boxed{x = \frac{1}{20}[32 - 4y + z]} \quad ...\ (A)$$

$$(2) \Rightarrow 2x + 17y + 4z = 35$$

$$17y = 35 - 2x - 4z$$

$$\boxed{y = \frac{1}{17}[35 - 2x - 4z]} \quad ...\ (B)$$

$$(3) \Rightarrow x + 3y + 10z = 24$$

$$10z = 24 - x - 3y$$

$$\boxed{z = \frac{1}{10}[24 - x - 3y]} \quad ...\ (C)$$

Iteration	$x = \frac{1}{20}[32 - 4y + z]$	$y = \frac{1}{17}[35 - 2x - 4z]$	$z = \frac{1}{10}[24 - x - y]$
0	0	0	0
1	1.600	1.871	1.679
2	1.210	1.510	1.816
3	1.389	1.463	1.821
4	1.397	1.466	1.820
5	1.398	1.466	1.820

Since 4^{th} and 5^{th} iteration values are same.

Hence the solution is $\boldsymbol{x = 1.398, y = 1.466, z = 1.820}$

Example 5: Solve the system by Gauss Seidel Iteration Method.

$$\boldsymbol{4x_1 + x_2 + x_3 = 6,\ x_1 + 4x_2 + x_3 = 6,\ x_1 + x_2 + 4x_2 = 6}$$

Solution:

Given system of equation

$$4x_1 + x_2 + x_3 = 6 \qquad ...(1)$$

$$x_1 + 4x_2 + x_3 = 6 \qquad ...(2)$$

$$x_1 + x_2 + 4x_3 = 6 \qquad ...(3)$$

The coefficient matrix is

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

As the coefficient of matrix is diagonally dominants

Solve for x, y, z

$$(1) \Rightarrow 4x_1 + x_2 + x_3 = 6$$

$$4x_1 = 6 - x_2 - x_3$$

$$\boxed{\boldsymbol{x_1 = \frac{1}{4}[6 - x_2 - x_3]}} \qquad ...\ (A)$$

$$(2) \Rightarrow x_1 + 4x_2 + x_3 = 6$$

$$4x_2 = 6 - x_1 - x_3$$

$$\boxed{x_2 = \frac{1}{4}[6 - x_1 - x_3]} \quad \ldots \text{(B)}$$

$$(3) \Rightarrow x_1 + x_2 + 4x_3 = 6$$

$$4x_3 = 6 - x_1 - x_2$$

$$\boxed{x_3 = \frac{1}{4}[6 - x_1 - x_2]} \quad \ldots \text{(C)}$$

Gauss Seidel Iteration table:

Iteration	$x_1 = \frac{1}{4}[6 - x_2 - x_3]$	$x_2 = \frac{1}{4}[6 - x_1 - x_2]$	$x_3 = \frac{1}{4}[6 - x_1 - x_2]$
0	0	0	0
1	1.500	1.125	0.844
2	1.008	1.037	0.989
3	0.994	1.004	1.001
4	0.999	1.000	1.000
5	1.000	1.000	1.000
6	1.000	1.000	1.000

Since 5^{th} and 6^{th} iteration values are same.

Hence the solution of the equation is

$$\boxed{x = 1, \ y = 1, \ z = 1}$$

Example 6: Solve the system by Gauss Jacobi Method
$\mathbf{20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25}$

[AU A/M 2023]

Solution:

Given system of equations

$$20x + y - 2z = 17 \quad ...(1)$$

$$3x + 20y - z = -18 \quad ...(2)$$

$$2x - 3y + 20z = 25 \quad ...(3)$$

The coefficient of matrix is

$$A = \begin{bmatrix} 20 & 1 & -2 \\ 3 & 20 & -1 \\ 2 & -3 & 20 \end{bmatrix}$$

As the coefficient of matrix is diagonally dominant solve for x, y, z

$$(1) \Rightarrow 20x + y - 2z = 17$$

$$20x = 17 - y + 2z$$

$$\boxed{x = \frac{1}{20}[17 - y + 2z]} \quad ... (A)$$

$$(2) \Rightarrow 3x + 20y - z = -18$$

$$20y = -18 - 3x + z$$

$$\boxed{y = \frac{1}{20}[-18 - 3x + z]} \quad ... (B)$$

$$(3) \Rightarrow 2x - 3y + 20z = 25$$

$$20z = 25 - 2x - 3y$$

$$\boxed{z = \frac{1}{20}[25 - 2x - 3y]} \quad ... (C)$$

Note: Using the values obtained in the previous step.

Gauss Jacobi iteration table

Iteration	$x = \frac{1}{20}[17 - y + 2z]$	$y = \frac{1}{20}[-18 - 3x + z]$	$z = \frac{1}{20}[25 - 2x - 3y]$
0	0	0	0
1	0.850	– 0.900	1.250
2	1.020	– 0.965	1.300
3	1.028	– 0.988	1.293
4	1.029	– 0.990	1.296
5	1.029	– 0.990	1.296
6	1.029	– 0.990	1.296

Since 5^{th} and 6^{th} iteration valus are same.

Hence the solution of the equation is

$$\mathbf{x = 1.009, \quad y = -0.990, \quad z = 1.296}$$

Example 7: Solve the system of equation by Gauss Seidel Iteration method

$$\mathbf{2x + y + 4z = 12;\ 8x - 3y + 2z = 20;\ 4x + 11y - z = 33.}$$

Solution:

Given system of equations

$$2x + y + 4z = 12$$

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

The coefficient of matrix is

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}$$

This is not diagonally dominant. Since rearrange the given system

$$8x - 3y + 2z = 20 \quad ...(1)$$

$$4x + 11y - z = 33 \quad ...(2)$$

$$2x + y + 4z = 12 \quad ...(3)$$

The co-efficient matrix

$$A = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 2 & 1 & 4 \end{bmatrix}$$

Now the co-efficient matrix is diagonally dominant

Solve for x, y, z

$$(1) \Rightarrow 8x - 3y + 2z = 20$$

$$8x = 20 + 3y - 2z$$

$$\boxed{\boldsymbol{x = \frac{1}{8}[20 + 3y - 2z]}} \quad ... (A)$$

$$(2) \Rightarrow 4x + 11y - z = 33$$

$$4x = 33 - 11y + z$$

$$\boxed{\boldsymbol{y = \frac{1}{11}[33 - 4x + z]}} \quad ... (B)$$

$$(3) \Rightarrow 2x + y + 4z = 12$$

$$4z = 12 - 2x - y$$

$$\boxed{z = \boldsymbol{\frac{1}{4}(12 - 2x - y)}}$$

Gauss Siedel Iteration table:

Iteration	$x = \frac{1}{8}[20 + 3y - 2z]$	$y = \frac{1}{11}[33 - 4z + z]$	$z = \frac{1}{4}[12 - 2x - y]$
0	0	0	0
1	2.500	2.091	1.182
2	3.057	2.070	0.954
3	3.038	2.077	0.962
4	3.038	2.077	0.962

Since 3rd and 4th iteration values are same.

Hence the solution of the equation is

$$\mathbf{x = 3.038 \quad y = 2.079 \quad z = 0.962}$$

Gauss Seidel Iteration table

Iteration	$x = \frac{1}{8}[20 + 3y - 2z]$	$y = \frac{1}{11}[33 - 4x + z]$	$z = \frac{1}{4}[12 - 2x - y]$
0	0	0	0
1	2.500	2.091	1.227
2	2.977	2.029	1.004
3	3.010	1.997	0.996
4	8.000	2.000	1.000

Hence the solution of the equation is

$$\boxed{x = 3, \; y = 2, \; z = 1}$$

Example 8: Using Gauss - Seidel method, solve the following system. Start with $x = 1, y = -2, z = 3$

$$\mathbf{41x - 2y + 3z = 65.46}$$

$$\mathbf{x - 27y + 2z = 71.31}$$

$$\mathbf{x + 3y + 52z = 173.61}$$

[AU, Apr/May 2004, MA 038]

Solution:

Given system of equation

$$41x - 2y + 3z = 65.46 \quad \text{...(1)}$$

$$x - 27y + 2z = 71.31 \quad \text{...(2)}$$

$$x + 3y + 52z = 173.61 \quad \text{...(3)}$$

The co-efficient matrix

$$A = \begin{bmatrix} 41 & -2 & 3 \\ 1 & -27 & 2 \\ 1 & 3 & 52 \end{bmatrix}$$

As the co-efficient matrix is diagonally dominant.

Solving for x, y, z from these equations, we get

$$(1) \Rightarrow x = \frac{1}{41}(65.46 + 2y - 3z)$$

$$(2) \Rightarrow y = -\frac{1}{27}(71.31 - x - 2z)$$

$$(3) \Rightarrow z = \frac{1}{52}(173.61 - x - 3y)$$

We start with values $x = 1, y = -2, z = 3$ and iterate, using the latest values.

The results are tabulated below:

Iteration	x	y	z
0	1	-2	3
1	1.27951	-2.37150	3.45087
2	1.22840	-2.33999	3.45003
3	1.23000	-2.34000	3.45000
4	1.23	-2.34	3.45

Since 3rd and 4th iteration value and same.

Hence the solution is $\boxed{x = 1.23,\ y = 2.34,\ z = 3.45}$

Example 9: Solve by Gauss-Seidel method.

$$\mathbf{2x + y + 6z = 9}$$

$$\mathbf{8x + 3y + 2z = 13}$$

$$\mathbf{x + 5y + z = 7}$$

Solution:

Given system

$$2x + y + 6z = 9$$

$$8x + 3y + 2z = 13$$

$$x + 5y + z = 7$$

The co-efficient matrix

$$\begin{bmatrix} 2 & 1 & 6 \\ 8 & 3 & 2 \\ 1 & 5 & 1 \end{bmatrix}$$

Since the given system of equations is not diagonally dominant, we rewrite the equations as follows:

$$8x + 3y + 2z = 13 \quad ...(1)$$

$$x + 5y + z = 7 \quad ...(2)$$

$$2x + y + 6z = 9 \quad ...(3)$$

Hence $(1) \Rightarrow x = \frac{1}{8}(13 - 3y - 2z)$

$(2) \Rightarrow y = \frac{1}{5}(7 - x - y)$

$(3) \Rightarrow z = \frac{1}{6}(9 - 2x - y)$

We start with $x = 0,\ y = 0,\ z = 0$

Iterations	x	y	z
0	0	0	0
1	1.62500	1.07500	0.77917
2	1.02708	1.03875	0.98452
3	0.98934	1.00523	1.00268
4	0.99737	0.99999	1.00088
5	0.99978	0.99987	1.00010
6	1.00002	0.99998	1.00000

Since 5^{th} and 6^{th} Iteration value are very close.

Hence the solution is $\boxed{x = 1,\ y = 1,\ z = 1}$

Example 10: Solve the following system of equations by Gauss-Seidel method

$10x - 2y + z = 12$; $x + 9y - z = 10$; $2x - y + 11z = 20$

Solution:

The given system of equation is

$$10x - 2y + z = 12 \quad ...(1)$$

$$x + 9y - z = 10 \quad ...(2)$$

$$2x - y + 11z = 20 \quad ...(3)$$

The co-efficient matrix

$$\begin{bmatrix} 10 & -2 & 1 \\ 1 & 9 & -1 \\ 2 & -1 & 11 \end{bmatrix}$$

The co-efficient matrix is diagonally dominant.

Solving for x, y, z, we get

$$(1) \Rightarrow x = \frac{1}{10}[12 + 2y - z]$$

$$(2) \Rightarrow y = \frac{1}{9}[10 - x + z]$$

$$(3) \Rightarrow z = \frac{1}{11}[20 - 2x + y]$$

We start with initial values $(x, y, z) = (0, 0, 0)$

The iteration values are tabulated as follows:

Iteration	x	y	z
0	0	0	0
1	1.2	0.978	1.689
2	1.227	1.162	1.701
3	1.262	1.160	1.694
4	1.263	1.160	1.694
5	1.263	1.160	1.694

Since 4^{th} and 5^{th} Iteration values are same.

$\therefore$ The solution is $\boxed{\mathbf{x = 1.263\ ;\ y = 1.16\ ;\ z = 1.694}}$

Example 11: Solve the following system of equations by Gauss-Seidel method

$$\mathbf{3x - 13y - 3z = 49\ ;\ 5x - 6y + 17z = 45\ ;\ 11x + 2y - 2z = -31}$$

Solution:

Given system of equations

$$3x - 13y - 3z = 49$$

$$5x - 6y + 17z = 45$$

$$11x + 2y - 2z = -31$$

The co-efficient matrix

$$\begin{bmatrix} 3 & -13 & -3 \\ 5 & -6 & 17 \\ 11 & 2 & -2 \end{bmatrix}$$

Since the co-efficient matrix is not diagonally dominant.

Rearranging the given system, we have

$$11x + 2y - 2z = -31 \quad ...(1)$$

$$3x - 13y - 3z = 49 \quad ...(2)$$

$$5x - 6y + 17z = 45 \quad ...(3)$$

The coefficient matrix of the rearranged system is diagonally dominant. Solving for x, y, z, we get

$$(1) \Rightarrow x = \frac{1}{11}[-31 - 2y + 2z]$$

$$(2) \Rightarrow y = \frac{-1}{13}[49 - 3x + 3z]$$

$$(3) \Rightarrow z = \frac{1}{17}[45 - 5x + 6y]$$

We start with initial values $(x, y, z) = (0, 0, 0)$.

The iteration values are tabulated as follows:

Iteration	x	y	z
0	0	0	0
1	− 2.818	− 4.419	1.916
2	− 1.666	− 4.596	1.515
3	− 1.707	− 4.513	1.556
4	− 1.713	− 4.524	1.555
5	− 1.713	− 4.523	1.555
6	− 1.713	− 4.523	1.555

Since 5th and 6th Iteration values are same.

$\therefore$ The solution is $x = -1.713$; $y = -4.523$; $z = 1.555$.

Example 12: Solve the following equations

$$\mathbf{13x_1 + 5x_2 - 3x_3 + x_4 = 18 \ ; \ 2x_1 + 12x_2 + x_3 - 4x_4 = 30 \ ;}$$

$$\mathbf{3x_1 - 4x_2 + 10x_3 + x_4 = 29 \ ; \ 2x_1 + x_2 - 3x_3 + 9x_4 = 31}$$

by Gauss-Seidel Method.

Solution:

Given system of equations

$$13x_1 + 15x_2 - 3x_3 + x_4 = 18 \quad \text{...(1)}$$

$$24x_1 + 12x_2 + x_3 - 4x_4 = 30 \quad \text{...(2)}$$

$$3x_1 - 4x_2 + 10x_3 + x_4 = 29 \quad \text{...(3)}$$

$$2x_1 + x_2 - 3x_3 + 9x_4 = 31 \quad \text{...(4)}$$

Given system of equations is diagonally dominant.

Solving for x_1, x_2, x_3, x_4, we have

$$(1) \Rightarrow x_1 = \frac{1}{13}\,[18 - 5x_2 + 3x_3 - x_4]$$

$$(2) \Rightarrow x_2 = \frac{1}{12}[30 - 2x_1 - x_3 + 4x_4]$$

$$(3) \Rightarrow x_3 = \frac{1}{10}[29 - 3x_1 + 4x_2 - x_4]$$

$$(4) \Rightarrow x_4 = \frac{1}{9}[31 - 2x_1 - x_2 + 3x_3]$$

We start with initial values $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$.

The iteration values are tabulated as follows

We Iterate using the latest available values.

Iteration	x_1	x_2	x_3	x_4
0	0	0	0	0
1	1.385	1.269	2.992	3.993
2	1.279	3.369	3.465	3.941
3	0.585	3.427	3.701	4.167
4	0.600	3.481	3.696	4.156
5	0.579	3.481	3.703	4.163
6	0.58	3.482	3.703	4.163
7	0.58	3.482	3.703	4.163

Since 6th and 7th Iteration values are same.

∴ The solution is

$$\boldsymbol{x_1 = 0.58,\ x_2 = 3.482,\ x_3 = 3.703,\ x_4 = 4.163.}$$

Example 13: Solve by Gauss-Jacobi method and Gauss-Seidel method.

$$\mathbf{10x_1 + x_2 + x_3 + x_4 = 21.09}$$

$$\mathbf{x_1 + 10x_2 + x_3 + x_4 = 31.08}$$

$$\mathbf{x_1 + x_2 + 10x_3 + x_4 = 41.07}$$

$$\mathbf{x_1 + x_2 + x_3 + 10x_4 = 51.06}$$

Solution:

Given system of equations

$$10x_1 + x_2 + x_3 + x_4 = 21.09 \quad ...(1)$$

$$x_1 + 10x_2 + x_3 + x_4 = 31.08 \quad ...(2)$$

$$x_1 + x_2 + 10x_3 + x_4 = 41.07 \quad ...(3)$$

$$x_1 + x_2 + x_3 + 10x_4 = 51.06 \quad ...(4)$$

The given system of equations is diagonally dominant. So we rewrite the equation as

$$(1) \Rightarrow x_1 = \frac{1}{10}(21.09 - x_2 - x_3 - x_4)$$

$$(2) \Rightarrow x_2 = \frac{1}{10}(31.08 - x_1 - x_3 - x_4)$$

$$(3) \Rightarrow x_3 = \frac{1}{10}(41.07 - x_1 - x_2 - x_4)$$

$$(4) \Rightarrow x_4 = \frac{1}{10}(51.06 - x_1 - x_2 - x_3)$$

We start with initial values $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$

Gauss-Jacobi method

We iterate using the values obtained in the previous step.

Iteration	x_1	x_2	x_3	x_4
0	0	0	0	0
1	2.109	3.108	4.107	5.106
2	0.8769	1.9758	3.0747	4.5133
3	1.1526	2.2615	3.3704	4.5133
4	1.0945	2.2044	3.3143	4.4276
5	1.1144	2.2244	3.3344	4.4447
6	1.1087	2.2187	3.3287	4.4387
7	1.1104	2.2204	3.3304	4.4404
8	1.1099	2.2199	3.3299	4.4399
9	1.1100	2.2200	3.3300	4.4400
10	1.11	2.22	3.33	4.44

Since 9^{th} and 10^{th} Iteration values are same.

Hence the solution is $\boxed{x_1 = 1.11,\ x_2 = 2.22,\ x_3 = 3.33,\ x_4 = 4.44}$

Gauss-Seidel method

We iterate using the latest available values.

Iteration	x_1	x_2	x_3	x_4
0	0	0	0	0
1	2.109	2.8971	3.6064	4.2448
2	1.0342	2.2195	3.3572	4.4450
3	1.1069	2.2171	3.3301	4.4406
4	1.1102	2.2199	3.3299	4.4400
5	1.11	2.22	3.33	4.44
6	1.11	2.22	3.33	4.44

Since 5^{th} and 6^{th} Iteration values are same.

Hence the solution is $\boxed{x_1 = 1.11, x_2 = 2.22, x_3 = 3.33, x_4 = 4.44}$

Example 14: Solve the following system of equations by Gauss-Jacobi method

$$4x_1 + x_2 + x_3 = 4$$

$$x_1 + 4x_2 - 2x_3 = 4$$

$$3x_1 + 2x_2 - 4x_3 = 6$$

Solution : Step 1:

Since initial approximation is not given assume

$$x_1^0 = x_2^0 = x_3^0 = 0$$

The equations satisfy diagonally dominant condition.

Step 2

$$x_1 = \frac{1}{4}(4 - x_2 - x_3)$$

$$x_2 = \frac{1}{4}(4 - x_1 + 2x_3)$$

$$x_3 = \frac{-1}{4}(6 - 3x_1 - 2x_2)$$

I Iteration

$$x_1^{(1)} = \frac{1}{4}(4 - x_2^0 - x_3^0) = \frac{1}{4}(4 - 0 - 0) = 1$$

$$x_2^{(1)} = \frac{1}{4}(4 - x_1^0 - 2x_3^0) = \frac{1}{4}(4 - 0 - 0) = 1$$

$$x_3^{(1)} = \frac{-1}{4}(6 - 3x_1^0 - 2x_2^0) = \frac{-1}{4}(6 - 0 - 0) = \frac{-3}{2}$$

II Iteration

$$x_1^{(2)} = \frac{1}{4}(4 - x_2^{(1)} - x_3^{(1)}) = \frac{1}{4}\left(4 - 1 - \left(\frac{-3}{2}\right)\right) = \frac{9}{8} = 1.125$$

$$x_2^{(2)} = \frac{1}{4}(4 - x_1^{(1)} + 2x_3^{(1)}) = \frac{1}{4}\left(4 - 1 - 2\left(\frac{-3}{2}\right)\right) = \frac{3}{2} = 1.5$$

$$x_3^{(2)} = \frac{-1}{4}\,6 - 3x_1^{(1)} - 2x_2^{(1)} = \frac{1}{4}(6 - 3(1) - 2(1)) = \frac{1}{4} = -0.25$$

III Iteration

$$x_1^{(3)} = \frac{1}{4}(4 - 1.5 + 0.25) = 0.6875$$

$$x_2^{(3)} = \frac{1}{4}(4 - 1.125 - 2(-0.25) = 0.84375$$

$$x_3^{(3)} = \frac{-1}{4}(6 - 3(1.125) - 2(1.5) = 0.09375$$

IV Iteration

$$x_1^{(4)} = \frac{1}{4}(4 - 0.84375 - 0.09375) = 0.76563$$

$$x_2^{(4)} = \frac{1}{4}(4 - 0.6875 - 2(0.09375) = 0.78125$$

$$x_3^{(4)} = \frac{-1}{4}(6 - 3(0.6875) - 2(0.84375) = -0.5625$$

V Iteration

$$x_1^{(5)} = \frac{1}{4}(4 - 0.78125 + 0.5625) = 0.94531$$

$$x_2^{(5)} = \frac{1}{4}(4 - 0.76563 - 2(-0.5625)) = 1.08984$$

$$x_3^{(5)} = \frac{-1}{4}(6 - 3 - (0.76563) - 2(0.78125)) = -0.53515$$

VI Iteration

$$x_1^{(6)} = \frac{1}{4}(4 - 1.08984 + 0.53515) = 0.86133$$

$$x_2^{(6)} = \frac{1}{4}(4 - 0.94531 + 2(-0.53515)) = 0.49610$$

$$x_3^{(6)} = \frac{-1}{4}(6 - 3(0.94531) - 2(1.08984)) = -0.24610$$

VII Iteration

$$x_1^{(7)} = \frac{1}{4}(4 - 0.49610 + 0.24610)) = 0.9375$$

$$x_2^{(7)} = \frac{1}{4}(4 - 0.86133 + 2(-0.24610)) = 0.66162$$

$$x_3^{(7)} = \frac{-1}{4}(6 - 3(0.86133) - 2(0.49610)) = -0.60595$$

VIII Iteration

$$x_1^{(8)} = \frac{1}{4}(4 - 0.66162 + 0.60595)) = 0.98608$$

$$x_2^{(8)} = \frac{1}{4}(4 - 0.9375 + 2(-0.60595)) = 0.46265$$

$$x_3^{(8)} = \frac{-1}{4}(6 - 3(0.9375) - 2(0.66162)) = -0.46607$$

IX Iteration

$$x_1 = \frac{1}{4}(4 - 0.46265 + (0.46607)) = 1.00086$$

$$x_2 = \frac{1}{4}(4 - 0.98608 + 2(-0.46607)) = 0.52045$$

$$x_3 = \frac{-1}{4}(6 - 3(0.98608) - 2(0.46265)) = -0.5290$$

Hence

$$\boxed{\begin{aligned} x_1 &= 1.00086 \\ x_2 &= 0.52045 \\ x_3 &= -0.5290 \end{aligned}}$$

Example 15: Solve by Gauss - Seidel method

$$8x + y + z = 8$$
$$2x + 4y + z = 4$$
$$x + 3y + 5z = 5$$

Solution:

Consider the system

$$8x + y + z = 8$$

$$2x + 4y + z = 4$$

$$x + 3y + 5z = 5$$

The equations are diagonally dominant. Initial approximation $x^{(0)} = y^{(0)} = z^{(0)} = 0.$

Step 1: Iteration 1

$$x^{(1)} = \frac{1}{8}[8 - y^{(0)} - z^{(0)}] = \frac{1}{8}[8 - 0 - 0] = 1$$

$$y^{(1)} = \frac{1}{4}[4 - 2x^{(1)} - z^{(0)}] = \frac{1}{8}[4 - 2(1) - 0] = 0.25$$

$$z^{(1)} = \frac{1}{5}[5 - 2x^{(1)} - 3y^{(1)}] = \frac{1}{5}[5 - 1 - 3(0.25)] = 0.65$$

Iteration 2

$$x^{(2)} = \frac{1}{8}[8 - y^{(1)} - z^{(1)}] = \frac{1}{8}[8 - 0.25 - 0.65] = 0.8875$$

$$y^{(2)} = \frac{1}{4}[4 - 2x^{(2)} - z^{(1)}] = \frac{1}{4}[4 - 2(0.8875) - 0.65] = 0.39375$$

$$z^{(2)} = \frac{1}{5}[5 - x^{(2)} - 3y^{(2)}] = \frac{1}{5}[5 - 0.8875 - 3(.39875)] = 0.58625$$

Iteration 3

$$x^{(3)} = \frac{1}{8}[8 - y^{(2)} - z^{(2)}] = \frac{1}{8}[8 - 0.39375 - 0.58625] = 0.8775$$

$$y^{(3)} = \frac{1}{4}[4 - 2x^{(3)} - z^{(2)}] = \frac{1}{4}[4 - 2(0.8775) - 0.58625] = 0.41469$$

$$z^{(3)} = \frac{1}{5}[5 - x^{(3)} - 3y^{(3)}] = \frac{1}{5}[5 - 0.8775 - 3(0.41469)] = 0.57569$$

Iteration 4

$$x^{(4)} = \frac{1}{8}[8 - y^{(3)} - z^{(3)}] = \frac{1}{8}[8 - 0.41469 - 0.57569] = 0.87620$$

$$y^{(4)} = \frac{1}{4}[4 - 2x^{(4)} - z^{(3)}] = \frac{1}{4}[4 - 2(0.87620) - 0.57569] = 0.41798$$

$$z^{(4)} = \frac{1}{5}[5 - x^{(4)} - 3y^{(4)}] = \frac{1}{5}[5 - 0.87620 - 3(0.41798)] = 0.57397$$

Iteration 5

$$x^{(5)} = \frac{1}{8}[8 - y^{(4)} - z^{(4)}] = \frac{1}{8}[8 - 0.41798 - 0.57397] = 0.87601$$

$$y^{(5)} = \frac{1}{4}[4 - 2x^{(5)} - z^{(4)}] = \frac{1}{4}[4 - 2(0.87601) - 0.57397] = 0.41850$$

$$z^{(5)} = \frac{1}{5}[5 - x^{(5)} - 3y^{(5)}] = \frac{1}{5}[5 - 0.87601 - 3(.41850)] = 0.57370$$

Iteration 6

$$x^{(6)} = \frac{1}{8}[8 - y^{(5)} - z^{(5)}] = \frac{1}{8}[8 - 0.41850 - 0.57370] = 0.87598$$

$$y^{(6)} = \frac{1}{4}[4 - 2x^{(6)} - z^{(5)}] = \frac{1}{4}[4 - 2(0.87598) - 0.5737] = 0.41859$$

$$z^{(6)} = \frac{1}{5}[5 - x^{(6)} - 3y^{(6)}] = \frac{1}{5}[5 - 0.87598 - 3(.41859)] = 0.57365$$

Iteration 7

$$x^{(7)} = \frac{1}{8}[8 - y^{(6)} - z^{(6)}] = \frac{1}{8}[8 - 0.41859 - 0.57365] = 0.87597$$

$$y^{(7)} = \frac{1}{4}[4 - 2x^{(7)} - z^{(6)}] = \frac{1}{4}[4 - 2(0.87597) - 0.57365] = 0.41860$$

$$z^{(7)} = \frac{1}{5}[5 - x^{(7)} - 3y^{(7)}] = \frac{1}{5}[5 - 0.87597 - 3(0.41860)] = 0.57365$$

$$\boxed{\begin{aligned} x &= \mathbf{0.87597} \\ y &= \mathbf{0.41860} \\ z &= \mathbf{0.57365} \end{aligned}}$$

EXERCISES

Solve the following set of equations by Gauss-Seidels Jacobi Method

1. $2x_1 + x_2 = 1$

$x_1 + 2x_2 + x_3 = 2$

$x_2 + x_3 = 4$ [**Ans.** $x_1 = 2.616,\ x_2 = -4.424,\ x_3 = 8.424$]

2. $2x_1 - x_2 = 7$

$-x_1 + 2x_2 - x_3 = 1$

$-x_2 + 2x_3 = 1$ [**Ans.** $x_1 = 5.3115,\ x_2 = -4.3125,\ x_3 = 2.6563$]

3. $4x_1 + x_2 + 2x_3 = 4$

$3x_1 + 5x_2 + x_3 = 7$

$x_1 + x_2 + 3x_3 = 3$ [**Ans.** $x_1 = \frac{13}{25},\ x_2 = \frac{124}{125},\ x_3 = \frac{62}{125}$

4. $10x_1 - 2x_2 - x_3 - x_4 = 3$

$-2x_1 + 10x_2 - x_3 - x_4 = 15$

$-x_1 - x_2 + 10x_3 - 2x_4 = 27$

$x_1 - x_2 - 2x_3 + 10x_4 = -9$

[**Ans.** $x_1 = 1,\ x_2 = +2,\ x_3 = 3,\ x_4 = 0$] ($7^{th}$ iteration)

5. $$\begin{pmatrix} 17 & 65 & -13 & 50 \\ 12 & 16 & 37 & 18 \\ 56 & 23 & 11 & -19 \\ 3 & -5 & 47 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 84 \\ 25 \\ 36 \\ 18 \end{pmatrix}$$

[**Ans.** $x_1 = 4.84,\ x_2 = -4.70,\ x_3 = -1.64,\ x_4 = 5.72$]

6. $8x - 3y + 2z = 20$

$4x + 11y - z = 33$

$6x + 3y + 12z = 35$ [**Ans.** $x = 3.0168,\ y = 1.9859,\ z = 0.9118$]

3.6 EIGEN VALUES OF A MATRIX BY POWER METHOD

Eigen Values and Eigen Vectors

Let A be any square matrix of order n then for any scalar λ, we can form a matrix $(A - \lambda I)$ where I is the n^{th} order unit matrix.

The determined of this matrix equated to zero is called the characteristic equation of A.

The characteristic equation of A is

$$|A - \lambda I| = 0$$

For 3×3 matrix $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$. Clearly the characteristic polynomial is degree n in λ having n roots $\lambda_1, \lambda_2, \ldots \lambda_n$. These values $\lambda_1, \lambda_2 \ldots \lambda_n$ are called Eigen values of the given matrix A.

For each of there Eigen values the system of Equation $(A - \lambda I) X = 0$ has a non-trivial solution for the vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$ this solution is called latent vector (or) Eigen vector corresponding to the Eigen value λ.

Note

When we can use power method? If n is too large, It is difficult to find the exact roots of the characteristic equation and hence eigen values are difficult to find. The above is the reason for moving to power method & Jacobi rotation method. In this situation we can use some numerical methods for such cases.

The list of method (numerical) available for finding Eigen values and Eigen vectors are

1. Power method
2. Jacobi iteration method

Power method

The power method is used to find the numerically largest Eigen value of a square matrix (also called dominant Eigen value). Let $\lambda_1, \lambda_2 \ldots \lambda_n$ be the Eigen value of the matrix A and let us take λ, be the largest (dominant) Eigen value then

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$$

their corresponding eigen vectors are $x_0, x_1, \ldots x_n$.

Note

1. If the Eigen values of A and $-3, 1, 2$ then -3 is the dominant Eigen value.
2. Power method will work satisfactorily only if matrix A has a dominant Eigen value.

WORKED EXAMPLES

Example 1. Find the largest eigen value of the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ by power method correct to 2 decimal places choose $[1, 1]^T$ as a initial Eigen vector.

Solution:

Given Matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$

Choose the initial eigen vector $= [1, 1]^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = X_0$

To find Dominant eigen value and the corresponding eigen vector.

$$AX_0 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+1 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.80 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.80 \end{bmatrix} = \begin{bmatrix} 4+0.80 \\ 1+2.40 \end{bmatrix} = \begin{bmatrix} 4.80 \\ 3.40 \end{bmatrix} = 4.80 \begin{bmatrix} 1 \\ 0.71 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.71 \end{bmatrix} = \begin{bmatrix} 4.71 \\ 3.13 \end{bmatrix} = 4.71 \begin{bmatrix} 1 \\ 0.67 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 4.67 \\ 3.01 \end{bmatrix} = 4.67\begin{bmatrix} 1 \\ 0.65 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.65 \end{bmatrix} = \begin{bmatrix} 4.65 \\ 2.95 \end{bmatrix} = 4.65\begin{bmatrix} 1 \\ 0.63 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4.63 \\ 2.89 \end{bmatrix} = 4.63\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \begin{bmatrix} 4.62 \\ 2.86 \end{bmatrix} = 4.62\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \lambda_6 X_6$$

The dominant Eigen value = 4.62

The corresponding Eigen vector $= \begin{bmatrix} \mathbf{1} \\ \mathbf{0.62} \end{bmatrix}$

Example 2: Find the dominant eigen value of the matrix $\begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}$ **by power method, correct to 2 decimal places choose** $[1, 0]^T$ **as a initial eigen vector.**

Solution:

Given matrix $A = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}$

Choose the initial Eigen vector $[1, 0]^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = X_0$ to find dominant eigen values corresponding eigen vector.

$$AX_0 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4+0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} = 4\begin{bmatrix} -1 \\ 40.25 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 4-1.25 \\ -1+0.50 \end{bmatrix} = \begin{bmatrix} 2.75 \\ -0.50 \end{bmatrix}$$

$$= 2.75\begin{bmatrix} 1 \\ -0.10 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.18 \end{bmatrix} = \begin{bmatrix} -4+0.91 \\ 1-0.36 \end{bmatrix} = \begin{bmatrix} -3.09 \\ 0.64 \end{bmatrix}$$

$$= 3.09\begin{bmatrix} -1 \\ 0.01 \end{bmatrix} = \lambda_3 X_4$$

$$AX_3 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0.21 \end{bmatrix} = \begin{bmatrix} 4 - 1.04 \\ -1 + 0.42 \end{bmatrix} = \begin{bmatrix} 2.96 \\ -0.58 \end{bmatrix}$$

$$= 2.96 \begin{bmatrix} 1 \\ -0.20 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.20 \end{bmatrix} = \begin{bmatrix} -4 + 0.98 \\ 1 - 0.40 \end{bmatrix} = \begin{bmatrix} -3.02 \\ 0.60 \end{bmatrix}$$

$$= 3.02 \begin{bmatrix} -1 \\ 0.20 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0.20 \end{bmatrix} = \begin{bmatrix} 4 - 1 \\ -1 + 0.40 \end{bmatrix} = \begin{bmatrix} 3 \\ -0.60 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ -0.20 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.20 \end{bmatrix} = \begin{bmatrix} -4 + 1 \\ 1 - 0.40 \end{bmatrix} = \begin{bmatrix} -3 \\ 0.60 \end{bmatrix}$$

$$= 3 \begin{bmatrix} -1 \\ 0.20 \end{bmatrix} = \lambda_2 X_3$$

The dominant eigen value = 3

The corresponding eigen vector = $\begin{bmatrix} \mathbf{-1} \\ \mathbf{0.20} \end{bmatrix}$

Example 3: Find the dominant eigen value of the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ **by power method, correct to 2 decimal places choose** $[\mathbf{1, 1}]^T$ **as a initial eigen vector.**

Solution:

Given Matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Choose the initial Eigen vector $= [1, 1]^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To find dominant eigen value and corresponding eigen vector.

$$AX_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_2 X_2$$

The dominant eigen value $= 2$

The corresponding eigen vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example 4: Apply power method to find the dominant eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix}$

Solution

Given $A = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix}$

Let $X_0 = (1, 1, 1)$ then

$$AX_0 = \begin{bmatrix} 9 & 1 & 3 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 17 \end{bmatrix} = 18 \begin{bmatrix} 1.00 \\ 0.67 \\ 0.94 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.67 \\ 0.94 \end{bmatrix} = \begin{bmatrix} 17.19 \\ 10.62 \\ 14.15 \end{bmatrix} = 17.19 \begin{bmatrix} 1.00 \\ 0.62 \\ 0.32 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.62 \\ 0.82 \end{bmatrix} = \begin{bmatrix} 16.18 \\ 10.30 \\ 10.72 \end{bmatrix} = 16.18 \begin{bmatrix} 1.00 \\ 0.64 \\ 0.79 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.64 \\ 0.79 \end{bmatrix} = \begin{bmatrix} 15.96 \\ 10.35 \\ 12.59 \end{bmatrix} = 15.96 \begin{bmatrix} 1.00 \\ 0.65 \\ 0.79 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 9 & 1 & 8 \\ 7 & 4 & 1 \\ 1 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.65 \\ 0.79 \end{bmatrix} = \begin{bmatrix} 15.97 \\ 10.39 \\ 12.66 \end{bmatrix} = 1.597 \begin{bmatrix} 1.00 \\ 0.65 \\ 0.79 \end{bmatrix} = \lambda_5 X_5$$

Hence the dominant eigen value is 15.97 and the corresponding eigen vector is $[1.000, 0.65, 0.39]^T$

Example 5: Find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} \mathbf{4} & \mathbf{6} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} & \mathbf{3} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

Solution:

$$\text{Given } A = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix}$$

Let $X_0 = (1, 1, 1)^T$ then

$$AX_0 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix} = 10 \begin{bmatrix} 1.0 \\ 0.8 \\ 0.5 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.8 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8.8 \\ 5.5 \\ 3.5 \end{bmatrix} = 8.8 \begin{bmatrix} 1.00 \\ 0.62 \\ 0.40 \end{bmatrix} = \lambda_2 X_2$$

$$AX_1 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.8 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8.8 \\ 5.5 \\ 3.5 \end{bmatrix} = 8.8 \begin{bmatrix} 1.00 \\ 0.62 \\ 0.40 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.62 \\ 0.00 \end{bmatrix} = \begin{bmatrix} 7.72 \\ 4.50 \\ 3.00 \end{bmatrix} = 7.72 \begin{bmatrix} 1.00 \\ 0.56 \\ 0.41 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.56 \\ 0.41 \end{bmatrix} = \begin{bmatrix} 7.36 \\ 4.03 \\ 3.23 \end{bmatrix} = 7.36 \begin{bmatrix} 1.00 \\ 0.55 \\ 0.44 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.55 \\ 0.49 \end{bmatrix} = \begin{bmatrix} 7.30 \\ 4.07 \\ 3.32 \end{bmatrix} = 7.30 \begin{bmatrix} 1.00 \\ 0.56 \\ 0.45 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.56 \\ 0.44 \end{bmatrix} = \begin{bmatrix} 7.36 \\ 4.15 \\ 3.35 \end{bmatrix} = 7.36 \begin{bmatrix} 1.00 \\ 0.56 \\ 0.45 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 4 & 6 & 0 \\ 0 & 5 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1.00 \\ 0.56 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 7.36 \\ 4.15 \\ 8.35 \end{bmatrix} = 7.36 \begin{bmatrix} 1.00 \\ 0.56 \\ 0.45 \end{bmatrix} = \lambda_7 X_7$$

The largest eigen value is 7.36 and corresponding eigen vector is $[1.00, 0.56, 0.45]^T$

Example 6: Determine the dominant eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ using power method.

Solution:

Let $X_0 = (1, 1, 1)^T$ be the initial vector, then

$$AX_0 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 22 \\ 5 \end{bmatrix} = 22 \begin{bmatrix} 0.227 \\ 1.000 \\ 0.227 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.227 \\ 1.000 \\ 0.227 \end{bmatrix} = \begin{bmatrix} 1.908 \\ 20.454 \\ 1.903 \end{bmatrix} = 20.454 \begin{bmatrix} 0.093 \\ 1.000 \\ 0.093 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.093 \\ 1.000 \\ 0.093 \end{bmatrix} = \begin{bmatrix} 1.372 \\ 20.186 \\ 1.372 \end{bmatrix} = 20.106 \begin{bmatrix} 0.066 \\ 1.000 \\ 0.068 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.068 \\ 1.000 \\ 0.068 \end{bmatrix} = \begin{bmatrix} 1.273 \\ 20.136 \\ 1.272 \end{bmatrix} = 20.136 \begin{bmatrix} 0.063 \\ 1.000 \\ 0.063 \end{bmatrix} - \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.063 \\ 1.000 \\ 0.063 \end{bmatrix} = \begin{bmatrix} 1.252 \\ 20.126 \\ 1.252 \end{bmatrix} = 20.126 \begin{bmatrix} 0.662 \\ 1.000 \\ 0.062 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.062 \\ 1.000 \\ 0.062 \end{bmatrix} = \begin{bmatrix} 1.248 \\ 20.128 \\ 1.248 \end{bmatrix} = 20.124 \begin{bmatrix} 0.062 \\ 1.000 \\ 0.062 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0.062 \\ 1.000 \\ 0.062 \end{bmatrix} = \begin{bmatrix} 1.248 \\ 20.128 \\ 1.248 \end{bmatrix} = 30.124 \begin{bmatrix} 0.062 \\ 1.000 \\ 0.062 \end{bmatrix} = \lambda_7 X_7$$

The largest eigen value is 20.124 and the corresponding eigen vector is $[0.062, 1.000, 0.062]^T$

Example 7: Find numerically the largest eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ **with initial vector** $[1, 0, 1]^T$ **upto 2 decimal places.**

Solution:

$$AX_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\begin{bmatrix} 100 \\ 0.04 \\ 0.08 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.04 \\ 0.08 \end{bmatrix} = 25\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix} = 252\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix} = 1.00\begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix}$$

The largest eigen value is 25.18 and the corresponding eigen vector is $(1,0.04, 0.07)^T$

Example 8: Find the Largest eigen value and the respective eigen vector by Power method for the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Solution:

Initial Eigen vector

$$X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$AX_0 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ \frac{-1}{2} \\ 0 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -2 \\ 0.5 \end{pmatrix} = 2.5 \begin{pmatrix} 1 \\ -0.8 \\ 0.2 \end{pmatrix}$$

$$AX_2 = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 2.8 \\ -2.8 \\ 1.2 \end{pmatrix} = 2.8 \begin{pmatrix} 1 \\ -1 \\ 0.4286 \end{pmatrix}$$

$$AX_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0.4286 \end{pmatrix} = \begin{pmatrix} 3 \\ -3.4286 \\ 1.8572 \end{pmatrix} = 3.4286 \begin{pmatrix} +0.8750 \\ -1 \\ 0.5417 \end{pmatrix}$$

$$AX_4 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +0.8750 \\ -1 \\ 0.5417 \end{pmatrix} = \begin{pmatrix} 2.75 \\ -3.4167 \\ 2.0834 \end{pmatrix} = 3.4167 \begin{pmatrix} +0.8049 \\ -1 \\ 0.6098 \end{pmatrix}$$

$$AX_5 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +0.8049 \\ -1 \\ 0.6098 \end{pmatrix} = \begin{pmatrix} 2.0698 \\ -3.4147 \\ 2.2196 \end{pmatrix} = 3.4147 \begin{pmatrix} 0.6061 \\ -1 \\ 0.6500 \end{pmatrix}$$

$$AX_6 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} +0.7643 \\ -1 \\ 0.65 \end{pmatrix} = \begin{pmatrix} 2.5286 \\ -3.4143 \\ 2.3 \end{pmatrix} = 3.4143 \begin{pmatrix} 0.7406 \\ -1 \\ 0.6736 \end{pmatrix}$$

$$AX_7 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0.7406 \\ -1 \\ 0.6736 \end{pmatrix} = \begin{pmatrix} 2.4812 \\ -3.4142 \\ 2.3472 \end{pmatrix} = 2.4142 \begin{pmatrix} 0.7267 \\ -1 \\ 0.6875 \end{pmatrix}$$

$$AX_8 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0.7267 \\ -1 \\ 0.6875 \end{pmatrix} = \begin{pmatrix} 2.4534 \\ -3.4142 \\ 2.375 \end{pmatrix} = 3.4142 \begin{pmatrix} 0.7186 \\ -1 \\ 0.6956 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0.7186 \\ -1 \\ 0.6956 \end{pmatrix} = \begin{pmatrix} 2.4372 \\ -3.4142 \\ 2.3912 \end{pmatrix} = 3.4142\begin{pmatrix} 0.7138 \\ -1 \\ 0.70037 \end{pmatrix}$$

Largest Eigen value (in magnitude) $= 3.4142$

$$\text{Eigen vector } = \begin{pmatrix} 0.7138 \\ -1 \\ 0.70037 \end{pmatrix}$$

Example 9: Find the largest (in magnitude) eigen value of the following matrices.

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-1} \\ \mathbf{3} & \mathbf{2} & \mathbf{4} \\ \mathbf{-1} & \mathbf{4} & \mathbf{10} \end{pmatrix}$$

Solution

$$\text{Initial Eigen vector } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = X^{(0)}$$

$$AX_0 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ 1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 13 \end{pmatrix} = 13\begin{pmatrix} 0.2308 \\ 0.6923 \\ 1 \end{pmatrix}$$

$$\text{Eigen vector } = \begin{pmatrix} 0.2308 \\ 0.6923 \\ 1 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ 1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.2308 \\ 0.6923 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.3077 \\ 6.0770 \\ 12.5384 \end{pmatrix} = 12.5384 = \begin{pmatrix} 0.1043 \\ 0.4847 \\ 1 \end{pmatrix}$$

$$AX_2 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.1043 \\ 0.4847 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5584 \\ 5.2823 \\ 11.8345 \end{pmatrix} = 11.8345\begin{pmatrix} 0.0472 \\ 0.4463 \\ 1 \end{pmatrix}$$

$$AX_3 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0472 \\ 0.4463 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3861 \\ 5.0342 \\ 11.738 \end{pmatrix} = 11.738\begin{pmatrix} 0.0329 \\ 0.4289 \\ 1 \end{pmatrix}$$

$$AX_4 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0329 \\ 0.4289 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3196 \\ 4.9565 \\ 11.6827 \end{pmatrix} = 11.6827\begin{pmatrix} 0.0274 \\ 0.4243 \\ 1 \end{pmatrix}$$

$$AX_5 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0274 \\ 0.4243 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3003 \\ 4.9308 \\ 11.6698 \end{pmatrix} = 11.6698\begin{pmatrix} 0.0257 \\ 0.4225 \\ 1 \end{pmatrix}$$

$$AX_6 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.0257 \\ 0.4225 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2932 \\ 4.9221 \\ 11.6643 \end{pmatrix} = 11.6643\begin{pmatrix} 0.0251 \\ 0.4220 \\ 1 \end{pmatrix}$$

Largest (in magnitude) Eigen Value $= 11.6643$

$$\text{Eigen vector } = \begin{pmatrix} 0.0251 \\ 0.4220 \\ 1 \end{pmatrix}$$

Example 10: Find the smallest (in magnitude) eigen value and eigen vector for the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ by power method.

Solution

$$\text{Initial value of eigen vector } = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = X_0$$

$$AX_0 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1.3333 \\ 1 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1.333 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.6666 \\ 9.3332 \\ 6.6666 \end{pmatrix} = 6.6666\begin{pmatrix} 1 \\ 1.4000 \\ 1 \end{pmatrix}$$

$$AX_2 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.8 \\ 9.6 \\ 6.8 \end{pmatrix} = 6.8\begin{pmatrix} 1 \\ 1.4118 \\ 1 \end{pmatrix}$$

$$AX_3 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1.4118 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.8236 \\ 9.6472 \\ 6.8236 \end{pmatrix} = 6.8236\begin{pmatrix} 1 \\ 1.4138 \\ 1 \end{pmatrix}$$

$$AX_4 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1.4138 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.8276 \\ 9.6552 \\ 6.8276 \end{pmatrix} = 6.8276 \begin{pmatrix} 1 \\ 1.4141 \\ 1 \end{pmatrix}$$

$$AX_5 = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1.4141 \\ 1 \end{pmatrix} = \begin{pmatrix} 6.8282 \\ 9.6564 \\ 6.8282 \end{pmatrix} = 6.8282 \begin{pmatrix} 1 \\ 1.4141 \\ 1 \end{pmatrix}$$

Smallest (in magnitude) eigen value $= 6.8282$

$$\text{Eigen vector} = \begin{pmatrix} 1 \\ 1.4141 \\ 1 \end{pmatrix}$$

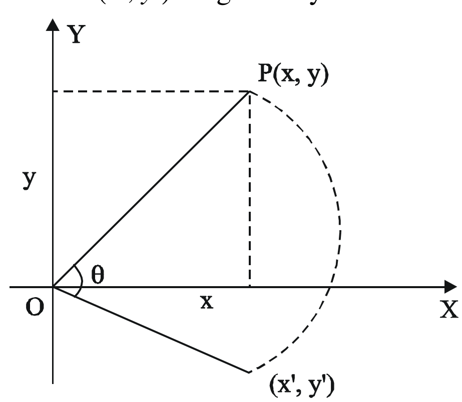
3.7 EIGEN VALUE OF A MATRIX BY JACOBI METHOD FOR SYMMETRIC MATRIX

Let A be a given real symmetric matrix. Its eigen values are real and there exists a real orthogonal matrix B. Such that $B^{-1}AB$ is a diagonal matrix D.

Jacobi's method consists of diagonalising A by applying a series of orthogonal transformations $B_1, B_2, \ldots, B_r$ such that their product B satisfies the equation $D = B^{-1}AB$

Rotation matrix

If $P(x, y)$ is any point in the xy plane and if OP is rotated (O is the origin) in the clockwise direction through an angle θ, then the new position of $P(x', y')$ is given by



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

i.e., $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$$

where $$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Hence P is called a Rotation matrix in the xy plane.

Here P is also an orthogonal matrix, since $PP^T = 1$.

3.7.1 Eigen values of 2×2 matrix by Jacobi method

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a symmetric matrix of order 2.

where $a_{12} = a_{21}$

Step 1. Assume the most general orthogonal Rotation matrix of order 2 is $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Let $B = P^T AP$ be the similar transformation. B is also symmetric

Then $$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ gives}$$

$$b_{11} = a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta \qquad \text{... (1)}$$

$$b_{12} = b_{21} = \frac{1}{2}(a_{22} - a_{11})\sin 2\theta) + a_{12}\cos 2\theta \qquad \text{... (2)}$$

$$b_{22} = a_{11}\sin^2\theta - a_{12}\sin 2\theta + a_{22}\cos^2\theta \qquad \text{... (3)}$$

Since A and B are similar and symmetric matrices

$$a_{11} + a_{22} = b_{11} + b_{22}$$

Step 2. To make B as a diagonal matrix.

Therefore, select θ so that $b_{12} = b_{21} = 0$

i.e., $$\frac{1}{2}(a_{22} - a_{11}) \sin 2\theta + a_{12} \cos 2\theta = 0$$

$$\Rightarrow \frac{1}{2}(a_{11} - a_{22}) \sin 2\theta = a_{12} \cos 2\theta$$

$$\cot 2\theta = \frac{a_{11} - a_{22}}{2a_{12}} = \alpha$$

Then $\cot\theta = \beta = \alpha \pm \sqrt{\alpha^2 + 1}$ and find

$$\sin\theta = \frac{1}{\sqrt{1+\beta^2}}, \quad \cos\theta = \frac{\beta}{1+\beta^2} \text{ if } \beta > 0$$

(OR) $$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}}\right) \text{if } a_{11} \neq a_{22}$$

$$= \frac{\pi}{4} \text{ if } a_{11} = a_{22} \text{ and } a_{12} > 0$$

$$= -\frac{\pi}{4} \text{ if } a_{11} = a_{22} \text{ and } a_{12} < 0$$

Step 3. Write down $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ using the value of θ and

Step 4. Get $D = P^T AP$

The diagonal elements of D are the Eigen value.

The columns of P are Eigen vectors.

3.7.2 Finding Eigen value of symmetric matrix by Jacobi iteration method

The matrix A is said to symmetric matrix

If $A = A^T$

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

The procedure for Jacobi method

Step 1: Write the given symmetric matrix A

Step 2: Find θ where $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}}\right)$

Step 3: Find the rotation matrix P where $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Step 4: Find the diagonal matrix D where $D = P^T AP$

Step 5: Write the conclusion.

WORKED EXAMPLES

Example 1: Use jacobi method to find the eigen value and the corresponding eigen vector of $A = \begin{bmatrix} \mathbf{6} & \sqrt{\mathbf{3}} \\ \sqrt{\mathbf{3}} & \mathbf{4} \end{bmatrix}$

Solution:

Given Matrix $A = \begin{bmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}$ ... (1)

$$A^T = \begin{bmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}$$

$$A = A^T$$

A is a symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{... (2)}$$

Comparing (1) and (2).

$$a_{11} = 6, a_{12} = \sqrt{3},\ a_{21} = \sqrt{3},\ a_{22} = 4$$

To find θ

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{3}}{6-4}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}(\sqrt{3})$$

$$= \frac{1}{2}\cdot\frac{\pi}{3}$$

$$\boldsymbol{\theta = \frac{\pi}{6}}$$

To find rotation matrix *P*

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

Find $D = P^T AP$

$$= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 6 & \sqrt{3} \\ \sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

$$D = \begin{pmatrix} 7 & 0 \\ 0 & \sqrt{3} \end{pmatrix}$$

$$7 \Rightarrow \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

$$3 \Rightarrow \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

Eigen value	Eigen vector
7	$\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$
3	$\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$

Example 2: Use Jacobi method to the find the eigen value and the corresponding eigen vector of matrix $A = \begin{bmatrix} \mathbf{4} & \mathbf{1} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}$

Solution:

Given

$$\text{Matrix } A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \quad ...(1)$$

$$A^T = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

$$A = A^T$$

A is symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad ... (2)$$

Comparing (1) and (2)

$$a_{11} = 4, \qquad a_{12} = 1$$

$$a_{21} = 1, \qquad a_{22} = 4$$

To find θ

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2a_{12}}{a_{11} - a_{22}} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{2a_{12}}{a_{11} - a_{22}} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{2\,(1)}{4 - 4} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{2}{0} \right)$$

$$= \frac{1}{2} \tan^{-1} (\infty)$$

$$= \frac{1}{2} \frac{\pi}{2}$$

$$\boldsymbol{\theta = \frac{\pi}{4}}$$

To find rotation matrix P.

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$D = P^T AP$$

$$D = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$$

$$5 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$3 \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Eigen value	Eigen vector
5	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
3	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Example 3: Use Jacobi method to find the eigen value, corresponding eigen vector $\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$

Solution:

Given Matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ ...(1)

$$A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A = A^T$$

Given matrix is symmetric matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{...(2)}$$

Comparing (1) and (2)

$$a_{11} = 2 \qquad a_{12} = 1$$

$$a_{21} = 1 \qquad a_{22} = 2$$

To find θ

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2a_{12}}{a_{11} - a_{22}} \right)$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{2\,(1)}{2 - 2} \right)$$

$$= \frac{1}{2} \tan^{-1} (\infty)$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

To find Rotation matrix P.

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix} = \frac{\pi}{4}$$

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$D = P^T AP$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Eigen value	**Eigen vector**
3	1
1	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Example 11: Using power method, find all the eigen values of

$$A = \begin{pmatrix} \mathbf{5} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{5} \end{pmatrix}$$

[AU May/June 2009 MA 1251]

Solution:

Let $X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be the initial vector. Then

$$AX = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix}$$

$$\therefore \quad AX_1 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 5.2 \\ 0 \\ 2 \end{pmatrix}$$

$$= 5.2\begin{pmatrix} 1 \\ 0 \\ 0.3846 \end{pmatrix}$$

Repeating thus, we have

$$AX_2 = \begin{pmatrix} 5.3846 \\ 0 \\ 2.9321 \end{pmatrix} = 5.3846\begin{pmatrix} 1 \\ 0 \\ 0.5429 \end{pmatrix}$$

$$AX_3 = \begin{pmatrix} 5.5429 \\ 0 \\ 3.7143 \end{pmatrix} = 5.5429\begin{pmatrix} 1 \\ 0 \\ 0.6701 \end{pmatrix}$$

$$AX_4 = \begin{pmatrix} 5.6701 \\ 0 \\ 2.4305 \end{pmatrix} = 5.6701\begin{pmatrix} 1 \\ 0 \\ 0.7672 \end{pmatrix}$$

$$AX_5 = \begin{pmatrix} 5.7642 \\ 0 \\ 4.8360 \end{pmatrix} = 5.7642\begin{pmatrix} 1 \\ 0 \\ 0.8389 \end{pmatrix}$$

Continuing in the same way, we can observe that 15^{th} and 16^{th} iterations are equal. In that case

$$AX_{16} = \begin{pmatrix} 5.997 \\ 0 \\ 5.985 \end{pmatrix} = 5.997\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ the eigen value $\lambda_1 = 6$ and eigen vector $X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Now consider $B = A - 6I$

$\therefore \quad B = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Now take the initial vector of B as $Y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \; BY = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now $$BY_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$BY_2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\therefore$ the dominant eigen value of $B = -2$

$\therefore$ the smallest eigen values of $A = -2 + 6 = 4$

By using the property,

Sum of the eigen values $=$ Trace of A

$$= 5 - 2 + 5 = 8$$

$\therefore$ the third eigen value is

$$\lambda_1 + \lambda_2 + \lambda_3 = 8$$

$$6 + 4 + \lambda_3 = 8$$

$$\therefore \; \lambda_3 = -2$$

$\therefore$ the eigen values are $6, 4, -2$.

Example 4: Using Jacobi's method, find all the eigen values and the eigen vectors of the matrix

$$\begin{bmatrix} \mathbf{1} & \sqrt{\mathbf{2}} & \mathbf{2} \\ \sqrt{\mathbf{2}} & \mathbf{3} & \sqrt{\mathbf{2}} \\ \mathbf{2} & \sqrt{\mathbf{2}} & \mathbf{1} \end{bmatrix}$$

Solution:

Here the largest non-diagonal element is $a_{13} = a_{31} = 2$. Also $a_{11} = 1$ and $a_{33} = 1$.

$$\therefore \quad \tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2 \times 2}{1-1} \to \infty$$

$$\text{i.e.,} \quad 2\theta = \frac{\pi}{2} \text{ (or) } \theta = \frac{\pi}{4}$$

$$\text{Then } B_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore$ The first transformation gives

$$D_1 = B_1^{-1} A B_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now the largest non-diagonal element is $a_{12} = a_{21} = 2$. Also $a_{11} = 3$ and $a_{22} = 3$.

$$\therefore \ \tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2 \times 2}{0} \to \infty$$

$$2\theta = \frac{\pi}{2} \text{ or } \theta = \frac{\pi}{4}$$

Then

$$B_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The second transformation gives

$$B_1^{-1} D_1 B_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hence the eigen values of the given matrix are $5, 1, -1$ and the corresponding eigen-vectors are the columns of

$$B = B_1 B_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

i.e., the eigen values of A are 5, 1 and -1.

The corresponding eigen vectors are given by

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

EXERCISES

PART - A

1. State fixed point theorem and the fixed point iteration formula.
2. What is the condition for the convergence of the iteration method for solving $x = \phi(x)$?
3. Solve by iteration method to determine the real root of $x^3 - 9x + 1 = 0$ correct to three decimal places.
4. State Newton's formula and order of convergence of that method. *(Nov/Dec 2009 MA 1251)*
5. Find an iterative formula for finding $\sqrt{N}$ where N is a real number, using Newton-Raphson formula. *(April/May 2008 MA 1251)*
6. What are the merits of Newton's method of iteration? *(Nov/Dec 2008 MA 1251)*
7. Write the convergence condition and order of convergence of Newton-Raphson method. *(May/June 2009 MA 1251) (May/June 2004 MA 038)*
8. Show that the iterative formula for finding the reciprocal of N is $x_{n+1} = x_n (2 - Nx_n)$. *(May/June 2006 MA 038)*
9. Give two direct methods to solve a system of linear equations. *(Nov/Dec 2008 MA 1251)*

10. Using Gauss elimination method, solve $x+y=2, 2x+3y=5$.
(May/June 2009 MA 1251)

11. Solve $3x-y=2, x+3y=4$ using Gauss elimination method.
(May/June 2009 MA 038)

12. What is the condition for the convergence of Gauss – Jacobi method in solving a system of simultaneous linear equations?
(May/June 2009 MA 1251)

13. State the condition for convergence of Seidel iterative method for solving a system of equations. *(April/May 2008 MA 1251)*

14. Solve $3x+y=2,\ x+3y=-2$ by Gauss Seidel iteration method.
(Nov/Dec 2005 MA 038)

15. State whether the following statement is true or false and justify: The convergence in Gauss - Seidel method is thrice as fast as in Jacobi's method. *(May/June 2006 MA 038)*

16. Are the first iteration values same if the equation $4x+y=8$ and $2x+3y=7$ are solved by Gauss Seidel and by Jacobi methods? *(Nov/Dec 2003 MA 038)*

17. Compare Gauss elimination with Gauss Seidel method.
(May/June 2009 MA 1251)

18. If 'A' is the largest eigen value of the matrix A, write the suitable method mathematically which will yield the smallest eigen value. *(May/June 2006 MA 038)*

19. Find the dominant eigen value of $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by power method *(Nov/Dec 2004 MA 038)*

20. Explain power method of finding the eigen values of a matrix.
(Nov/Dec 2005 MA 038)

PART - B

1. Solve the following by **iteration** method

(a) $x^3+x^2-1=0$ [**Ans:** 0.7549]

(b) $e^x = 3x$ [**Ans:** 0.6170]

(c) $3x - \sqrt{1+\sin x} = 0$ [**Ans:** 0.3919]

(d) $3x + \sin x - e^x$ [**Ans:** 0.3604]

2. Find the root of the equation $\cos x - 3x + 2 = 0$, lying between 0 and 1, by **fixed point iteration** method. [**Ans:** 0.8792]

3. Find the positive root of the equation $x^3 - 8x + 89 = 0$, lying between 1 and 1.5 by simple **iteration** method.

[**Ans:** 1.2361]

4. Solve for positive root using **Newton Raphson method,** upto 4 decimal places.

(a) $x^3 = 6x - 4$ [**Ans:** 0.7320]

(b) $x - \cos x = 0$ [**Ans:** 0.7391]

(c) $e^x = 2x + 1$ [**Ans:** 1.2564]

(d) $3x^3 - 9x^2 + 8 = 0$ [**Ans:** 1.2261]

(e) $x^3 - x - 1 = 0$ [**Ans:** 1.3247]

5. Find the negative root of $x^4 + 12x + 7 = 0$, by **Newton - Raphson method,** correct to 4 decimal places, given that the root lines between -2 and -3. [**Ans:** -2.0472]

6. Find the square root of 15 using **Newton - Raphson** method. [**Ans:** 3.87298]

7. Find $\sqrt[3]{17}$ using Newton - Raphson method correct to 4 decimal places. [**Ans:** 2.5713]

8. Find $\frac{1}{11}$ using Newton - Raphson method, correct to 4 decimal places. [**Ans:** 0.0909]

9. Solve the following system of equations by **Gauss elimination** method and **Gauss - Jordan method.**

(a) $3x - y = 5$

$x + y = -1$ [**Ans:** $x = 1, y = -2$]

(b) $2x + 3y - z = 5$

$4x + 4y - 3z = 3$

$2x - 3y + 2z = 2$ [**Ans:** $x = 1, y = 2, z = 3$]

(c) $10x_1 + x_2 - x_3 = 11.19$

$x_1 + 10x_2 + x_3 = 20.08$

$-x_1 + x_2 + 10x_3 = 35.61$

[**Ans:** $x_1 = 1.3200, x_2 = 1.5219, x_3 = 3.5408$]

(d) $2x - 4y + 6z + 8w = 9$

$6x - 2y + 4z + 10w = 19$

$2x + 4y - 5z + w = 15$

$4x + 2y - z + 3w = 12$

[**Ans:** $x = -0.5, y = 2, z = -1, w = 3$]

(e) $2x - 7y + 4z =$

$x + 9y - 6z = 1$

$-3x + 8y + 5z = 6$ [**Ans:** $x = 4, y = 1, z = 2$]

(f) Solve by Gauss elimination method.

$3x + 4y + 5z = 18$

$2x - y + 8z = 13$

$5x - 2y + 7z = 20$ [**Ans:** $x = 3, y = 1, z = 1$]

(g) $5x_1 - x_2 = 9$

$-x_1 + 5x_2 - x_3 = 4$

$-x_2 + 5x_3 = -6$ [**Ans:** $x_1 = 2, x_2 = 1, x_3 = -1$]

(h) $2x + y = 3, \quad 7x - 3y = 4$ [**Ans:** $x = 1, y = 1$]

(i) $2x_1 + x_2 - x_3 + 3x_1 = 8$

$x_1 + x_2 + x_3 - x_4 + 2 = 0$

$3x_1 + 2x_2 - 1, x_3 = -2, x_4 = 1)$

$4x_2 + 3x_3 + 2x_4 + 8 = 0$

[**Ans:** $x_1 = 2, x_2 = -1, x_3 = -2, x_4 = 1$]

(j) $3x + y + 3z = 3$

$2x - 3y - z = -3$

$2x + y + z = 4$ [**Ans:** $1.83333, y = 3.1666, z = -2.83$]

10. Solve the following system of equations by **Gauss – Jacobi** method and **Gauss – Seidel method**

(a) $10z - 5y - 2z = 3$

$4x - 10y + 3z = -3$

$x + 6y + 10z = -3$ [**Ans:** $x = 0.342, y = 0.285, z = -0.505$]

(b) $28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$

[**Ans:** $x = 0.9936, y = 1.5069, z = 1.8486$]

(c) $5x - 2y + z = -4$

$x + 6y - 2z = -1$

$3x + y + 5z = 13$ [**Ans:** $x = -1, y = 1, z = 3$]

(d) $10x - 2y + z = 12$

$x + 9y - z = 10$

$2x - y + 11z = 20$ [**Ans:** $x = 1.263, y = 1.16, z = 1.694$]

(e) $10x + 2y + z = 9$

$x + 10y - z = -22$

$-2x + 3y + 10z = 22$ [**Ans:** $x = 1, y = 2, z = 3$]

11. Find the inverse of the following matrices by **Gauss -Jordan method.**

(a) $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$ [**Ans:** $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$]

(b) $\begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$ [**Ans:** $\begin{bmatrix} 1 & -0.2 & -0.4 \\ 1 & -0.2 & -1.4 \\ -1 & 0.4 & 0.8 \end{bmatrix}$]

(c) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ [**Ans:** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$]

[**Ans:** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$]

(e) $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 1 & 4 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 2 & 5 & 3 & 2 \end{bmatrix}$

[**Ans:** $\begin{bmatrix} -\frac{5}{3} & -\frac{14}{9} & \frac{4}{3} & \frac{13}{9} \\ \frac{1}{3} & \frac{7}{9} & -\frac{2}{3} & \frac{-2}{9} \\ 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} \end{bmatrix}$]

12. Find the dominant eigen value and the corresponding vector of the matrices. [Power method]

(i) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ [**Ans:** $6 ; \begin{pmatrix} 4 \\ 1 \end{pmatrix}$]

(ii) $\begin{pmatrix} -4 & -5 \\ 1 & 2 \end{pmatrix}$ [**Ans:** $-3 ; \begin{pmatrix} 1 \\ -0.2 \end{pmatrix}$]

(iii) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ [**Ans:** $5.38 ; \begin{pmatrix} 0.46 \\ 1 \end{pmatrix}$]

(iv) $\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ [**Ans:** $25.182 ; \begin{pmatrix} 1 \\ 0.045 \\ 0.068 \end{pmatrix}$]

(v) $\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}$ [**Ans:** $-1 ; \begin{pmatrix} 1 \\ -0.33 \\ 0 \end{pmatrix}$]

(vi) $\begin{pmatrix} 3 & 2 & 4 \\ -1 & 4 & 10 \\ 1 & 3 & -1 \end{pmatrix}$ [**Ans:** $7.4 ; \begin{pmatrix} 0.8 \\ 1 \\ 0.4 \end{pmatrix}$]

(vii) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix}$ [**Ans:** $7 ; \begin{pmatrix} 0.56 \\ 0.18 \\ 1 \end{pmatrix}$]

(viii) $\begin{pmatrix} 10 & 0 & 0 \\ 1 & -3 & -7 \\ 0 & 2 & 6 \end{pmatrix}$ [**Ans:** $10 ; \begin{pmatrix} 1 \\ 0.06 \\ 0.3 \end{pmatrix}$]

13. Using **Jacobi's method** find the eigen values and eigen vectors of the following matrices.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ [**Ans:** $\begin{bmatrix} 3, & -1 \\ (3, 0), & (0, -1) \end{bmatrix}$]

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ [**Ans:** $\begin{bmatrix} 1, & -1 \\ (1, 0)^T, & (0, -1)^T \end{bmatrix}$]

14. Find all the eigen values and eigen vectors of the following matrices, by Jacobi's method.

(a) $\begin{pmatrix} -2 & -2 & 6 \\ -2 & 3 & 4 \\ 6 & 4 & -1 \end{pmatrix}$

[**Ans:** $\begin{bmatrix} -9 & 6 & 3 \\ (-2, -1, 2)^T, & (2, -2, 1)^T, & (1, 2, 2)^T \end{bmatrix}$]

(b) $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

[**Ans:** $\begin{bmatrix} 4 & -2 & 6 \\ (1, 0, -1)^T, & (0, 1, 0)^T, & (1, 0, 1)^T \end{bmatrix}$]

(c) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

[**Ans:** $\begin{pmatrix} 0.6340, 2.2652, 3.1007\ ; \\ (0.6280, 0.6280, 0.4592)^T \\ (-0.7726, 0.4319, 0.4655)^T \\ (0.0938, -0.6474, 0.7564)^T \end{pmatrix}$]

(d) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

[**Ans:** $\begin{matrix} 1, & 1, & -1 \\ (1, 0, 1)^T, & (0, 1, 0)^T, & (-1, 0, 2)^T \end{matrix}$]

SHORT QUESTIONS AND ANSWERS

1. What are the merits of newton-Raphson method?

- A better and closer approximation to the root can be found from this method.
- From this method we are able to find the roots of the Equations [Transcendental].

2. Mention the order and condition for the convergence of N-R method?

Condition of convergence: $|f(x) f''(x)| < |f'(x)|^2$

Order of convergence: 2 (quadratic convergence)

3. Using newton-Raphson method, find the iteration formula to compute $\sqrt{N}$.

$$X_{n+1} = \frac{1}{2}\left[X_n + \frac{N}{X_n}\right], \; n = 0, 1, 2$$

4. Write the procedure involved in Gauss-Elimination method?

- Make the matrix into upper triangular matrix.
- Apply back-substitution method
- Find the solution

5. Compare Gauss Jacobi with Gauss seidal.

Jacobi	**Seidal**
Indirect method	Indirect method
Solve the system of Equations	Solve the system of Equations
One set of value is used to find the next approximations	New values of unknowns are used immediately to find the next values.

6. Explain power method to determine the Eigen values of the matrix.

- The method is used to find the largest (dominant) Eigen values of the matrix and the corresponding Eigen vector.
- Formula

$Y_{i+1} = AX_i \quad i = 0, 1, 2 \ldots$

7. Solve the following system of equations using Gauss-Jordan Method. $2x + y = 3, x - 2y = -1$

$$(A, B) \sim \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & -2 & -1 \end{array}\right] \qquad \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 1 \end{array}\right] \div 5$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & 1 & 3 \end{array}\right] \qquad \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]^{R_1 \to R_1 + + 2R_2}$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 5 & 5 \end{array}\right]_{R_2 \to R_2 - 2R_1}$$

Solution

$\boldsymbol{x = 1}$

$\boldsymbol{y = 1}$

8. What are indirect and direct methods?

$$\left.\begin{array}{l}\text{Gauss Jacobi} \\ \text{Gauss Seidel}\end{array}\right\} \to \text{Indirect}$$

$$\left.\begin{array}{l}\text{Gauss Elimination} \\ \text{Gauss Jordan}\end{array}\right\} \to \text{Direct}$$

9. Compare Gauss Elimination with Gauss-Jordon Elimination.

Elimination	**Jordon**
Make the matrix into upper triangular matrix	Make the matrix into diagonal matrix
Back substitution	Direct substitution

10. Find the dominant eigen value of the matrix by power method.

$\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$ **(Solve by power method)**

Solution

Let $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$AX_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 0.43 \\ 1 \end{bmatrix} = 7X_2$$

$$AX_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.43 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.43 \\ 5.29 \end{bmatrix} = 5.29 \begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = 5.29X_3$$

$$AX_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.46 \\ 5.38 \end{bmatrix} = 5.38 \begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = 5.38X_4$$

$$AX_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.46 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.46 \\ 5.38 \end{bmatrix} = 5.38 \begin{bmatrix} 0.46 \\ 1 \end{bmatrix}$$

$\therefore$ Dominant Eigen value is 5.

11. What is meant by fixed point iteration method?

This method is otherwise called as method of successive approximation (or) Iteration method. It is used to solve Algebraic and Transcendental Equations.

12. Write down the formula for N-R method? Another name of N-R method? (Method of Tangents)

$$x_{n+1} = x_n - \frac{f(\alpha_n)}{f'(x_n)}$$

13. Compare Gauss elimination and Gauss-Jacobi method.

Gauss Elimination	**Gauss Jacobi**
Direct method	**Iterative method**
We get exact solution	Approximate solution
No condition of convergence	Diagonally dominant
Back substitution	Successive approximation

14. Solve the equations $A + B + C = 6$, $3A + 3B + 4C = 20$, $2A + B + 3C = 13$ using Gauss Elimination method.

Hint: Use back substitution method.

$$A = 3, B = 1, C = 2$$

15. Define a diagonally dominant system of equation.

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

16. Using Newton – Raphson method, find the iteration formula to compute N.

Let $x = \sqrt{N}$

$$x^2 - N = 0$$

Let $f(x) = x^2 - N$

$$f'(x) = 2x$$

We know that,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left[\frac{x_n^2 - N}{2x_n}\right]$$

$$= \frac{x_n}{2} + \frac{N}{2x_n}$$

$$\boxed{\sqrt{N} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]}$$

17. Obtain the iterative formula to find $\frac{1}{N}$ using Newton Raphson method.

Let $x = \frac{1}{N}$

$$N = \frac{1}{x}$$

$$f(x) = \frac{1}{x} - N$$

$$f'(x) = -\frac{1}{x^2}$$

We know that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left[\frac{1}{x_n} - N\right]}{\left[-\frac{1}{x_n^2}\right]}$$

$$= x_n + \left[\frac{1}{x_n} - N\right] x_n^2$$

$$= x_n + \frac{x_n^2}{x_n} - N x_n^2$$

$$= x_n + x_n - N x_n^2$$

$$= 2x_n - N x_n^2$$

$$\boxed{\frac{1}{N} = x_n (2 - N x_n)}$$

UNIT - IV

Interpolation, Numerical Differentiation, Numerical Interpolation

4.0 INTERPOLATION

Definition

Interpolation is a process of estimating the value of a function at an intermediate point when its values are known only at certain specified points.

Types of interpolation

- Interpolation with equal intervals
- Gregory - Newton's forward interpolation
    - ❖ Gregory - Newton's backward interpolation
- Interpolation with unequal intervals
    - ❖ Newton's divided - difference interpolation
- For both equal & unequal intervals
    - ❖ Lagrange's interpolation

4.1 NEWTON'S FORWARD INTERPOLATION

Formulas

The forward difference operator is Δ (delta)

$$P = \frac{x - x_0}{h}$$

x – Unknown

x_0 – Initial value

h – Step size

	x_0	x_1	x_2	x_3	x_4	x_5
x	10	12	14	16	18	20
$f(x)$ or y	22	28	24	23	32	12
	y_0	y_1	y_2	y_3	y_4	y_5

Newton's Forward Difference Interpolation Formula

$$y_p(x) = y_0 + P\,\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \ldots$$

WORKED EXAMPLES

Example 1: The following table gives the population of a town during the last six census. Estimate the increase in the population in the year 1925.

Year	**1911**	**1921**	**1941**	**1941**	**1951**	**1961**
Population (in thousands)	**12**	**13**	**20**	**27**	**39**	**52**

Solution:

$x_0 = 1911$	$y_0 = 12$
$x_1 = 1921$	$y_1 = 13$
$x_2 = 1931$	$y_2 = 20$
$x_3 = 1941$	$y_3 = 27$
$x_4 = 1951$	$y_4 = 39$
$x_5 = 1961$	$y_5 = 52$

$$p = \frac{x - x_0}{h}, \; h = 10$$

$$x = 1925, \ x_0 = 1911$$

$$p = \frac{1925 - 1911}{10}$$

$$= \frac{14}{10}$$

$$\boldsymbol{p = 1.4}$$

$$y_p(x) = y_0 + P\,\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \ldots$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911	12					
		1				
1921	13		6			
		7		− 6		
1931	20		0		11	
		7		5		− 20
1941	27		5		− 9	
		12		− 4		
1951	39		1			
		13				
1961	52					

$$y_p(1925) = 12 + (1.4 \times 1) + \left[\frac{1.4\,(1.4-1)}{2} \times (6)\right] + \left[\frac{(1.4)\,(1.4-1)\,(1.4-2)}{6}(-6)\right] + \ldots$$

$$= 12 + 1.4 + (1.68) + (0.336)$$

$$= 15.416 \ \text{(in thousands)}$$

Hence the population in 1925 is **15.416** (in thousands)

Example 2: Find the value of y at $x = 5$ from the following table

x	4	6	8	10
y	1	3	8	10

The interval is equal we can use Newton's forward in interpolation. *[AU M/J 2012]*

Solution:

$x_0 = 4$	$y_0 = 1$
$x_1 = 6$	$y_1 = 3$
$x_2 = 8$	$y_2 = 8$
$x_3 = 10$	$y_3 = 10$

$$p = \frac{x - x_0}{h}$$

$$x = 5, x_0 = 4, h = 2$$

$$p = \frac{5 - 4}{2}$$

$$= \frac{1}{2}$$

$$\boxed{p = 0.5}$$

$$y_p(x) = y_0 + P \Delta y_0 + \frac{P(P-1)}{2!} \Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!} \Delta^3 y_0 + \ldots$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		-6
8	8		-3	
		2		
10	10			

$$y_p(5) = 1 + 0.5\,(2) + \frac{(0.5)\,(0.5-1)}{2}\,(3) + \frac{(0.5)\,(0.5-1)\,(0.5-2)\,(-6)}{6}$$

$$= 1 + 1 - \frac{0.75}{2} - \frac{2.25}{6}$$

$$= 2 - 0.375 - 0.375$$

$$= 1.25$$

Hence the value of y at $x = 5$ is **1.25.**

Example 3: At a particular city the age of a person and their sugar level is given in the following table. Find the sugar level of the person whose age is 43. *[AU A/U 2019]*

x **(Age)**	**40**	**50**	**60**	**70**	**80**	**90**
f **(*x*) (Sugar Level)**	**184**	**204**	**226**	**250**	**276**	**304**

Solution:

The interval is equal we can use Newton's forward interpolation.

$x_0 = 40$	$y_0 = 184$
$x_1 = 50$	$y_1 = 204$
$x_2 = 60$	$y_2 = 226$
$x_3 = 70$	$y_3 = 250$
$x_4 = 80$	$y_4 = 276$
$x_5 = 90$	$y_5 = 304$

$$p = \frac{x - x_0}{h}$$

$x = 43,\ x_0 = 40,\ h = 10$

$$p = \frac{43 - 40}{10}$$

$$= \frac{3}{10}$$

$$\boxed{p = 0.3}$$

$$y_p\,(x) = y_0 + P\,\Delta\,y_0 + \frac{P\,(P-1)}{2!}\,\Delta^2\,y_0 + \frac{P\,(P-1)\,(P-2)}{3!}\,\Delta^3\,y_0 + \ldots$$

Forward difference table

x	***y***	$\Delta\,y$	$\Delta^2\,y$	$\Delta^3\,y$
40	184			
		20		
50	204		2	
		22		0
60	226		2	
		24		0
70	250		2	
		26		0
80	216		2	
		28		
90	304			

$$y_p\,(43) = 184 + (0.3)\,(20) + \frac{(0.3)\,(0.3-1)}{2}\,(2) + \ldots$$

$$= 184 + 6 - \frac{0.42}{2}$$

$$y_p\,(43) = 189.79.$$

The sugar level is **189.79**.

Example 4: Find *y* (22) from the table.

x	20	25	30	35	40	45
y	354	332	291	260	231	204

The interval is equal we can use Newton's forward interpolation.

Solution:

$x_0 = 20$	$y_0 = 354$
$x_1 = 25$	$y_1 = 332$
$x_2 = 30$	$y_2 = 291$
$x_3 = 35$	$y_3 = 260$
$x_4 = 40$	$y_4 = 231$
$x_5 = 45$	$y_5 = 204$

$$p = \frac{x - x_0}{h}$$

$$x = 22,\ \ x_0 = 20,\ h\ = 5$$

$$p = \frac{22 - 20}{5} = \frac{2}{5}$$

$$\boxed{\mathbf{p = 0.4}}$$

$$y_p\,(x) = y_0 + P\,\Delta\,y_0 + \frac{P\,(P-1)}{2!}\,\Delta^2\,y_0 + \frac{P\,(P-1)\,(P-2)}{3!}\,\Delta^3\,y_0 + \ldots$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20	354					
		– 22				
25	332		– 19			
		– 41		29		
30	291		10		– 37	
		– 31		– 8		45
35	260		2		8	
		– 29		0		
40	231		2			
		– 27				
45	204					

$$y_p\,(22) = 354 + (0.4)\,(-22) + \frac{(0.4)\,(0.4-1)}{2}\,(-19)$$

$$+ \frac{(0.4)\,(0.4-1)\,(0.4-2)\,(29)}{6} + \ldots$$

$$= 354 - 8.8 + \frac{4.56}{2} + \frac{11.136}{6}$$

$$= 354 - 8.8 + 2.28 + 1.856$$

$$\boxed{y_p\,(2) = 349.336}$$

Example 5: Find $f\,(0.2)$ from the table given below

x	0	1	2	3	4	5	6
$f\,(x)$	176	185	194	203	212	220	229

The interval is equal we can use Newton's forward interpolation.

Solution:

$x_0 = 0$	$y_0 = 176$
$x_1 = 1$	$y_1 = 185$
$x_2 = 2$	$y_2 = 194$
$x_3 = 3$	$y_3 = 203$
$x_4 = 4$	$y_4 = 212$
$x_5 = 5$	$y_5 = 220$
$x_6 = 6$	$y_0 = 229$

$$p = \frac{x - x_0}{h}$$

$$x = 0,\ x_0 = 0.2,\ h = 1$$

$$p = \frac{0.2 - 0}{1}$$

$$\boxed{p = 0.2}$$

$$y_p\,(x) = y_0 + P\,\Delta\,y_0 + \frac{P\,(P-1)}{2!}\,\Delta^2\,y_0 + \frac{P\,(P-1)\,(P-2)}{3!}\,\Delta^3\,y_0 + \ldots$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	176						
		9					
1	185		0				
		9		0			
2	194		0		0		
		9		0		– 1	
3	203		0		– 1		5
		9		– 1		4	
4	212		– 1		3		
		8		2			
5	220		1				
		9					
6	229						

$$y_p\,(0.2) = 176 + (0.2)\,(9) + \frac{(0.2)\,(0.2-1)}{2}\,(0) + 0$$

$$= 176 + 1.8 + 0$$

$$\boxed{y_p\,(0.2) = 177.8}$$

4.2 NEWTON'S BACKWARD INTERPOLATION: FORMULA

The Backward difference operator is ∇ (del)

$$y_q\,(x) = y_n + q\,\nabla\,y_n + \frac{q\,(q+1)}{2!}\,\nabla^2\,y_n + \frac{q\,(q+1)\,(q+2)}{3!}\,\nabla^3\,y_n + \ldots$$

$$\boxed{q = \frac{x - x_n}{h}}$$

WORKED EXAMPLES

Example 1: Find y when $x = 27$ from the following data.

x	10	15	20	25	30
y	35.4	32.2	29.1	26.0	23.1

Since the interval is equal we can use Newton's backward interpolation.

Solution:

$x_0 = 10$	$y_0 = 35.4$
$x_1 = 15$	$y_1 = 32.2$
$x_2 = 20$	$y_2 = 29.1$
$x_3 = 25$	$y_3 = 26.0$
$x_4 = 30$	$y_4 = 23.1$

$$\mathbf{x = 27}$$

$$x_n = 30$$

$$h = 5$$

$$q = \frac{x - x_n}{h}$$

$$= \frac{27 - 30}{5}$$

$$\boxed{\mathbf{q = -0.6}}$$

$$y_q(x) = y_n + q\,\nabla y_n + \frac{q\,(q+1)}{2!}\,\nabla^2 y_n + \frac{q\,(q+1)\,(q+2)}{3!}\,\nabla^3 y_n + \dots$$

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	35.4				
		–3.2			
15	32.2		0.1		
		–3.1		–0.1	
20	29.1		0		0.3
		–3.1		0.2	
25	26.0		0.2		
		–2.9			
30	23.1				

$$y_q\,(27) = 23.1 + (-\,0.6)\,(-\,2.9) + \frac{(-\,0.6)\,(-\,0.6+1)\,(0.2)}{2}$$

$$+ \frac{(-\,0.6)\,(-\,0.6-11)\,(-\,0.6+2)\,(0.2)}{6}$$

$$= 23.1 + 1.74 - 0.024 - 0.0112$$

$$\boxed{y_q\,(27) = 24.80}$$

Example 2: The following data gives the melting point of an alloy of lead and zinc where *T* is the temperature in degree. Centigrade and *P* is the percentage of lead in the alloy. Using Newton's formula find the melting point of alloy containing 84% of lead.

p	40	50	60	70	80	90
t	184	204	226	250	276	304

Since the Interval is equal we can used Newton's backward interpolation.

Solution:

$x_0 = 40$	$y_0 = 184$
$x_1 = 50$	$y_1 = 204$
$x_2 = 60$	$y_2 = 226$
$x_3 = 70$	$y_3 = 250$
$x_4 = 80$	$y_4 = 276$
$x_5 = 90$	$y_5 = 304$

$$q = \frac{x - x_n}{h}$$

$$= \frac{84 - 90}{10} = \frac{-6}{10}$$

$$\boxed{q = -0.6}$$

$$y_q(x) = y_n + q \nabla y_n + \frac{q(q+1)}{2!} \nabla^2 y_n + \frac{q(q+1)(q+2)}{3!} \nabla^3 y_n + \dots$$

Backward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
40	184			
50	204	20		
60	226	22	2	
70	250	24	2	0
80	276	26	2	0
90	304	28	2	0

$$y_q(84) = 304 + (-0.6)(28) + \frac{(-0.6)(-0.6+1)}{2}(2) + 0$$

$$= 304 - 16.8 - 0.24$$

$$\boxed{y_q(84) = 286.96}$$

Example 3: The following data are taken from the steam table. Find the pressure at the temperature $T = 142°C$ and $T = 175°C$.

Temperature °C	**140**	**150**	**160**	**170**	**180**
Pressure kg/cm^2	**3.685**	**4.854**	**6.302**	**8.076**	**0.225**

Solution:

Case (i): To find the pressure when the temperature

$T = 142°C$

Use Newton's Forward Interpolation

$$p = \frac{x - x_0}{h}$$

$$= \frac{142 - 140}{10}$$

$$= \frac{2}{10}$$

$$\boxed{p = 0.2}$$

$$y_p\,(x) = y_0 + P\,\Delta\,y_0 + \frac{P\,(P-1)}{2!}\,\Delta^2\,y_0 + \frac{P\,(P-1)\,(P-2)}{3!}\,\Delta^3\,y_0 + \ldots$$

Case (ii): To find the pressure when the temperature

$$T = 175°\text{C}$$

Use Newton's Backward Interpolation

$$q = \frac{x - x_n}{h}$$

$$= \frac{175 - 180}{10}$$

$$= \frac{-5}{10}$$

$$\boxed{q = -0.5}$$

$$y_q\,(x) = y_m + q\,\nabla\,y_n + \frac{q\,(q+1)}{2!}\,\nabla^2\,y_n + \frac{q\,(q+1)\,(q+2)}{3!}\,\nabla^3\,y_n + \ldots$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685	1.169			
150	4.854	1.448	0.279		
160	6.302	1.774	0.326	0.047	0.002
170	8.076	2.149	0.375	0.049	
180	10.225				

$$y_p\ (142°C) = 3.685 + (0.2)\ (1.169)\ \frac{(0.2)\ (0.2-1)}{2!}\ (0.279)$$

$$+\frac{(0.2)\ (0.2-1)\ (0.42)\ (0.042)}{6}$$

$$= 3.685 + 0.2338 - 0.02232 + 2.352 \times 10^{-3}$$

$$\boxed{y_p\ (142°C) = 3.89}$$

$$y_q\ (175°C) = 10.225 + (-0.5)\ (2.149) + \frac{(-0.5)\ (-0.5+1)}{2}\ (0.375)$$

$$+\frac{(-0.5)\ (-0.5+1)\ (-0.5+2)}{6}\ (0.049)^2$$

$$= 10.225 - 1.0745 - 0.046875 - 0.0030625$$

$$\boxed{y_q\ (175°C) = 9.100}$$

Example 4: From the data given below find the number of students whose weight is between 60 and 70. *[AU A/M 2022]*

Weights	**0-40**	**40-60**	**60-80**	**80-100**	**100-120**
No. of students	**250**	**120**	**100**	**70**	**50**

Solution:

Weights	Below 40	Below 60	Below 80	Below 100	Below 120
No. of students	250	250 + 120 = 370	370 + 100 = 470	470 + 70 = 540	540 + 50 = 590

$x_0 = 40,\ x = 70,\ h = 20$

$$p = \frac{x - x_0}{h}$$

$$= \frac{70 - 40}{20}$$

$$= \frac{30}{20} = \frac{3}{2}$$

$$\boxed{p = 1.5}$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
Below 60	$250 + 120 = 370$	120			
Below 80	$370 + 100 = 470$	100	– 20		
Below 100	$470 + 70 = 540$	70	– 30	– 10	
Below 120	$540 + 50 = 590$	50	– 20	10	20

∴ The number of students above weight is below 70 is 424.

$$y_p(x) = y_0 + P\,\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y_p(70) = 250 + 180 - 7.5 + 0.625 + 0.46875$$

$$= 423.59$$

$$y_p(70) = 424$$

∴ The number of students whose weight is between 60 and 70

$$= 424 - 370$$

$$= 54$$

Example 5: From the following table, find the value of tan 45°15′ by Newton's forward interpolation formula.

$x°$	**45**	**46**	**47**	**48**	**49**	**50**
tan $x°$	**1.00000**	**1.03553**	**1.07237**	**1.11061**	**1.15037**	**1.19175**

[AU N/D 2021]

Solution:

we can use forward interpolation $p = \frac{x - x_0}{h}$

$$x = 45°15', x_0 = 45°, h = 1°$$

$$p = \frac{(45°15' - 45°)}{(1°)} = 0.25$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
45°	1.00000					
		0.03553				
46°	1.03553		0.00131			
		0.03634		0.00009		
47°	1.07237		0.00140		0.00003	
		0.03824		0.00012		− 0.00005
48°	1.11061		0.00152		− 0.00002	
		0.03976		0.00010		
49°	1.15037		0.00162			
		0.04138				
50°	1.19175					

$$y_p(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

$$= 1.0000 + \frac{1}{4}(0.03553) + \frac{\left(\frac{1}{4}\right)\left(\frac{-3}{4}\right)}{2}(0.00131)$$

$$+ \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)}{6}(0.00009) + \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)\left(\frac{-11}{4}\right)}{24}(0.00003)$$

$$+ \frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)\left(\frac{-11}{4}\right)\left(\frac{-15}{4}\right)}{120}(-0.00005)$$

$$= 1.0000 + 0.0088825 - 0.0001228 + 0.0000049$$

$$\mathbf{y_p\,(\tan 45°\ 15') = 1.00876}$$

Example 6: From the following table of half yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46 and 63.

Age x	**45**	**50**	**55**	**60**	**65**
Premium y	**114.84**	**96.16**	**83.32**	**74.48**	**68.48**

Solution:

To find y at $x = 46$ use forward interpolation formula and to find y at $x = 63$ use backward interpolation formula.

$$P = \frac{x - x_0}{h} = \frac{46 - 45}{5} = \frac{1}{5}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	114.84				
		– 18.68			
50	96.16		5.84		
		– 12.84		– 1.84	
55	83.32		4.00		0.68
		– 8.84		– 1.16	
60	74.48		2.84		
		– 6			
65	68.48				

Using Newton's Forward Interpolation

$$y_p\,(x) = y_0 + p\,\Delta\,y_0 + \frac{p\,(p-1)}{2!}\,\Delta^2\,y_0 + \frac{p\,(p-1)\,(p-2)}{3!}\,\Delta^3\,y_0 + \ldots$$

$$y\,(46) = 114.84 + \frac{\left(\frac{1}{5}\right)}{1!}(-\,18.68) + \frac{\left(\frac{1}{5}\right)\left(\frac{1}{5} - 1\right)}{2!}(5.84)$$

$$+ \frac{\left(\frac{1}{5}\right)\left(\frac{1}{5} - 1\right)\left(\frac{1}{5} - 2\right)}{3!}(-\,1.84)$$

$$= 114.84 - 3.736 - 0.4672 - 0.08832$$

$$y_p\,(46) = 110.53$$

Using Newton's backward interpolation

$$y_q\,(x) = y_n + q\,\nabla\,y_n + \frac{q\,(q+1)}{2!}\,\nabla^2\,y_n + \frac{q\,(q+1)\,(q+2)}{3!}\,\nabla^3\,y_n + \ldots$$

$$q = \frac{x - x_n}{h} = \frac{63 - 65}{5} = \frac{2}{5}$$

$$y\,(63) = 68.48 + \frac{\left(\frac{-\,2}{5}\right)}{1!}(-\,6) + \frac{\left(\frac{-\,2}{5}\right)\left(\frac{-\,2}{5} + 1\right)}{2!}(2.84)$$

$$+ \frac{\left(\frac{-\,2}{5}\right)\left(\frac{-\,2}{5} + 1\right)\left(\frac{-\,2}{5} + 2\right)}{3!}(-\,1.16)$$

$$= 68.45 + 2.4 - 0.3408 + 0.07424$$

$$= 70.585152$$

$$\boxed{y_q\,(63) = 70.59}$$

4.3 LAGRANGE'S INTERPOLATION

Formula

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Note

Suppose 3 set of values $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ are given then the corresponding formula is

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

WORKED EXAMPLES

Example 1: Using Lagrange's interpolation calculate the profit in the year 2000 from the following data. *[AU M/J 2012]*

Year	1997	1999	2001	2002
Profit in Lakhs of Rs.	43	65	159	248

Solution:

Since the interval is not equal we can use Lagrange's interpolation.

$$\boxed{x = 2000}$$

$x_0 = 1997$	$y_0 = 43$
$x_1 = 1999$	$y_1 = 65$
$x_2 = 2001$	$y_2 = 159$
$x_3 = 2002$	$y_3 = 248$

$$y(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$y(2000) = \frac{(2000 - 1999)(2000 - 2001)(2000 - 2002)}{(1997 - 1999)(1997 - 2001)(1997 - 2002)} (43) +$$

$$\frac{(2000 - 1997)(2000 - 2001)(2000 - 2002)}{(1999 - 1997)(1999 - 2001)(1999 - 2002)} (65) +$$

$$\frac{(2000 - 1997)(2000 - 1999)(2000 - 2002)}{(2001 - 1997)(2001 - 1999)(2001 - 2002)} (159) +$$

$$\frac{(2000 - 1997)(2000 - 1999)(2000 - 2001)}{(2002 - 1997)(2002 - 1999)(2002 - 2001)} (248)$$

$$= \left[\frac{2}{-40} \times 43 \right] + \left[\frac{6}{12} \times 65 \right] + \left[\frac{-6}{-8} \times 159 \right] + \left[\frac{-3}{15} \times 248 \right]$$

$$= -2.15 + 32.5 + 119.25 - 49.6$$

$$= 100$$

Hence the profit in the year 2000 is 100 lakhs.

Example 2: Using Lagrange's method find *y* (9.5) from the following data:

x	**7**	**8**	**9**	**10**
y	**3**	**1**	**1**	**9**

Solution:

To find: *y* (9.5)

$$x = 9.5$$

Given data

$x_0 = 7$	$y_0 = 3$
$x_1 = 8$	$y_1 = 1$
$x_2 = 9$	$y_2 = 1$
$x_3 = 10$	$y_3 = 9$

Lagrange's Interpolation Formula

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(9.5-8)(4.5-9)(9.5-10)}{(7-8)(7-9)(7-10)}(3) + \frac{(4.5-7)(4.5-9)(4.5-10)}{(8-7)(8-9)(8-10)}(1)$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)(1)}{(9-7)(9-8)(9-10)} + \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)}(9)$$

$$= 0.1875 - 0.3125 + 0.9375 + 2.8125 \text{ (use calculator)}$$

$$\boxed{\mathbf{y(9.5) = 3.625}}$$

Example 3: Find $f(10)$ if $f(5) = 12$, $f(6) = 13$, $f(9) = 14$, $f(11) = 16$ *[AU N/D 2021]*

Solution

The given data can be written as

x	5	6	9	11
$f(x)$	12	13	14	16

$$\boxed{\mathbf{x = 10}}$$

To find: $f(10)$

$x_0 = 5$	$y_0 = 12$
$x_1 = 6$	$y_1 = 13$
$x_2 = 9$	$y_2 = 14$
$x_3 = 11$	$y_3 = 16$

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= 2 - 4.3 + 11.6 + 5.3$$

$$\boxed{\mathbf{y(10) = 14.66}}$$

Example 4: Find $y(27)$ if $y(14) = 68.7$, $y(17) = 64.0$, $y(3) = 44.0$ $y(35) = 39.1$

Solution

The given data can be written as

x	14	17	31	35
y	68.7	64.0	64.0	39.1

To find: $y(27)$

$$x = 27$$

$x_0 = 14$	$y_0 = 68.7$
$x_1 = 17$	$y_1 = 64.0$
$x_2 = 31$	$y_2 = 44.0$
$x_3 = 35$	$y_3 = 39.1$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(27-17)(27-31)(27-35)}{(14-17)(14-31)(14-35)}(68.7) + \frac{(27-14)(27-31)(27-35)}{(17-14)(17-31)(17-35)}(64.1)$$

$$+ \frac{(27-14)(27-17)(27-35)}{(31-14)(31-17)(31-35)}(44.0) + \frac{(27-14)(27-17)(27-31)}{(35-14)(35-17)(35-31)}(39.1)$$

$$= -20.526 + 35.216 + 48.067 - 13.447$$

$$\boxed{\mathbf{y(27) = 49.31}}$$

Example 5: Using Lagrange's interpolation formula, find the value corresponding to $x = 10$ from the following table:

x	5	6	9	11
y	12	13	14	16

Solution:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$x_0 = 5$	$y_0 = 12$
$x_1 = 6$	$y_1 = 13$
$x_2 = 9$	$y_2 = 14$
$x_3 = 1$	$y_3 = 16$

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{4.1(-1)}{(-1)(-4)(-6)}(12) + \frac{5.1(-1)}{(1)(-3)(-5)}(13) + \frac{5.4(-1)}{4.3(-2)}(14) + \frac{5.4.1}{6.5.2}(16)$$

$$= 14.66$$

Example 6: Apply Lagrange's formula to find $f(5)$, given that $f(1) = 2, f(2) = 4, f(3) = 8$ and $f(7) = 128$. *[AU M/J 2010]*

Solution:

The given data can be written as

x	1	2	3	7
$f(x)$	2	4	8	128

$x_0 = 1$	$y_0 = 2$
$x_1 = 2$	$y_1 = 4$
$x_2 = 3$	$y_2 = 8$
$x_3 = 7$	$y_3 = 128$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(5-2)(5-3)(5-7)}{(1-2)(1-3)(1-7)}(2) + \frac{(5-1)(5-3)(5-7)}{(2-1)(2-3)(2-7)}(4)$$

$$+ \frac{(5-1)(5-2)(5-7)}{(3-1)(3-2)(3-7)}(8) + \frac{(5-1)(5-2)(5-3)}{(7-1)(7-2)(7-3)}(128)$$

$$= 2 - 12.8 + 24 + 25.6$$

$$\boxed{y(5) = 38.8}$$

Example 7: Given $u_0 = 6, u_1 = 9,\ u_3 = 33$ and $u_7 = -15$. Find u_2

Solution:

Given data

$x_0 = 0$	$y_0 = 6$
$x_1 = 1$	$y_1 = 9$
$x_2 = 3$	$y_2 = 33$
$x_3 = 7$	$y_3 = -15$

where $y = u(x)$

To Find:

y when $x = 2$

$$u(x) = \frac{(x-1)(x-3)(x-7)}{(-1)(-3)(-7)}(6) + \frac{x(x-3)(x-7)}{x(-2)(-6)}(9) + \frac{x(x-1)(x-7)}{3 \times 2(-4)}(33) + \frac{x(x-1)(x-3)}{7 \times 6 \times 4}(-15)$$

$\because x = 2$

$$u_2 = \frac{-10}{7} + \frac{15}{2} + \frac{55}{4} + \frac{5}{28}$$

$\boxed{u_2 = 20}$

Example 8: Interpolate $y(12)$ for the data given below

x	10	15	20	25	30	35
$y(x)$	35	33	21	27	22	14

Solution:

$x_0 = 10$	$y_0 = 35$
$x_1 = 15$	$y_1 = 33$
$x_2 = 20$	$y_2 = 29$

$x_3 = 25$	$y_3 = 27$
$x_4 = 30$	$y_4 = 22$
$x_5 = 35$	$y_5 = 14$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_3)} y_0$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} y_2 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} y_3 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} y_4 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} y_5$$

$$y(12) = \frac{(12-15)(12-20)(12-25)(12-30)(12-35)}{(10-15)(10-20)(10-25)(10-30)(10-35)}(35)$$

$$+\frac{(12-10)(12-20)(12-25)(12-30)(12-35)}{(15-10)(15-20)(15-25)(15-30)(15-35)}(33)$$

$$+\frac{(12-10)(12-15)(12-25)(12-30)(12-35)}{(20-10)(20-15)(20-25)(20-30)(20-35)}(29)$$

$$+\frac{(12-10)(12-15)(12-20)(12-30)(12-35)}{(25-10)(25-15)(25-20)(25-30)(25-35)}(27)$$

$$+\frac{(12-10)(12-15)(12-20)(12-25)(12-35)}{(30-10)(30-15)(30-20)(30-25)(30-35)}(22)$$

$$+\frac{(12-10)(12-15)(12-20)(12-25)(12-30)}{(35-10)(35-15)(35-20)(35-25)(35-30)}(14)$$

$$= \frac{(-3)(-8)(-13)(-18)(-23)}{(-5)(-10)(-15)(-20)(-25)} 35 + \frac{2(-5)(-13)(-18)(-23)}{5(-5)(-10)(-15)(-20)} 33$$

$$+ \frac{2(-3)(-13)(-18)(-23)}{(10)(5)(-10)(-5)(-15)} 29 + \frac{2(-3)(-8)(-18)(-23)}{(15)(10)(5)(-5)(-10)} 27$$

$$2 \frac{(-3)(-8)(-13)(-23)}{(20)(15)(10)(5)(-5)} 22 + \frac{2(-3)(-8)(-13)(-18)}{(25)(20)(15)(10)(5)} 14$$

$$= 12.0557 + 37.8893 - 34.9725 + 14.3078 - 4.2099 + 0.4193$$

$$\boxed{\mathbf{y\ (12) = 35.49}}$$

Example 9: Using Lagrange's interpolation, find the polynomial through the points (0, 0), (1, 1) and (2, 2).

Solution:

The given data can be written as

x	0	1	2
y	0	1	2

$x_0 = 0$	$y_0 = 10$
$x_1 = 1$	$y_1 = 1$
$x_2 = 12$	$y_2 = 2$

By Lagrange's method

$$y(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_2)(x_1 - x_2)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_2)(x_2 - 1)} y_2$$

$$y(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)}(0) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(1) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(2)$$

$$= 0 - 1\,[(x-0)(x-2)] + 1\,[(x-0)(x-1)]$$

$$y(x) = -1\,[x(x-2)] + 1\,[x(x-1)]$$

$$= -1\,[x^2 - 2x] + 1\,[x^2 - x]$$

$$= -x^2 + 2x + x^2 - x$$

$$\boxed{y = x}$$

Example 10: Using Lagrange's interpolation fit a polynomial to the data given below.

x	**0**	**1**	**2**
y	**0**	**1**	**20**

Solution

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,y_2$$

$$y(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)}(0) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(1) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(20)$$

$$= 0 - 1\,[(x-0)(x-2)] + 10\,[(x-0)(x-1)]$$

$$= -1\,[x(x-2)] + 10\,[x(x-1)]$$

$$= -1\,[x^2 - 2x] + 10\,[x^2 - x]]$$

$$= -x^2 + 2x + 10x^2 - 10x$$

$$\boxed{y(x) = 9x^2 - 8x}$$

Example 11: Using Lagrange's interpolation, find the cubic polynomial for the data given below.

x	0	1	3	4
$f(x)$	– 12	0	6	12

Solution:

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0) +$$

$$\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(x-4)}(6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$$= 1\,[(x-1)(x-3)(x-4)] + 0 - 1\,[(x)(x-1)(x-4)]$$

$$+ 1\,[x(x-1)(x-3)]$$

$$x^2 - 3x - 1x + 3)(x-4)] - 1\,[(x^2-3)(x-4)] + 1\,[(x^2-x)(x-3)]$$

$$= [(x^2 - 4x + 3)(x-4)] - 1\,[x^3 - 4x^2 - x^2 + 4x] + 1\,[x^3 - 3x^2 - x^2 + 3x]$$

$$= [x^3 - 4x^2 - 4x^2 + 16x + 3x - 12 - 1x^3 + 4x^2 + x^2 - 4x + x^3 - 3x^2 - x^2 + 3x]$$

$$\boxed{f(x) = x^3 - 7x^2 + 18x - 12}$$

Example 12: Fit a polynomial for the data given below

x	1	3	5	7
$f(x)$	24	120	336	720

Solution:

$x_0 = 1$	$y_0 = 24$
$x_1 = 3$	$y_1 = 120$
$x_2 = 5$	$y_2 = 336$
$x_3 = 7$	$y_3 = 720$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-3)(x-5)(x-7)}{(1-3)(1-5)(1-7)}(24) + \frac{(x-1)(x-5)(x-7)}{(3-1)(3-5)(3-7)}(120) +$$

$$\frac{(x-1)(x-3)(x-7)}{(5-1)(5-3)(5-7)}(336) + \frac{(x-1)(x-3)(x-5)}{(7-1)(7-3)(7-5)}(720)$$

$$= -0.5\,[(x-3)(x-5)(x-7)] + 7.5\,[(x-1)(x-5)(x-7)]$$

$$-21\,[(x-1)(x-3)(x-7)] + 15\,[(x-1)(x-3)(x-5)]$$

$$= -0.5\,[(x^2-5x-3x+15)(x-7)] + 7.5\,[(x^2-5x-x+5)(x-7)]$$

$$-21\,[(x^2-3x-x+3)(x-7)] + 15\,[(x^2-3x-x+3)(x-5)]$$

$$= -0.5\,[(x^2-8x+15)(x-7)] + 7.5\,[(x^2-6x+5)(x-7)]$$

$$-21\,[(x^2-4x+3)(x-7)] + 15\,[(x^2-4x+3)(x-5)]$$

$$= -0.5\,[x^3-7x^2-8x^2+56x+15x-105]$$

$$+7.5\,[x^3-7x^2-6x^2+42x+5x^3-35]$$

$$-21\,[x^3-7x^2-4x^2+28x+3x-21] + 15\,[x^3-5x^2-4x^2+20x+3x-15]$$

$$= -0.5\,[x^3-15x^2+71x-105] + 7.5\,[x^3-13x^2+47x-35]$$

$$-21\,[x^3-11x^2+31x-21] + 15\,[x^3-9x^2+23x-15]$$

$$= -0.5x^3+7.5x^2-35.5x+52.5+7.5x^3-97.5x^2+352.5-13]$$

$$-262.5-21x^3+231x^2-651x+441+15x^3-135x^2+345x-225$$

$$\boxed{\mathbf{y(x) = x^3+6x^2+11x+6}}$$

Example 13: Find the second degree polynomial fitting the following data.

x	1	2	4
y	4	5	13

Solution:

Given data

$x_0 = 1$	$y_0 = 4$
$x_1 = 2$	$y_1 = 5$
$x_2 = 4$	$y_2 = 13$

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$= \frac{(x - 2)(x - 4)}{(-1)(-3)} (4) + \frac{(x - 1)(x - 4)}{(1)(-2)} (5) + \frac{(x - 1)(x - 2)}{(2)(3)} (13)$$

$$= \frac{1}{6} [8x^2 - 48x + 64 - 15x^2 + 75x - 60 + 13x^2 - 39x + 26]$$

$$= \frac{1}{6} [6x^2 - 12x + 30]$$

$$\boxed{f(x) = \frac{1}{6} [6x^2 - 12x + 30]}$$

Example 14: Given $f(2) = 5$; $f(2.5) = 5.5$ find the linear interpolating polynomial using lagrange interpolation.

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x	2	2.5
$f(x)$	5	5.5

Solution:

$$y = f(x) = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

$x_0 = 2$	$y_0 = 5$
$x_1 = 2.5$	$y_1 = 5.5$

$$f(x) = \frac{x - 2.5}{2 - 2.5}(5) + \frac{x - 2}{2.5 - 2}(5.5)$$

$$= \frac{(x - 2.5)}{(-0.5)}(5) + \frac{x - 2}{(0.5)}(5.5)$$

$$= -10(x - 2.5) + 11(x - 2)$$

$$= -10x + 25 + 11x - 22$$

$\boxed{y = x + 3}$ is the required polynomial.

Example 1: Apply Lagrange's formula inversely to obtain the root of the equation $f(x) = 0$ given that $f(0) = -4$, $f(1) = 1$, $f(3) = 29$ and $f(4) = 52$

Solution:

Given that $x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 4$

$$y_0 = -4,\ Y_1 = 1,\ y_2 = 29,\ y_3 = 52$$

To find x such that $f(x) = 0$.

Applying Lagrange's interpolation formula inversely, we get

$$x = \frac{(y - y_1)(y - y_2)(y - y_3)}{(y_0 - y_1)(y_0 - y_2)(y_0 - y_3)} x_0 + \frac{(y - y_0)(y - y_2)(y - y_3)}{(y_1 - y_0)(y_1 - y_2)(y_1 - y_3)} x_1$$

$$+ \frac{(y - y_0)(y - y_1)(y - y_3)}{(y_2 - y_0)(y_2 - y_1)(y_2 - y_3)} x_2 + \frac{(y - y_0)(y - y_1)(y - y_2)}{(y_3 - y_0)(y_3 - y_1)(y_3 - y_2)} x_3$$

Using the given data and $y = 0$, we have

$$x = \frac{(-1)(-29)(-52)}{(-5)(-23)(-56)}(0) + \frac{4(-29)(-52)}{5 \times (-28)(-51)}(1)$$

$$+ \frac{4(-1)(-52)}{33 \times (28)(-23)}(3) + \frac{4 \times (-1)(-29)}{(56)(51)(23)}(4)$$

$\Rightarrow x = 0.8448 - 0.0294 + 0.0071$

$\boxed{\therefore x = 0.8225}$

Example 2: Given the data

x :	3	5	7	9	11
y :	6	24	58	108	175

Find the value of x corresponding to $y = 100$

Solution:

Given $x_0 = 3,\ x_1 = 5,\ x_2 = 7,\ x_3 = 9,\ x_4 = 11$

$y_0 = 6,\ y_1 = 24,\ y_2 = 58,\ y_3 = 108,\ y_4 = 174$

By Lagrange's formula for inverse interpolation, we have

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} x_0$$

$$+ \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)} x_2$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_1)(y_3-y_4)} x_3$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} x_4$$

Using the given data and $y = 100$, we get

$$x = \frac{(76)(42)(-8)(-74)}{(-18)(-52)(-102)(-168)}(3) + \frac{(94)(42)(-8)(-74)}{(18)(-34)(-84)(-150)}(5)$$

$$+ \frac{(94)(76)(-8)(-74)}{(52)(34)(-50)(-116)}(7) + \frac{(94)(76)(42)(-74)}{(102)(84)(50)(-66)}(9)$$

$$+ \frac{(94)(76)(42)(-8)}{(168)(150)(116)(66)}(11)$$

$\Rightarrow x = 0.3534 - 1.5155 + 2.8870 + 7.0676 - 0.1369$

$\boxed{\mathbf{x = 8.656}}$

$\therefore$ The value of x corresponding to $y = 100$ is 8.656.

4.4 NEWTON'S DIVIDED DIFFERENCE METHOD (for unequal Intervals)

Properties of divided differences

- The divided differences are symmetrical in all their arguments.
- The n^{th} divided differences of a polynomial of the n^{th} degree are constant.

The divided difference operator is

Formula

$$f(x) = f(x_0) + (x - x_0)\, \Delta_1 f(x_0) + (x - x_0)(x - x_1) \Delta_1^2 f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\, \Delta_1^3 f(x_0) + \ldots$$

Example 1: Find the divided difference table for the following data

x	2	5	10
y	5	29	109

Solution:

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$
2	5		
		$\frac{29-5}{5-2} = 8$	
5	29		$\frac{16-8}{10-2} = 1$
		$\frac{109-29}{10-5} = 16$	
10	109		

Example 2: Form the divided difference Table

x	0	1	2	4	5
$f(x)$	1	14	15	5	6

Solution:

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$	$\Delta_4^4 f(x)$
0	1	$\frac{14-1}{1-0}=13$			
1	14	$\frac{15-14}{2-1}=1$	$\frac{1-13}{2-0}=-6$	$\frac{1+5}{3}=2$	
2	15	$\frac{5-15}{4-2}=-5$	$\frac{-5-1}{4-1}=-2$	$\frac{-2+6}{4-0}=1$	$\frac{1-1}{5-0}=0$
4	5	$\frac{6-5}{5-4}=1$		$\frac{2+2}{5-1}=1$	
5	6				

Example 3: Create the divided difference table

x	1	2	4	6
$f(x)$	– 26	12	256	844

Solution:

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$
1	– 26	$\frac{12+26}{2-1}=38$		
2	12	$\frac{256-12}{4-2}=122$	28	3
4	256	294	43	
6	844			

Example 4: Find the divided difference table

x	0	2	3	4	7	8
$y = f(x)$	4	26	58	112	460	668

Solution:

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$	$\Delta_1^4 f(x)$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		21		
		202			
8	668				

Example 5: Form the divided difference table.

x	0	1	3	4
$f(x)$	1	4	40	85

Solution:

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^2 f(x)$
0	1			
		3		
1	4		5	
		18		1
3	40		9	
		45		
4	85			

Example 6: Form the divided difference table.

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

Solution:

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$	$\Delta_1^4 f(x)$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

Example 7: Use Newton's divided difference formula to fit a polynomial to the data and hence find y when $x = 8$

x	3	7	9	10
y	168	120	72	63

Solution:

Given

$x_0 = 3$	$y_0 = 168$
$x_1 = 7$	$y_1 = 120$
$x_2 = 9$	$y_2 = 72$
$x_3 = 10$	$y_3 = 63$

By Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0) \Delta_1 f(x_0) + (x - x_0)(x - x_1)$$

$$\Delta_1^2 + f(x_0) + (x - x_0)(x - x_1)(x - x_2) \Delta_1^3 f(x_0) + \ldots$$

Divided difference table

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$
3 7 9 10	168 120 72 63	– 12 – 24 – 9	– 2 5	1

Polynomial

$$y = f(x) = 168 + (x - 3)(-12) + (x - 3)(x - 7)(-2) + (x - 3)(x - 7)(x - 9)(1)$$

$$= 168 - 12x + 36 + (x^2 - 7x - 3x + 21)(-2) + (x^2 - 7x - 3x + 21)(x - 9)$$

$$= 168 - 12x + 36 - 2x^2 + 20x - 42 + x^3 - 10x^2 + 21x - 9x^2 + 90x - 189$$

$$\boxed{y = f(x) = x^3 - 21x^2 + 119x - 7}$$

when $x = 8$

$y = f(x) = 8^3 - 21\,(8)^2 + 119\,(8) - 27$

$= 512 - 21\,(64) + 952 - 27$

$\boxed{f(8) = 93}$

Example 8: Use Newton's divided difference formula, find *y* when *x* = 6 for the following data

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

Solution:

Given:

$x_0 = 0$	$f(x_0) = 4$
$x_1 = 2$	$f(x_1) = 26$
$x_2 = 3$	$f(x_2) = 58$
$x_3 = 4$	$f(x_3) = 112$
$x_4 = 7$	$f(x_4) = 466$
$x_5 = 9$	$f(x_5) = 922$

Divided difference table

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$
0	4	11		
2	26	32	7	
3	58	54	11	1
4	112	118	16	
7	466	228	22	1
9	922			

By Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)\,\Delta_1 f(x_0) + (x - x_0)(x - x_1)\,\Delta_1^2 \times f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\,\Delta_1^3 f(x_0) + \ldots$$

$$y = f(0) = 4 + (6 - 0)\,11 + (6 - 0)(6 - 2)\,7 + (6 - 0)(6 - 2)(6 - 3)$$

$$= 4 + 66 + (24 \times 7) + 72$$

$$\boxed{f(0) = 310}$$

Example 9: If $f(0) = 0$, $f(1) = -0$, $f(2) = -12$, $f(4) = 0$, $f(5) = 600$ and $f(7) = 7308$. Find a polynomial that satisfies this data using Newton's divided difference interpolation formula. Hence find $f(0)$ and $f'(0)$.

x	**0**	**1**	**2**	**4**	**5**	**7**
$y = f(x)$	**0**	**0**	**– 12**	**0**	**600**	**7308**

Solution:

Given data

$x_0 = 0$	$f(x_0) = 0$
$x_1 = 1$	$f(x_1) = 0$
$x_2 = 2$	$f(x_2) = -12$
$x_3 = 4$	$f(x_3) = 0$
$x_4 = 5$	$f(x_4) = 600$
$x_5 = 7$	$f(x_5) = 7308$

By Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)\, \Delta_1 f(x_0) + (x - x_0)(x - x_1)\, \Delta_1^2 f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\, \Delta_1^3 f(x_0) + \ldots$$

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$	$\Delta_4^4 f(x)$	$\Delta_1^5 f(x)$
0	0	0	− 6	3	9	1
1	0	− 12	6	48	16	
2	− 12	6	198	144		
4	0	600	918			
5	600	3354				
7	7308					

Polynomial

$$y = f(x) = 0 + (x - 0)(0) + (x - 0)(x - 1)(-6) + (x - 0)(x - 1)(x - 2)\,3$$

$$+ (x - 0)(x - 1)(x - 2)(x - 4)\,9 + (x - 0)(x - 1)(x - 2)(x - 4)(x - 5)(1)$$

$$= x(x - 1)(-6) + (x)(x - 1)(x - 2)\,3 + (x)(x - 1)(x - 2)(x - 4)\,9$$

$$+ (x)(x - 1)(x - 2)(x - 4)(x - 5)$$

$$= (x^2 - x)(-6) + 3(x)(x^2 - 2x - x + 2) + (x)(x^2 - 2x - x + 2)$$

$$(x - 4)\,9 + (x)(x^2 - 2x - x + 2)(x^2 - 5x - 4x + 20)$$

$$= -6x^2 + 6x + 3x^3 - 6x^2 - 3x^2 + 6x + (x^3 - 2x^2 - x^2 + 2x)$$

$$(x - 4)\,9 + (x^4 - 5x^3 - 4x^3 + 20x^2 - 2x^3 + 10x^2$$

$$+ 8x^2 - 40x - x^3 + 5x^2 + 4x^2 - 20x + 2x^2 - 10x - 8x + 40)$$

$$= -6x^2 + 6x + 3x^3 - 6x^2 - 3x^2 + 6x + (x^4 - 2x^4 - x^3 + 2x^2 - 4x^3$$

$$+ 8x^2 + 4x^2 - 8x)\,9 + x^5 - 5x^4 - 4x^4 + 20x^3$$

$$- 2x^4 + 10x^3 + 8x^3 - 40x^2 - x^4 + 5x^3 + 4x^3 - 20x^2$$

$$+ 2x^3 - 10x^2 - 8x^2 + 40x$$

Simplifying we get,

$$f(x) = x^5 - 3x^4 - 11x^3 + 33x^2 - 20x$$

$$f(6) = (6)^5 - 3\,(6)^4 - 11\,(6)^3 + 33\,(6)^2 - 20\,(6)^2$$

$$\boxed{\mathbf{f(6) = 2580}}$$

$$f'(x) = 5x^4 - 12x^3 + 33x^2 + 66x - 20$$

$$f'(6) = 5\,(6)^4 - 12\,(6)^3 + 33\,(6)^2 + 66\,(6) - 20$$

$$\boxed{\mathbf{f(6) = 5452}}$$

Example 10: Using Newton's divided difference formula, find $u(3)$ given $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$.

[A.U. Apr/May 2004]

Solution:

We from the divided difference table, since the intervals are unequal

x	$u(x)$	$\Delta u(x)$	$\Delta^2 u(x)$	$\Delta^3 u(x)$
1	-26			
		$\frac{12+26}{2-1} = 38$		
2	12		$\frac{122-38}{4-1} = 28$	
		$\frac{256-12}{4-2} = 122$		$\frac{43-28}{6-1} = 3$
4	256		$\frac{294-122}{6-2} = 43$	
		$\frac{844-256}{6-4} = 294$		
6	844			

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x - x_0)\,\Delta_1 f(x_0) + (x - x_0)(x - x_1)\,\Delta_1^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,\Delta_1^3 f(x_0) + \ldots$$

Here,

$$u(x) = u(x_0) + (x - x_0) \Delta_1 u(x_0) + (x - x_0)(x - x_1) \Delta_1^2 u(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2) \Delta_1^3 u(0) + \ldots$$

$$x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 6$$

$$u(x_0) = -26, \Delta_1 u(x_0) = 38, \Delta_1^2 u(x_0) = 28, \Delta_1^3 u(x_0) = 3$$

$$\therefore u(x) = -26 + (x - 1)\,38 + (x - 1)(x - 2)\,28 + (x - 1)(x - 2)(x - 4)\,3$$

$$u(3) = -26 + (2)(38)(2)(1)(28) + (2)(1)(-1)(3)$$

$$= -26 + 76 + 56 - 6$$

$$\boldsymbol{u(3) = 100}$$

Example 11: Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula also find $f(3)$.

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

[A.U N/D 2004, N/D 2011, M/J 2014, A/M 2017 R13]

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{(-1)-(-4)} = -404$			
-1	33		$\frac{-28-(-404)}{0-(-4)} = 94$		
		$\frac{5-33}{0-(-1)} = -28$		$\frac{10-94}{2-(-4)} = -14$	
0	5		$\frac{2-(-28)}{2-(-1)} = 10$		$\frac{13+14}{5-(-4)} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5-(-1)} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x - x_0) \Delta_1 f(x_0) + (x - x_0)(x - x_1) \Delta_1^2 f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\,\Delta_1^3 f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta_1^4 f(x_0)$$

Here, $x_0 = -4, \quad x_1 = -1, \quad x_2 = 0, \quad x_3 = 2, \quad x_4 = 5$

$f(x_0) = 1245, \quad \Delta_1 f(x_0) = -404, \quad \Delta_1^2 f(x_0) = 94,$

$\Delta_1^3 f(x_0) = -14, \quad \Delta_1^4 f(x_0) = 3$

$$f(x) = 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$

$$+ (x + 4)(x + 1)(x)(-14) + (x + 4)(x + 1)(x)(x - 2)(3)$$

$$= 1245 - 404x - 1616 + (94)[x^2 + 5x + 4] - 14x[x^2 + 5x + 4]$$

$$+ 3x[x^3 - 2x^2 + 5x^2 - 10x + 4x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x[x^3 + 3x^2 - 6x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x^4 + 9x^3 - 18x^2 - 24x$$

$$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

$$f(3) = 3(3^4) - 5(3)^3 + 6(3^{2)} - 14(3) + 5$$

$$\mathbf{f(3) = 125}$$

4.5 NUMERICAL INTEGRATION (Trapezridal and Simpson's Rule)

Introduction

The process of finding the value of a definite integral $\int_a^b y\,dx$ from a set of valus (x_n, y_n), $n = 0, 1, 2, \ldots$ where $x_0 = a$, $x_n = b$ of the function $y = f(x)$ is called Numberical Interation.

Note

The order of error in trapezoidal rule is h^2.

Trapezoidal Rule

$$I = \frac{h}{2}\left[\begin{pmatrix}\text{Sum of first} \\ \text{and last ordinates}\end{pmatrix} + \begin{pmatrix}\text{Sum of} \\ \text{remaining ordinates}\end{pmatrix}\right]$$

Simpson's 1/3 Rule

This rule is applicable only when the number of intervals is **even**.

$$I = \frac{h}{3}\left[\begin{pmatrix}\text{Sum of first and} \\ \text{last ordinates}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of} \\ \text{odd ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of} \\ \text{even ordinates}\end{pmatrix}\right]$$

Order	Error
h^4	$\lvert E\rvert < \frac{(b-a)\,h^4}{180} M$

Simpson's 3/8 Rule

This rule is applicable only when the number of intervals is **multiple of 3**.

$$I = \frac{3h}{8}\left[\begin{pmatrix}\text{Sum of first and} \\ \text{last ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of} \\ \text{multiple of 3}\end{pmatrix} + 3\begin{pmatrix}\text{Sum of} \\ \text{remaining} \\ \text{values}\end{pmatrix}\right]$$

Order:	Error
h^2	$\frac{-3}{8} h^2 y^2 (\zeta)$

WORKED EXAMPLES

Example 1: By dividing the Range into 8 equal parts evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using trapezoidal rule and simpson's 1/3 rule and simpson's 3/8 rule. *[AU N/D 2021]*

Solution:

Given:

$$I = \int_{-1}^{1} \frac{dx}{1+x^2}$$

$$n = 8, a = -1, b = 1$$

$$h = \frac{b - a}{n}$$

$$h = \frac{1 + 1}{8} = \frac{2}{8} = \frac{1}{4} = 0.25$$

From the table

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
$y = \frac{1}{1 + x^2}$	0.5	0.64	0.8	0.9412	1	0.12	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_n

(i) Trapezoidal rule

$$I = \frac{h}{2}\left[\begin{pmatrix}\text{Sum of first and} \\ \text{last ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of} \\ \text{remaining ordinates}\end{pmatrix}\right]$$

$$= \frac{0.25}{2} [(0.5 + 0.5) + 2 (0.64 + 0.8 + 0.9412 + 1 + 0.9412 + 0.8 + 0.64)]$$

$$= 0.125 [(1) + 2 (5.7624)]$$

$$\boxed{\boldsymbol{I = 1.5656}}$$

(ii) Simpson's 1/3 rule

$$I = \frac{h}{3}\left[\begin{pmatrix}\text{Sum of first} \\ \text{and} \\ \text{last ordinates}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of} \\ \text{odd ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of} \\ \text{even ordinates}\end{pmatrix}\right]$$

$$= \frac{0.25}{3} [(0.5 + 0.5) + 4 (0.64 + 0.9412 + 0.9412) + 2 (0.8 + 1 + 0.8)]$$

$$= 0.0833 [(1) + 4 (3.1624) + 2 (2.6)]$$

$$\boxed{\boldsymbol{I = 1.57017}}$$

(iii) Simpson's 3/8 rule

Since the interval is even in the given problem. ∴ Simpson's 3/8 rule not applicable.

Example 2: Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by taking $h = 1$ using Simpson's 3/8 rule.

Solution:

Given

$$I = \int_0^6 \frac{dx}{1+x^2}$$

$$h = \frac{b-a}{3}$$

$$h = 1, a = 3, b = 6$$

Simpson's 3/8 rule

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.03846	0.02702
	y_0	y_1	y_2	y_3	y_4	y_5	yn

$$I = \frac{3h}{8}\left[\begin{pmatrix}\text{Sum of first}\\ \text{and}\\ \text{last ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{multiple of 3}\end{pmatrix} + 3\begin{pmatrix}\text{Sum of}\\ \text{remaining}\\ \text{values}\end{pmatrix}\right]$$

$$= \frac{3 \times 1}{8}\,[(1 + 0.02702) + 2\,(0.1) + 3\,(0.5 + 0.2 + 0.0588 + 0.03846)$$

$$= \frac{3}{8}\,[1.02702 + 0.2 + 3\,(0.79726)]$$

$$\boxed{\boldsymbol{I = 1.357085}}$$

Example 3: Dividing the Range into 10 equal parts evaluate $\int_0^{\pi/2} \sin dx$ using Simpson's 1/3 Rule.

Solution:

$$I = \int_0^{\pi/2} \sin x \, dx$$

$$y = \sin x$$

$$a = 0,\ b = \frac{\pi}{2}\ n = 10$$

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{10}$$

$$h = \frac{\pi}{20}$$

x	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
$y = \sin x$	0	0.1564	0.3090	0.4539	0.5877	0.7071	0.8090	0.8910	0.9510	0.9876	1

$$I = \frac{h}{3}\left[\begin{pmatrix}\text{Sum of first and}\\ \text{last ordinates}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{odd ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{even}\\ \text{ordinates}\end{pmatrix}\right]$$

$$= \frac{\pi}{20 \times 3}\left[\begin{array}{l}(0+1) + 4\,(0.1564 + 0.4539 + 0.7071 + 0.8910 + 0.9876)\\ +\,2\,(0.3090 + 0.5877 + 0.8090 + 0.9510)\end{array}\right]$$

$$= \frac{\pi}{60}\,[(1) + 12.784 + 5.3134]$$

$$= 0.9999$$

$$\boxed{I = 1}$$

Example 4: Evaluate $\int_{4}^{5.2} \log_e x\, dx$ (or) $\int_{4}^{5.2} \ln x\, dx$ using trapezoidal Simpson's 1/3 Rule and Simpson's 3/8 rule.

Solution:

Given:

$$I = \int_{4}^{5.2} \log_e x \text{ or } \int_{4}^{5.2} \ln x\, dx$$

$$y = \log_e x \text{ or } \ln x$$

$$h = \frac{b-a}{n}$$

$$= \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$$

Form the table

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.4816	1.5201	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_7

Form for Trapezoidal Rule

$$I = \frac{h}{2}\left[\begin{pmatrix}\text{Sum of first and}\\ \text{last ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of remaining}\\ \text{ordinates}\end{pmatrix}\right]$$

$$= \frac{0.2}{2}\,[(1.3863 + 1.6487) + 2\,(1.4351 + 1.4816 + 1.5686 + 1.5261 + 1.6094)]$$

$$\boxed{I = 1.8277}$$

Formula for Simpson's 1/3 Rule

$$I = \frac{h}{3}\left[\begin{pmatrix}\text{Sum of first and}\\ \text{last ordinates}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{odd values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{even values}\end{pmatrix}\right]$$

$$= \frac{0.2}{3}\,[(1.3863 + 1.6487) + 4\,(1.4354 + 1.5261 + 1.6094)$$

$$+ 2\,(1.4816 + 1.5686)]$$

$$\boxed{I = 1.8279}$$

Simposon's 3/8 rule

$$I = \frac{3h}{8}\left[\begin{pmatrix}\text{Sum of first and}\\ \text{last ordinates}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{multiple of 3}\end{pmatrix} + 3\begin{pmatrix}\text{Sum of}\\ \text{remaining}\\ \text{values}\end{pmatrix}\right]$$

$$= \frac{3\,(0.2)}{8}\,[(1.3863 + 1.6487) + 2\,(1.5261) + 3\,(1.4351 + 1.4816$$

$$+ 1.5686 + 1.6094]$$

$$\boxed{I = 1.8278}$$

Example 5: Evaluate $\int_0^{\pi} \frac{\sin x}{x}\,dx$ by dividing the interval into 6 equal parts using simpson's 1/3 rule and 3/8 rule.

Solution:

Given:

$$I = \int_0^{\pi} \frac{\sin x}{x}\,dx$$

$$y = \frac{\sin x}{x},\ n = 6$$

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

Form the table

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$
y	1	0.9549	0.8270	0.6366	0.4135	0.1910	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1/3 Rule

$$I = \frac{\pi/6}{3} [(1 + 0) + 4 (0.9549 + 0.6366 + 0.1910) + 2 (0.8270 + 0.4135)]$$

$$= \frac{\pi}{8} [(1) + 4 (1.7825) + 2 (1.2405)]$$

$$\boxed{I = 1.8520}$$

Simpson's 3/8 rule

$$I = \frac{3 \times \pi/6}{8} \frac{3h}{8} \left[\begin{pmatrix} \text{Sum of first} \\ \text{and} \\ \text{last ordinates} \end{pmatrix} + 2 \begin{pmatrix} \text{Sum of} \\ \text{multiple} \\ \text{of 3} \end{pmatrix} + 3 \begin{pmatrix} \text{Sum of} \\ \text{remaining} \\ \text{values} \end{pmatrix} \right]$$

$$= \frac{3 \times \pi/6}{8} [(1 + 0) + 2 (0.6366) + 3 (0.9549 + 0.8270 + 0.4135 + 0.1910)]$$

$$= \frac{3\pi}{48} [(1) + 2 (0.6366) + 3 (2.3864)]$$

$$\boxed{I = 1.8520}$$

Example 6: Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the interval into 4 equal parts using trapezoidal rule.

Solution:

$$I = \int_0^1 e^{-x^2} dx$$

$$y = e^{-x^2}, \ n = 4$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$$

x	0	0.25	0.5	0.75	1
y	1	0.9394	0.7788	0.5698	0.3678
	y_0	y_1	y_2	y_3	y_4

Trapezoidal Rule

$$I = \frac{0.25}{2}\ [(1 + 0.3678) + 2\ (0.9394 + 0.7788 + 0.5698)]$$

$$\boxed{I = 0.7428}$$

Example 7: Evaluate $\int_0^2 \frac{dx}{x^2 + x + 1}$ correct to three decimal places dividing the range into 8 equal parts using Simpson's 1/3 Rule.

Solution:

Given:

$$I = \int_0^2 \frac{dx}{x^2 + x + 1}$$

$$y = \frac{1}{x^2 + x + 1}$$

$$h = \frac{b-a}{n} = \frac{2-0}{8} = \frac{2}{8} = \frac{1}{4} = 0.25$$

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	1	0.7619	0.5714	0.4324	0.3333	0.2623	0.2105	0.1720	0.1429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_n

Simpson's 1/3 rule

$$I = \frac{h}{3}\left[\begin{pmatrix}\text{Sum of first and}\\ \text{last ordinates}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{odd values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{even values}\end{pmatrix}\right]$$

$$= \frac{h}{3}\left[\begin{array}{l}(1 + 0.1429) + 4\,(0.7619 + 0.4324 + 0.2623 + 0.1720)\\ + 2\,(0.5714 + 0.3333 + 0.2105)\end{array}\right]$$

$$= \frac{0.25}{3}\,(9.8877)$$

$$\boxed{I = 0.8239}$$

Example 8: Using Trapezoidal rule evaluate $\int_{0.6}^{2} y\,dx$ from the following table.

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

Solution:

$$h = 0.2$$

Trapezoidal rule

$$\int_{0.6}^{2} ydx = \frac{h}{2}\,[(y_0 + y_1) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)]$$

$$= \frac{0.2}{2}\,[13.68 + 2\,(1.58 + 2.03 + 4.32 + 6.25 + 8.36 + 10.23)]$$

$$= 0.1\,[79.22]$$

$$\boxed{I = 7.922}$$

Example 9: Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3}\,dx$ using Simpson's 1/3 rule with $h = 0.25$

Solution:

x	0	0.25	0.5	0.75	1.0
$y = \frac{x^2}{1+x^3}$	0	0.06154	0.22222	0.39560	0.5000
	y_0	y_1	y_2	y_3	y_4

Simpson's 1/3 Rule

$$\int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{h}{3}[(y_0 + y_4) + 2y_2 + 4\,(y_1 + y_3)]$$

$$= \frac{0.25}{3}[(0 + 0.5) + 2\,(0.22222) + 4\,(0.06154 + 0.39560)]$$

$$= \frac{0.25}{3}[0.5 + 0.44444 + 1.82856]$$

$$\boxed{I = 0.231083}$$

Example 10: Given $e^0 = 1, e^1 = 2.72, e^2 = 7.39, e^3 = 20.09$, $e^4 = 54.60$. Use Simpson's rule to find an approximate value of $\int_0^4 e^x\,dx$. Also compare your result with the exact value of the integral.

Solution:

x	0	1	2	3	4
$y = e^x$	1	2.72	7.39	20.09	54.60
	y_0	y_1	y_2	y_3	y_4

Simpson's rule

$$\int_0^4 e^x\,dx = \frac{h}{3}[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)]$$

$$= \frac{1}{3}[55.60 + 14.78 + 4(2.72 + 20.09)]$$

$$= \frac{1}{3}[70.38 + 91.24]$$

$$\boxed{I = 53.8733}$$

Actual Integration

$$I = \int_{\infty}^{4} e^x d_n = [e^x]_0^4 = [e^4 - e^0]$$

$$\boxed{I = 53.8733}$$

Example 11: Evaluate $I = \int_0^1 \frac{1}{1+x}dx$, correct to three decimal places using trapezoidal rule with $h = 0.25$

x	0	0.25	0.8	0.75	1
$f(x)$	1	0.8	0.6667	0.5714	0.5
	y_0	y_1	y_2	y_3	y_4

Solution:

Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[(1 + 0.5) + 2(0.8 + 0.666 + 0.5714)]$$

$$= \frac{0.25}{2}(5.5748)$$

$$\boxed{I = 0.696}$$

Example 12: Evaluate, $\int_0^6 \frac{1}{1+x^2}\,dx$ by (i) Trapezoidal rule

(ii) Simpson's rule. Also check up the results by actual integration.
[AU N/D 2024, A.U CBT N/D 2010, CBT A/M 2011, Tvli A/M 2011, N/D 2013]

Solution:

Here, $b-a=6-0=6.$

Divide into 6 equal parts

$h=\frac{6}{6}=1.$ Hence, the table is

$x:$	0	1	2	3	4	5	6
$\frac{1}{1+x^2}=f(x):$	1.00	0.500	0.200	0.100	0.058824	0.038462	0.027027
	y_0	y_1	y_2	y_3	y_4	y_5	$y_6=y_n$

There are 7 ordinates $(n=6)$ We can use all the formula.

(i) By Trapezoidal rule,

$$I=\int_0^6 \frac{1}{1+x^2}\,dx=\frac{1}{2}\left[(1+0.027027)+2\,(0.5+0.2+0.1+0.058824+0.038462)\right]$$

$$=1.41079950$$

(ii) By Simpson's one-third rule,

$$\therefore\ I=\frac{h}{3}\left[(y_0+y_6)+2\,(y_2+y_4)+4\,(y_1+y_3+y_5)\right]$$

$$I=\frac{1}{3}\left[(1+0.027027)+2\,(0.2+058824)+4\,(0.5+0.1+0.038462)\right]$$

$$=\frac{1}{3}(1.027027+0.517648+2.553848)$$

$$=1.36617433$$

(iii) By actual integration,

$$I = \int_0^6 \frac{1}{1+x^2}\,dx = (\tan^{-1} x)_0^6 = \tan^{-1} 6 \;= 1.40564765$$

Conclusion: Here, the value by trapezoidal rule is closer to the actual value than the value by Simpson's rule.

Example 13: A rocket is launched from the ground. Its acceleration during the first 80 seconds is given by the following data:

***t* :**	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**
***a* :**	**30**	**31.63**	**33.44**	**35.47**	**37.75**	**40.33**	**43.29**	**46.69**	**50.87**

Find the velocity at $t = 80$ seconds using Simpson's rule.

Solution:

$v =$ velocity and $a =$ acceleration

$$\therefore\ a = \frac{dv}{dt} \Rightarrow v = \int a\,dt$$

$\therefore$ The velocity at $t = 80$ seconds is given by

$$v = \int_0^{80} a\,dt. \text{ Here } h = 10$$

$$= \frac{10}{3}\,[\,(30 + 50.67) + 4\,(31.63 + 35.47 + 40.33 + 46.69)$$

$$+ 2\,(33.44 + 37.75 + 43.29)\,]$$

$$= 3086.77$$

Example 14: Using Simpson's rule of integration, find the area bounded by the x-axis, the lines $x=1$ and $x=4$ and the curve which is drawn passes through the points given below. Also find the volume of solid of revolution got by revolving this area about the $x-$ axis.

x :	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y :	2	2.4	2.7	2.8	3.0	2.6	2.1

Solution:

Given the ordinates $y_0 = 2, y_1 = 2.4, y_2 = 2.7,$

$y_3 = 2.8, y_4 = 3, y = 2.6, y_6 = 2.1$

By Simpson's 1/3 rule, the required area

$$\int_1^4 ydx = \frac{h}{3}\left[(y_0 + y_6) + 4\,(y_1 + y_3 + y_5) + 2\,(y_2 + y_4) \right]$$

$$= \frac{0.5}{3}\,[4.1 + 4\,(2.4 + 2.8 + 2.6) + 2\,(2.7 + 3)]$$

$$= 7.7833 \text{ square units.}$$

$$\text{Volume } = \pi \int_1^4 y^2\,dx$$

$$\int_1^4 y^2\,dx = \frac{0.5}{3}\left[\left[2^2 + (2.1)^2 \right] + 4\left[(2.4)^2 + (2.8)^2 + (2.6)^2 \right] + \left[(2.7)^2 + 3^2 \right]\right]$$

$$= 20.3916$$

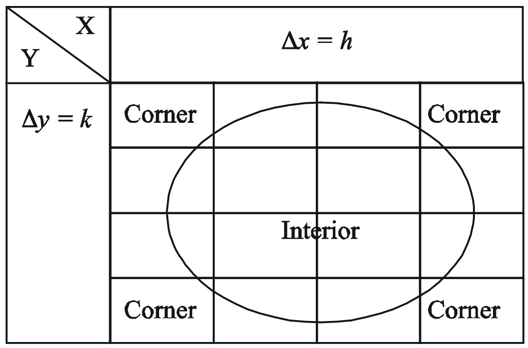
$$\text{Required volume } = \pi \times 20.3916$$

$$= 64.088 \text{ cubic units}$$

4.6 DOUBLE INTEGRATION

(i) Trapezoidal rule

(ii) Simpson's rule



Formula for Trapezoidal rule

$$I = \frac{hk}{4}\left[1\begin{pmatrix}\text{Sum of}\\ \text{Corner values}\\ \text{in the first}\\ \text{boundary}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{Remaining}\\ \text{values in the}\\ \text{first boundary}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{interior}\\ \text{values}\\ \text{(inside)}\end{pmatrix}\right]$$

Formula for Simpson's rule: (For all Except 5×5)

$$I = \frac{hk}{9}\left[1\begin{pmatrix}\text{Sum of}\\ \text{Corner values}\\ \text{in the first}\\ \text{boundary}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{Remaining}\\ \text{values in}\\ \text{the first}\\ \text{boundary}\end{pmatrix} + 16\begin{pmatrix}\text{Sum of}\\ \text{interior}\\ \text{values}\end{pmatrix}\right]$$

$$[h = k = 0.25]$$

WORKED EXAMPLES

Example 1: Evaluate $\int_0^1 \int_0^1 e^{x+y}\, dx\, dy$ **using the trapezoidal and Simpson's Rule by taking $h = k = 0.5$ (or) $\Delta x = \Delta y = 0.5$**

Solution:

$$I = \int_0^1 \int_0^1 e^{x+y}\, dx\, dy$$

Limits

$$x \Rightarrow 0 \text{ to } 1$$

$$y \Rightarrow 0 \text{ to } 1$$

$$h = 0.5$$

$$k = 0.5$$

$$f(x, y) = e^{x+y}$$

(i) Trapezoidal rule

$$I = \frac{hk}{4}\left[1\begin{pmatrix}\text{Sum of}\\ \text{Corner values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{Remaining}\\ \text{values in the}\\ \text{first boundary}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{interior}\\ \text{values}\end{pmatrix}\right]$$

Form the table

Y \ X	**0**	**0.5**	**1**
0	(1)	1.64	(2.71)
0.5	1.64	2.71	4.48
1	(2.71)	4.48	(7.38)

$$I = \frac{0.5 \times 0.5}{4}[1\,(1 + 2.71 + 7.38 + 2.71) + 2\,(1.64 + 4.48 + 4.48 + 1.64) + 4\,(2.17)]$$

$$= 0.0625\,[13.8 + 24.48 + 10.84]$$

$$\boxed{I = 3.0762}$$

(ii) Simpson's rule

$$I = \frac{hk}{9}\left[1\begin{pmatrix}\text{Sum of}\\ \text{Corner values}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{Remaining}\\ \text{values}\end{pmatrix} + 16\begin{pmatrix}\text{Sum of}\\ \text{interior}\\ \text{values}\end{pmatrix}\right]$$

$$= \frac{0.5 \times 0.5}{9}[1\,(1 + 2.71 + 2.71 + 7.38) + 4\,(1.64 + 1.64 + 4.48 + 4.48) + 16\,(2.71)]$$

$$\boxed{I = 2.9543}$$

Example 2: Evaluate $\int_1^2 \int_3^4 \frac{dx\,dy}{(x+y)^2}$ take $\Delta x = \Delta y = 0.5$ by Trapezoidal rule and Simpson's rule.

Solution:

$$I = \int_1^2 \int_3^4 \frac{dx\,dy}{(x+y)^2}$$

Limits

$$x \Rightarrow 3 \text{ to } 4$$

$$y \Rightarrow 1 \text{ to } 2$$

$$\Delta x = 0.5$$

$$\Delta y = 0.5$$

$$f(x, y) = \frac{1}{(x+y)^2}$$

$$\Delta x = h$$

$\Delta y = k$

Y \ X	**3**	**3.5**	**4**
1	0.0625	0.0493	0.04
1.5	0.0493	0.04	0.0330
2	0.04	0.0330	0.0277

(i) Trapezoidal rule

$$I = \frac{hk}{4}\left[1\begin{pmatrix}\text{Sum of}\\ \text{Corner values}\\ \text{in the first}\\ \text{boundary}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{Remaining}\\ \text{values in the}\\ \text{boundary}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{interior}\\ \text{values}\end{pmatrix}\right]$$

$$= \frac{0.5 \times 0.5}{4}\left[\begin{matrix}1\,(0.0625 + 0.4 + 0.04 + 0.0277)\\ +\,2\,(0.0493 + 0.0330 + 0.0330 + 0.0493) + 4\,(0.04)\end{matrix}\right]$$

$$= 0.0625\,[0.1702 + 0.3292 + 0.16]$$

$$\boxed{I = 0.0412}$$

(ii) Simpson's rule

$$I = \frac{hk}{9}\left[1\begin{pmatrix}\text{Sum of}\\ \text{Corner values}\\ \text{in the firt}\\ \text{boundary}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{remaining}\\ \text{values in the}\\ \text{boundary}\end{pmatrix} + 16\begin{pmatrix}\text{Sum of}\\ \text{interior}\\ \text{values}\end{pmatrix}\right]$$

$$= \frac{0.5 \times 0.5}{9}\left[\begin{matrix}1\,(0.0625 + 0.04 + 0.04 + 0.0277)\\ +\,4\,(0.0493 + 0.0330 + 0.0330 + 0.0493) + 16\,(0.04)\end{matrix}\right]$$

$$\boxed{\boldsymbol{I = 0.0407}}$$

Example 3: Using Trapezoidal rule evaluate $\displaystyle\int_1^2 \int_1^2 \frac{dx\,dy}{x+y}$ **by taking $h = 0.25$ and $k = 0.5$**

Solution:

$$I = \int_1^2 \int_1^2 \frac{dx\,dy}{x+y}$$

Limits

$$x \Rightarrow 1 \text{ to } 2$$

$$y \Rightarrow 1 \text{ to } 2$$

$$h = 0.25$$

$$k = 0.5$$

$$f(x, y) = \frac{1}{x+y}$$

Y \ X	**1**	**1.25**	**1.5**	**1.75**	**2**
1	0.5	0.4444	0.4	0.3636	0.3333
1.5	0.4	0.3636	0.3333	0.3076	0.2857
2	0.3333	0.3076	0.2857	0.26660	0.25

Trapezoidal rule

$$I = \frac{hk}{4}\left[1\begin{pmatrix}\text{Sum of}\\ \text{Corner values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of}\\ \text{remaining}\\ \text{values}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of}\\ \text{interior}\\ \text{values}\end{pmatrix}\right]$$

$$= \frac{0.25 \times 0.5}{4}\left[\begin{array}{l} 1\,(0.5 + 0.3333 + 0.25 + 0.3333) + 2\,(0.4444 + 0.4 \\ + 0.3636 + 0.2857 + 0.2666 + 0.2857 + 0.3076 + 0.4) \\ + 4\,(0.3636 + 0.3333 + 0.3076) \end{array}\right]$$

$$\boxed{\mathbf{I = 0.34065}}$$

Simpson's rule for 5×5 matrix:

$$I = \frac{hk}{9}\left[1\begin{pmatrix}\text{Sum of}\\ \text{first 4 corner}\\ \text{values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of third}\\ \text{position value}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of second}\\ \text{fourth position}\\ \text{value of}\\ \text{middle value}\end{pmatrix}\right.$$
$$\left. + 8\begin{pmatrix}\text{Sum of}\\ +\text{ values}\end{pmatrix} + 16\begin{pmatrix}\text{Sum of}\\ \text{remaining}\\ \text{values}\end{pmatrix}\right]$$

Example 4: Evaluate $\displaystyle\int_0^1 \int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\,dx\,dy$ using Simpson's rule by taking $h = k = 0.25$

Solution:

$$I = \int_0^1 \int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\,dx\,dy$$

Limits

x varies from 1 to 2, $h = 0.25$

y varies from 0 to 1, $k = 0.25$

$$f(x, y) = \frac{2xy}{(1+x^2)(1+y^2)}$$

Y \ X	**1**	**1.25**	**1.5**	**1.75**	**2**
0	1 0	0	0 2	0	0 1
0.25	0.2353	0.2295	0.2172	0.2027	0.1882
0.5	2 0.4	0.3902	0.3692	0.3446	0.32 2
0.75	0.48	0.4683	0.4431	0.4135	0.384
1	1 0.5	0.4878	0.4615 2	0.4308	0.4 1

Simpson's rule

$$I = \frac{hk}{9}\left[1\begin{pmatrix}\text{Sum of}\\ \text{first 4 corner}\\ \text{values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of values of odd}\\ \text{position in the boundary}\end{pmatrix}\right.$$

$$\left. + 4\begin{pmatrix}\text{Sum of values of}\\ \text{even position +}\\ \text{centre value}\end{pmatrix} + 8\begin{pmatrix}\text{Sum of values}\\ \text{of odd position}\end{pmatrix} + 16\begin{pmatrix}\text{Sum of}\\ \text{values of}\\ \text{even}\\ \text{position}\end{pmatrix}\right]$$

$$= \frac{0.25 \times 0.25}{9}\left[\begin{array}{l} 1\,(0 + 0 + 0.4 + 0.5) + 2\,(0 + 0.32 + 0.4615 + 0.4) \\ + 4\,(0 + 0 + 0.1882 + 0.384 + 0.4308 + 0.4878 \\ \ + 0.48 + 0.2353 + 0.3692) \\ + 8\,(0.3902 + 0.2172 + 0.3446 + 0.4431) \\ + 16\,(0.2295 + 0.2027 + 0.4135 + 0.4683) \end{array}\right]$$

$$= \frac{0.25 \times 0.25}{9}\,[1\,(0.9) + 2\,(1.1815) + 4\,(2.5753) + 8\,(1.3951) + 16\,(1.3140)]$$

$$= \frac{0.0625}{9}\,(45.7490)$$

$$\boxed{\boldsymbol{I = 0.3171}}$$

Example 5: Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$ using Simpson's rule by taking $h = k = 0.1$.

Solution:

$$I = \int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$$

$$f(x, y) = \frac{1}{xy}$$

Limits

x varies from 2 to 2.4

y varies from 1 to 1.4

$$h = 0.1, \; k = 0.1$$

Y \ X	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

Simpson's rule formula

$$I = \frac{hk}{9}\left[1\begin{pmatrix}\text{Sum of}\\ \text{first 4 corner}\\ \text{values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of values}\\ \text{of odd position}\\ \text{in the boundary}\end{pmatrix} + 4\begin{pmatrix}\text{Sum of values of even}\\ \text{position + centre value}\end{pmatrix} + 8\begin{pmatrix}\text{Sum of values of}\\ \text{odd position}\end{pmatrix} + 16\begin{pmatrix}\text{Sum of values of}\\ \text{even position}\end{pmatrix}\right]$$

$$= \frac{0.1 \times 0.1}{9} \left[\begin{array}{l} 1\,(0.5 + 0.4167 + 0.3571 + 0.2976 + 0.3571) \\ + 2\,(0.4167 + 0.4545 + 0.3472 + 0.3247) + 4\,(0.4762 \\ + 0.4348 + 0.3788 + 0.3205 + 0.3106 + 0.3401 \\ + 0.3846 + 0.4545 + 0.3788) \end{array} \right.$$

$$\left. \begin{array}{l} + 8\,(0.4132 + 0.3497 + 0.3623 + 0.3623 + 0.3968) \\ + 16\,(0.4329 + 0.3953 + 0.3344 + 0.3663) \end{array} \right]$$

$$= \frac{(0.1) \times (0.1)}{9} [1\,(1.9285) + 2\,(1.5431) + 4\,(3.4789) + 8\,(1.4860)$$

$$+ 16\,(1.5289)]$$

$$= \frac{(0.1)\,(0.1)}{9} [55.2807]$$

$$\boxed{I = 0.0613}$$

Example 6: Evaluate $\int_1^2 \int_1^2 \frac{dx\,dy}{x+y}$ **with** $h = k = 0.25$ **using simpson's rule.**

Solution:

$$I = \int_1^2 \int_1^2 \frac{dx\,dy}{x+y}$$

$$f(x, y) = \frac{1}{x+y}$$

Limits

x varies from 1 to 2

y varies from 1 to 2

$h = 0.25, k = 0.25$

Y \ X	**1.0**	**1.25**	**1.5**	**1.75**	**2.0**
1.0	0.5	0.444	0.4	0.3636	0.333
1.25	0.444	0.40	0.3636	0.333	0.3077
1.5	0.40	0.3636	0.333	0.3077	0.2857
1.75	0.3636	0.333	0.3077	0.2857	0.2667
2.0	0.333	0.3077	0.2857	0.2667	0.25

Simpson's rule

$$I = \frac{hk}{9}\left[1\begin{pmatrix}\text{Sum of}\\ \text{first 4 corner}\\ \text{values}\end{pmatrix} + 2\begin{pmatrix}\text{Sum of values}\\ \text{of odd position}\\ \text{in the boundary}\end{pmatrix}\right.$$

$$+ 4\begin{pmatrix}\text{Sum of values of}\\ \text{even position + centre}\\ \text{values}\end{pmatrix} + 8\begin{pmatrix}\text{Sum of values of}\\ \text{odd position}\end{pmatrix}$$

$$\left. + 16\begin{pmatrix}\text{Sum of values of}\\ \text{even position}\end{pmatrix}\right]$$

$$= \frac{0.25 \times 0.25}{9}\begin{bmatrix} 1\,(0.5 + 0.3333 + 0.25 + 0.3333) + 2\,(0.4 + \\ 0.2857 + 0.2857 + 0.40) + 4\,(0.444 + \\ 0.3636 + 0.2667 + 0.3077 + 0.3636 + \\ 0.444 + 0.3333) + 8\,(0.3636 + 0.3077 \\ + 0.3077 + 0.3636) + 16\,(0.40 + 0.333) \\ + 0.2857 + 0.3333) \end{bmatrix}$$

$$= \frac{(0.25)\,(0.25)}{9}\begin{bmatrix} 1\,(1.4166) + 2\,(1.3714) + 4\,(3.0973) \\ + 8\,(1.3426) + 16\,(1.3523) \end{bmatrix}$$

$$= \frac{(0.25)\,(0.25)}{9}\,(48.9262)$$

$$\boxed{I = 0.3397}$$

Example 6: Evaluate $\int_1^2 \int_1^2 \frac{1}{x^2+y^2}\, dx\, dy$**, numerically with** $h = 0.2$ **along** x**-direction and** $k = 0.25$ **along** y**–direction.**

[A.U. Nov. 1996, M/J 2012] [A.U A/M 2015 (R8-10)]

Solution:

y \ x	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2**
1	0.5	0.4098	0.3378	0.2809	0.2359	0.2
1.25	0.3902	0.3331	0.2839	0.2426	0.2082	0.1798
1.5	0.3077	0.2710	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1679	0.1524	0.1381	0.125

By Trapezoidal rule

$$\int_1^2 \int_1^2 \frac{1}{x^2+y^2}\, dx\, dy = \frac{hk}{4} \text{ [Sum of values of } f \text{ at the four corners}$$

$$+ 2 \text{ (Sum of the values of } f \text{ at the remaining nodes on the boundary)}$$

$$+ 4 \text{ Sum of the values of } f \text{ at the interior nodes)]}$$

$$= \frac{(0.2)\,(0.25)}{4} [0.5 + 0.2 + 0.125 + 0.2)$$

$$+ 2\,(0.2462 + 0.3077 + 0.3902 + 0.4098 + 0.3378$$

$$+ 0.2809 + 0.2359 + 0.1798 + 0.16 + 0.1416$$

$$+ 0.1381 + 0.1524 + 0.1679 + 0.1838)$$

$$+ 4\,(0.3331 + 0.2839 + 0.2426 + 0.2802 + 0.2710$$
$$+ 0.2375 + 0.2079 + 0.1821 + 0.2221 + 0.1991$$
$$+ 0.1779 + 0.1587)]$$

$$= \frac{(0.2)\,(0.25)}{4}\,[1.025 + 6.6642 + 10.8964] = \frac{(0.2)\,(0.25)}{4\,[18.5856]}$$

$$= 0.2323$$

Example 7: Evaluate, $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$, **by using Trapezoidal rule. Simpson's rule and also by actual integration.**
[A.U A/M 2017 R-8]

Solution:

Divide the range on x and y direction into 2 equal parts and obtain the values of $f = \sin(x+y)$ at each node.

Here, $h = \frac{\pi}{4} = k$

y \ x	**0**	$\frac{\pi}{4}$	$\frac{\pi}{2}$
0	0	0.7071	1
$\frac{\pi}{4}$	0.7071	1	0.7071
$\frac{\pi}{2}$	1	0.7071	0

Case 1: By Trapezoidal rule,

$$I = \frac{\frac{\pi}{4} \times \frac{\pi}{4}}{4}\,[(0 + 1 + 1 + 0) + 2\,(0.7071 + 0.7071 + 0.7071 + 0.7071) + 4\,(1)]$$

$= 0.1542\ (11.6568)$

$= 1.7975$

Case 2: By Simpson's rule,

$$I = \frac{\frac{\pi}{4} \times \frac{\pi}{4}}{9} [(0 + 1 + 1 + 0) + 4\ (0.7071 + 0.7071 + 0.7071 + 0.7071) + 16\ (1)]$$

$= (0.0685)\ (29.3136)$

$= 2.0080$

Case 3: By actual integration,

$$\int_0^{\pi/2} \int_0^{\pi/2} \sin\,(x + y)\ dx\ dy = \int_0^{\pi/2} \int_0^{\pi/2} (\sin x \cos y + \cos x \sin y) dx\ dy$$

$$= \left(\int_0^{\pi/2} \sin x\ dx \right) \left(\int_0^{\pi/2} \cos y\ dy \right) + \left(\int_0^{\pi/2} \cos x\ dx \right) \left(\int_0^{\pi/2} \sin y\ dy \right)$$

$$= \left(-\cos x \right)_0^{\pi/2} \left(\sin y \right)_0^{\pi/2} \left(\sin x \right)_0^{\pi/2} \left(-\cos y \right)_0^{\pi/2}$$

$= 1 \times 1 + 1 \times 1$

$= 2$

The value got by Simpson's rule differs from the exact value are by 0.008, while the error in the Trapezoidal rule is 0.2125.

Example 8: Evaluate, $\boldsymbol{I = \int_0^{1/2} \int_0^{1/2} \frac{\sin\,(xy)}{1 + xy} dx\ dy}$ **using Simpson's rule with** $\boldsymbol{h = k = \frac{1}{4}.}$

[A.U Tvli N/D 2011, A.U M/J 2012, M/J 2014]

Solution:

Let $f\,(x, y) = \dfrac{\sin\,(xy)}{1 + xy}$

The values of $f(x, y)$ at the nodal points are given in the following table.

y \ x	**0**	**1/4**	**1/2**
0	0	0	0
1/4	0	0.0588 ($4 \times 4 = 16$)	0.1108
1/2	0	0.1108	0.1979

By Simpson's rule,

$$I = \left[\frac{\left(\frac{1}{4}\right)}{3}\right]\left[\frac{\left(\frac{1}{4}\right)}{3}\right]\{(0 + 0 + 0.1979 + 0)$$

$$+ 4\,(0 + 0 + 0.1108 + 0.1108) + 16\,(0.0588)\}$$

$$= \left(\frac{1}{144}\right)(2.0251)$$

$$= 0.0141$$

4.7 NUMERICAL DIFFERENTIATION

$f(x)$ - function	$\Rightarrow y$
$f'(x) =$ first derivative	$\Rightarrow y'$
$f''(x) =$ second derivative	$\Rightarrow y''$

Formula for forward

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \dots\right]$$

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\right]$$

Formula for backward

$$f'(x_0) = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \dots\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \dots\right]$$

$$f'''(x_0) = \frac{1}{h^2}\left[\nabla^3 y_0 + \frac{3}{2}\nabla^4 y_0 + \dots\right]$$

WORKED EXAMPLES

Example 1: Compute $f'(0)$ and $f''(x)$ from the following data:

x	0	1	2	3	4
$f(x)$	1	2.718	7.381	20.086	54.598

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	1.718			
1	2.718	4.663	2.945		
2	7.381	12.705	8.042	5.097	
3	20.086	34.512	21.807	13.765	8.668
4	54.598				

Here we have to find derivatives at $x = 0$ which is initial value of the table. Therefore by Newton's forward difference formula for derivatives at $x = x_0$, we have

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\right]$$

$x_0 = 0, y_0 = 1, h = 1$

$$\therefore f'(0) = \frac{1}{1}\left[1.718 - \frac{1}{2}(2.945) + \frac{1}{3}(5.097) - \frac{1}{4}(8.068) \right]$$

$$= [1.718 - 1.4725 + 1.699 - 2.017]$$

$$f'(0) = -0.0725$$

To find derivative at $x = 4$, which is the last value of the table, we use backward difference formula.

$$f''(x_0) = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \dots \right]$$

$$= \frac{1}{1}\left[21.807 + 13.765 + \frac{11}{12}(8.668) \right]$$

$$= 21.807 + 13.765 + 7.9457$$

$$f''(4) = 43.5177$$

Example 2: Find the first and second derivative of y at $x = 15$ from the table below.

x :	15	17	19	21	23	25
y :	**3.873**	**4.123**	**4.359**	**4.583**	**4.796**	**5.000**

Solution:

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15	3.873					
		0.250				
17	4.123		0.014			
		0.236		0.002		
19	4.359		–0.0012		–0.001	
		0.224		0.001		0.002
21	4.583		–0.011		0.001	
		0.213		0.002		
23	4.796		–0.009			
		0.204				
25	5.000					

From the above table,

$\Delta y_0 = 0.250; \Delta^2 y_0 = -0.014; \Delta^3 y_0 = 0.002;$ $\Delta^4 y_0 = -0.001;$ $\Delta^5 y_0 = 0.002$

From the tabular values, we have

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^2 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\right]$$

Here $h = 2$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}\, x=15} = \frac{1}{2}\left[0.250 + \frac{0.014}{2} + \frac{0.002}{3} + \frac{0.001}{4} + \frac{0.002}{5}\right]$$

$$= 0.1250 + 0.0035 + 0.0003 + 0.000125 + 0.0002$$

$$= 0.1291$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}\left[-0.014 - 0.002 + \frac{11}{12}(-0.001) - \frac{5}{6}90.002)\right]$$

$$= -\frac{1}{4}[0.014 + 0.002 + 0.0009 + 0.0017]$$

$$= -0.0046$$

Example 3: Find the two derivatives of y at $x = 54$ from the following data

x :	50	51	52	53	54
y :	3.6840	3.7084	3.7325	3.7563	3.7798

Solution:

Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
50	3.6840				
		0.0244			
51	3.7084		– 0.0003		
		0.0241		0	
52	3.7325		– 0.0003		0
		0.0238		0	
53	3.7563		– 0.0003		
		0.0235			
54	3.7798				

By Newton's backward difference formula,

$$\left(\frac{dy}{dx}\right)_{\text{at } x = x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y^n}{3} + \dots\right]$$

Here $h = 1$; $\nabla y_n = 0.0235$; $\nabla^2 y_n = -0.0003$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } x = 54} = 0.0235 - \frac{0.0003}{2}$$

$$= 0.02335$$

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } x = 54} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \dots\right]$$

$$= -0.0003$$

Example 4: Find the first and second derivatives of the function tabulated below, at the power $x = 1.5$ and $x = 4$.

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2.0	7.000		3.000		
		6.625		0.75	
2.5	13.625		3.750		0
		10.375		0.75	
3.0	24.00		4.500		0
		14.875		0.75	
3.5	38.875		5.250		
		20.125			
4.0	59.000				

Since $x = 1.5$ is a tabular value near the beginning of the given data, the derivatives are given by

$$\left(\frac{dy}{dx}\right)_{\text{at } x = x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots\right] \quad \text{...(1)}$$

$$\left(\frac{d^y}{dx^2}\right)_{\text{at } x = x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 Y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\right] \quad \text{...(2)}$$

From (1),

$$\left(\frac{dy}{dx}\right)_{\text{at } x = 1.5} = \frac{1}{0.5}\left[3.625 - \frac{3.000}{2} + \frac{0.75}{3}\right]$$

$$= 4.75$$

From (2),

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } x = 1.5} = \frac{1}{0.25}[3.000 - 0.75]$$

$$= 9.000$$

The value $x = 4$ is at the end of the data.

For the tabular values at the end of the data,

$$\left(\frac{dy}{dx}\right)_{\text{at } x = x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \dots\right] \text{ and}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } x = x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } x = 4} = \frac{1}{0.5}\left[20.125 + \frac{5.250}{2} + \frac{0.750}{3}\right] = 46$$

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } x = 4} = \frac{1}{0.25}[5.250 + 0.75] = 24$$

EXERCISES

Using Lagrange's interpolation formula, solve the problems below.

1. Find $f(0)$ given

x :	-1	-2	2	4
$f(x)$:	-1	-9	11	69

[**Ans.** $f(0) = 1$]

2. Find $f(x)$ given the gable. *[A.U. A/M 2011]*

x :	0	1	4	5
$f(x)$:	4	3	24	39

[**Ans.** $f(x) = 2x^2 - 3x + 4$]

3. Find $f(x)$, also find $f(2)$.

x :	0	1	3	4
y :	-12	0	6	12

[**Ans.** $f(x) = x^3 - 7x^2 + 18x - 12,\ f(2) = 4$]

4. Find $y(1.50)$

x :	1.0	1.2	1.4	1.6	1.8	2.0
y :	0.2420	0.1942	0.1497	0.1109	0.0790	0.0540

[**Ans.** $y(1.50) = 0.1295$]

5. Find $f(2)$

x :	0	1	3	4
y :	5	6	50	105

[**Ans.** $f(2) = 19$]

6. Find $f(6)$

x :	2	5	7	10	12
$f(x)$:	18	180	448	1210	2028

[**Ans.** $f(6) = 294$]

Newton's Divided Differences Interpolation

Using Newton's divided differences formula, solve the following problems.

1. Find $f(x)$

x :	0	1	2	4	5	7
$f(x)$:	0	0	– 12	0	600	7308

[**Ans.** $f(x) = x(x-1)(x-4)(x^2+2x-5)$]

2. Find the polynomial equation of degree four passing through the points. (8, 1515), (7, 778), (5, 138), (4, 43) and (2, 3)

[**Ans.** $y = x^4 - 10x^3 + 36x^2 - 36x - 5$]

3. If $y(0) = -18$, $y(1) = 0$, $y(5) = -248$, $y(6) = 0$ and $y(9) = 13104$, find $y = f(x)$.

[**Ans.** $f(x) = (x-1)(x-3)(x-6)(x^2+x+1)$]

4. Find $f(5)$

x :	0	1	3	6
$f(x)$:	1	4	88	1309

[**Ans.** $f(5) = 636$]

5. Find the pressure of steam at 142°C using Newton's general formula

Temp °C :	140	150	160	170	180
Pressure kgf/cm^2 :	3.685	4.854	6.302	8.076	10.225

[**Ans.** 3.8986688]

6. Obtain the value of $\log_{10}656$ given $\log_{10}654 = 2.8156$, $\log_{10}658 = 2.8182$, $\log_{10}659 = 2.8189$ and $\log_{10}666 = 2.8202$

[**Ans.** 2.8169]

Newton Forward and Backward Difference Formula

1. Find the values of y at $x = 21$ and $x = 28$ from the following data: **[A.U N/D 2016 (R13)]**

x :	20	23	26	29
y :	0.3420	0.3907	0.4384	0.4848

[**Ans.** $y(21) = 0.3583$, $y(28) = 0.4695$]

2. Find the value of $f(x)$ at $x = 9$ given the table:

x :	2	5	8	11
$f(x)$:	94.8	87.9	81.3	75.1

[**Ans.** 79.2]

3. Find $y(1.02)$ given

x :	1.00	1.05	1.10	1.15	1.20
y :	0.3413	0.3531	0.3643	0.3749	0.3849

[**Ans.** 0.34614]

4. Estimate $\sin 38°$ from the data given below:

x :	0	10	20	30	40
$\sin x$:	0	0.17365	0.34202	0.50000	0.6479

[A.U A/M 201 (R8-19)] [A.U M/J 2016 (R13)]

[**Ans.** 0.61566]

5. Using Newton's backward formula, find the polynomial of degree 3 passing thorough (3, 0), (4, 24), (5, 60) and (6, 120).

[**Ans.** $y = 2x^3 - 18x^2 + 76x - 120$]

Derivatives from Difference

1. Find the first and second derivative of $\sqrt{x}$ at $x = 15$ from the table below.

x :	15	17	19	21	23	25
$\sqrt{x}$:	3.873	4.123	4.359	4.583	4.796	5.000

[**Ans.** 0.1289, – 0.004]

2. Obtain the second derivative of y at $x = 0.96$ from the data.

x :	0.96	0.98	1.00	1.02	1.04
y :	0.7825	0.7739	0.7651	0.7563	0.7473

[**Ans.** – 1.91666]

3. Find the first three derivatives of the function at $x = 1.5$ from the table below.

x :	1.5	2.0	2.5	3.0	3.5	4.0
y :	3.375	7.0	13.625	24.0	38.875	59.0

[**Ans.** 4.75, 9.0, 6.0]

4. The following data give the corresponding values for pressure and specific volume of a superheated steam.

Volume v :	2	4	6	8	10
Pressure p :	105	42.7	25.3	16.7	13.0

[**Ans.** – 52.4]

5. Find the minimum value of $f(x)$ which has values.

x :	0	2	4	6
$f(x)$:	3	3	11	27

[**Ans.** 2.25]

6. Find the extremum values of y given the table below:

x :	2	3	4	5	6
y :	31.1875	12.0275	2.8655	3.7052	14.5440

[**Ans.** $x = 4.42$ min $y = 2$]

Trapezoidal and Simpson's 1/3, Rule Single Integral

1. Compute, the value of $\int_{1}^{2} \frac{1}{x}\,dx$ using Simpson's rule and Trapezoidal rule. Take $h = 0.25$ [**Ans.** 0.6931, 0.6971]

2. Evaluate, $\int_{0}^{1} \sqrt{\sin x + \cos x}\,dx$ correct to two decimal places,using seven ordinates. [**Ans.** 1.3935]

3. Find the value of $\log 2^{1/3}$ from $\int_{0}^{1} \frac{x^2}{1+x^3}\,dx$ using Simpson's one-third rule with $h = 0.025$ [**Ans.** 0.23108]

4. The velocity v of a particle at distance s from a point units path is given by the table below.

s in metre	0	10	20	30	40	50	60
v m/sec	47	58	64	65	61	52	38

Estimate the time taken to travel 60 metres by using Simpson's one-third rule. [**Ans.** A.U N/D 2014]

[Hint: $v = \frac{ds}{dt}$, $t = \int_{0}^{60} \frac{1}{v}\,dx$; Take $y = \frac{1}{v}$] [**Ans.** 1.0635 sec]

5. When a train is moving at 30 m/sec. steam is shut off and brakes are applied. The speed of the train per second after t seconds is given by

Time (t) :	0	5	10	15	20	25	30	35	40
Speed (v) :	30	24	19.5	16	13.6	11.7	10.0	8.5	7.0

[**Ans.** 606.66 mm]

6. Evaluate $\int_0^2 \frac{1}{x^2 + x + 1} dx$ to three decimals, dividing the range of integration into 8 equal parts using Simpson's rule.
[**Ans.** 0.8145]

7. A solid of revolution is formed by rotating about the X-axis, the area between X-axis, $x = 0, x = 1$ and the curve through the points (0, 1), (0.25 0.9896), (0.75, 0.9089) and (1, 0.8415). Find the volume of solid. [**Ans.** 2.819]

8. Given $e^0 = 1, e^1 = 2.72, e^2 = 7.39, e^3 = 20.09, e^4 = 54.60$, use Simpson's rule to evaluate $\int_0^4 e^x dx$. Compare your result with exact value.] [**Ans.** 53.8733, 53.598]

9. Evaluate, $\int_1^{1.4} e^{-x^2} dx$ by taking $h = 0.1$ using Simpson's rule.
[**Ans.** 0.972]

10. Evaluate, $\int_0^{10} \frac{1}{1 + x} dx$ by dividing the range into 8 equal parts. [**Ans.** 1.299]

11. Calculate, $\int_0^{\pi} \sin^3 x \, dx$ taking $h = \frac{\pi}{6}$ [**Ans.** 1.305]

Double Integrals using Trapezoidal and Simpson's Rules

1. Evaluate, $\int_1^2 \int_3^4 \frac{1}{(x + y)^2} dx\, dy$, taking $h = k = 0.5$ by both Trapezoidal rule and Simpson's rule.
[**Ans.** T.R. 0.0413, S.R. 0.0408]

2. Evaluate, $\int_{1}^{1.2}\int_{1}^{1.4} \frac{1}{x+y}\,dx\,dy$, Trapezoidal rule and Simpson's rule. [**Ans.** T.R: 0.0349, S.R; 0.0349]

3. Evaluate, $\int_{1.}^{2}\int_{1}^{1.5} \log(x+2y)\,dx\,dy$, taking $\Delta x = 0.15$, $\Delta y = 0.25$. [**Ans.** T.R: 0.4292, S.R.: 0.4296]

4. Evaluate, the integral $\int_{2}^{3.2}\int_{1}^{2.6} \frac{1}{x+y}\,dy\,dx$ [**Ans.** 0.4899]

5. Evaluate, the integral $I = \int_{1}^{2}\int_{1}^{2} \frac{1}{x+1}\,dx\,dy$, using the Trapezoidal rule with $h = k = 0.5$ and $h = k = 0.25$
[**Ans.** 0.343304, 0.340668]

6. The following table gives the values of $f(x, y) = e^{y} \sin x$, over the interval $0 \le x \le 0.2$, $0 \le y \le 0.2$.

y \ x	0	0.1	0.2
0	0	0.0998	0.1987
0.1	0	0.1103	0.2196
0.2	0	0.1219	0.2427

Evaluate the integral of $f(x, y)$, over the interval $0 \le x \le 0.2$, $0 \le y \le 0.2$ (a) by the Trapezoidal rule with $h = 0.2$ (b) by the Trapezoidal rule with $h = 0.1$ (c) by Simpson's rule with $h = 0.1$

[**Ans.** (a) 0.004414, (b) 0.004413, (c) 0.004413]

SHORT QUESTIONS AND ANSWERS

1. State any two properties of divided differences.

- The divided difference are symmetrical in all their arguments.
- The operator $\Delta\!\!\!|$ is linear.
- The nth divided differences of a polynomial of nth degree are constant.

2. State Lagrange formula to find $y(x)$ if three sets of values (x_0, y_0), (x_1, y_1) and (x_2, y_2) are given

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

3. State Lagranges formula to find $y(x)$ if four sets of values (x_0, y_0), (x_1, y_1) (x_2, y_2) and (x_3, y_3) are given.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

4. If $f(3) = 5$ and $f(5) = 3$, what is the form of $f(x)$ by Lagrange's formula?

x	3	5
$f(x)$	5	3

$$y = \frac{(x-5)}{3-5}(5) + \frac{(x-3)}{5-3}(3)$$

$$y = \frac{-5}{2}(x-5) + \frac{3}{2}(x-3)$$

5. Write the formula for forward difference.

$$y_p = y_0 + p\,\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

where $P = \dfrac{x - x_0}{h}$, $h -$ step size.

6. Write down the backward difference formula.

$$y_q = y_n + q\,\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \dots$$

where $q = \dfrac{x - x_n}{h}$, $h -$ step size.

7. Find the divided difference table for the data.

x	2	5	10
y	5	29	109

X	Y	$\Delta_1 y$	$\Delta_1^2 y$
2	5	$\frac{29-5}{5-2} = 8$	
5	29	$\frac{109-29}{10-5} = 16$	$\frac{16-8}{10-2} = \frac{8}{8} = 1$
10	109		

8. State the newton's divided difference interpolation formula.

$$f(x) = f(x_0) + (x - x_0)\,\Delta_1 f(x_0) + (x - x_0)(x - x_1)\,\Delta_1^2 + \Delta_2 f(x_0) + f(x_0) + \dots$$

9. Find the divided differences of $f(x) = x^3 + x + 2$ for the arguments 1, 3, 6, 11

$x = 1, 3, 6, 11 \Rightarrow f(x) = 4, 32, 224, 1344$

X	Y	$\Delta_1 y$	$\Delta_1^2 y$	$\Delta_1^3 y$
1	4	$\frac{32-1}{3-1}=14$		
3	32	$\frac{224-32}{3}=\frac{192}{3}=64$	$\frac{64-14}{6-1}=10$	$\frac{20-10}{11-1}=1$
6	224	$\frac{1344-224}{5}=224$	$\frac{224-64}{11-3}=20$	
11	1344			

10. State the Trapezoidal rule with error order.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2 \text{ sum of remaining values}) \right]$$

Error order: h^2

11. State the Simpson's 1/3 rule and 3/8 rule with error order.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + \begin{matrix}\text{4 (sum of}\\ \text{odd values)}\end{matrix} + \begin{matrix}\text{2 (sum of}\\ \text{Even values}\end{matrix} \right]$$

Error order: h^4.

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + \begin{matrix}\text{2 (sum of}\\ \text{multiples of 3)}\end{matrix} + \begin{matrix}\text{3 (sum of}\\ \text{remaining values}\end{matrix} \right]$$

12. State double integral formula for Trapezoidal rule $\int_a^b \int_c^d f(x, y)\,dx\,dy$

$$I = \frac{hk}{4}\left[1 \begin{pmatrix}\text{sum of first}\\ \text{4 corner values}\end{pmatrix} + 2 \begin{pmatrix}\text{sum of remaining}\\ \text{values in the 1st corner}\end{pmatrix} + 4 \begin{pmatrix}\text{sum of}\\ \text{remaining values}\end{pmatrix} \right]$$

13. What is the order of error in Trapezoidal and Simpson's 1/3rd rule?

Trapezoidal rule: Error $= \frac{-(b-a)}{12} h^2 y''(\varepsilon)$; $a \le x \le b$

Order of the order $= h^2$

Simpson's rule: Error $= \frac{-(b-a)}{180} h^4 y^{iv}(\varepsilon)$; $a \le x \le b$

Order of the error $= h^4$

14. Given $f(2) = 5, f(2.5) = 5.5$ find the linear interpolating polynomial using Lagrange interpolation.

x	2	2.5
y	5	5.5

$$Y = f(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

$$= \frac{x - 2.5}{2 - 2.5}(5) + \frac{x - 2}{2.5 - 2}(5.5)$$

$$= \frac{x - 2.5}{-0.5}(5) + \frac{x - 2}{0.5}(5.5)$$

$$= -(x - 2.5)(10) + (x - 2)\, 11$$

$$= -10x + 25 + 11x - 22$$

$$\boxed{f(x) = x + 3}$$

UNIT - V

Numerical Solution of Ordinary Differential Equations

5.0 INTRODUCTION

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations involved in physical problems do not belong to any of these standard types and one has to resort to numerical methods.

Solving an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the **finite form of solution**. In the absence of such a solution, we have shifted to numerical methods of solution. The differential equation together with the initial conditions is called an **initial value problem**.

Let us consider the first order differential equation.

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \qquad \text{... (1)}$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The method of Taylor series belongs to the former class of solutions. In this method, y in (1) is approximated by a truncated series, each term of which is a function of x. As such, this is referred to as **single-step method**. The methods of Euler, Runge - Kutta, Milne, Adams - Bashforth etc. belong to the latter class of solutions. In these methods, the next point is evaluated by performing iterations till sufficient accuracy is achieved. As such, these methods are called **step-by-step or multi-step methods**.

5.1 SINGLE STEP METHODS

In one-step method, we use the data of just one preceding step.

Suppose the ordinary differential equation has given, we can find the numerical solution by using.

1. Taylors series method.
2. Euler method.
3. Euler modified method.
4. Rung-Kutta 4^{th} order method.

These methods are called single steps method.

5.1.1 Taylor Series Method

Consider the ordinary differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x)_0 = y_0$ then

$$y(x) = y_0 + (x - x_0)\, y'_0 + \frac{(x - x_0)^2}{2!} y'_0 + \frac{(x - x_0)^3}{3!} y''_0 + \frac{(x - x_0)^4}{4!} y^{iv} + \dots$$

(OR)

Note: Replace $x - x_0 = h$

$$y(x) = (y_0) + (h)\, y'_0 + \frac{(h)^2}{2!} y'_0 + \frac{(h)^3}{3!} y''_0 + \frac{(h)^4}{4!} y^{iv} + \dots$$

5.1.2 Solution of ODE by Taylor series method

AIM: To find the numerical solution of the equation

$$\frac{dy}{dx} = f(x, y) \qquad \dots (1)$$

given the initial condition $y(x_0) = y_0$. ... (2)

Now, we expand $y(x)$ about the point $x = x_0$ in a Taylor's series in powers of $(x - x_0)$. That is,

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots \qquad \ldots (3)$$

where $y^{(r)}(x_0) = \left(\dfrac{d^2 y}{dx^2} \right)_{x = x_0}$

i.e., $y(x) = y_0 + \dfrac{(x-x_0)}{1!} y_0' + \dfrac{(x-x_0)^2}{2!} y_0'' + \ldots$

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \qquad \ldots (4)$$

where $h = x_1 - x_0$ or $x_1 = x_0 + h$

To find $y_0', y_0'', \ldots$ we use (1) and its derivatives at $x = x_0$. Though the series (4) is an infinite series, we can truncate it at any convenient term, if h is small and the accuracy is obtained. Now, having got y_1, we can calculate

$$y_1', y_1'', y_1''', \ldots \text{ etc., by using } y' = f(x, y)$$

Now expanding $y(x)$, in a Taylor's series about the point $x = x_1$, we get

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \qquad \ldots (5)$$

Proceeding in the same way, we get

$$\boxed{y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \ldots} \qquad \ldots (6)$$

WORKED EXAMPLES

Example 1: Use Taylor's series method to solve $\dfrac{dy}{dx} = x + y$; $y(1) = 0$ numerically upto $x = 1.2$ with $h = 0.1$.

Solution:

Given $y(1) = 0 \Rightarrow x_0 = 1, \; y_0 = 0, \; h = 0.1$

$\Rightarrow$	$\frac{dy}{dx} = y' = x + y$	$y_0' = x_0 + y_0 = 1$
$\Rightarrow$	$y'' = 1 + y'$	$y_0'' = 1 + y_0' = 2$
$\Rightarrow$	$y''' = y''$	$y_0''' = y_0'' = 2$
$\Rightarrow$	$y^{(iv)} = y'''$	$y_0^{(iv)} = 2$
$\Rightarrow$	$y^{(v)} = y^{(iv)}$	$y_0^{(v)} = 2$

By Taylor's series, we have

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(iv)} + \ldots\ldots$$

$$y_1 = 0 + (0.1)\,(1) + \frac{(0.1)^2}{2!}\,(2) + \frac{(0.1)^3}{3!}\,(2) + \frac{(0.1)^4}{4!}\,(2) + \ldots\ldots$$

$\Rightarrow\ y\,(1, 1) = 0.1103081 = 0.110\,(\text{app.})$

Also, $x_1 = x_0 + h = 1.1$

Again, $y_1 = x_1 + y_1 = 1.1 + 0.11 = 1.21$

$$y_1'' = 1 + y_1' = 1 + 1.21 = 2.21$$

$$y_1''' = y_1'' = 2.21$$

$$y_1^{(iv)} = 2.21,\ \ y_1^{(v)} = 2.21$$

Now, $y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots\ldots$

$$= 0.11 + (0.1)\,(1.21) + \frac{(0.1)^2}{2}\,(2.21) + \ldots\ldots$$

$\Rightarrow y\,(1.2) = 0.232$ (app.)

Example 2: Using Taylor series method, find $y\,(1.1)$ and $y\,(1.2)$ correct to four decimal places given $\frac{dy}{dx} = xy^{1/3}$ and $y\,(1) = 1.$ *(AU 21)*

Solution:

Given $y\,(1) = 1 \Rightarrow x_0 = 1, y_0 = 1, h = 0.1$

$y' = xy^{1/3}$	$y_0' = 1\,(1)^{1/3} = 1$
$y'' = \frac{1}{3}xy^{-2/3}\,y' + y^{1/3}$	$y_0'' = \frac{1}{3}x_0\,y_0^{-2/3}\,y_0' + y_0^{1/3} = 4/3$
$= \frac{1}{3}x^2\,y^{-1/3} + y^{1/3}$	$y''' = \left[\frac{x^2}{3}\left(-\frac{1}{3}\right)y^{-4/3}\,y' + \frac{2x}{3}\,y^{1/3} + \frac{1}{3}\,y^{-2/3}\,y'\right]$

By Taylor series,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \ldots$$

$$y_1 = y\,(1.1) = 1 + (0.1)\,(1) + \frac{(.1)^2}{2}\left(\frac{4}{3}\right) + \frac{(.1)^3}{6}\left(\frac{8}{9}\right) + \ldots$$

$$= 1 + 0.1 + 0.00666 + 0.000148 + \ldots$$

$$y\,(1.1) = 1.10681$$

We start with (x_1, y_1) as the starting value.

$$y_1 = 1.10681$$

$$x_1 = 1.1, \quad y_1' = x_1\,y_1^{1/3} = (1.1)\,(1.10681)^{1/3} = 1.13785$$

$$y_1'' = \frac{1}{3}x_1\,y_1^{-2/3}\,y_1' + y_1^{1/3}$$

$$= \frac{1}{3}(1.1)\,(1.10681)^{-2/3}\,(1.13785) + (1.10681)^{1/3}$$

$$= 0.38992 + 1.03441 = 1.42433$$

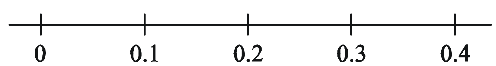
$$y_2''' = 0.929787$$

$$\therefore \quad y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \ldots$$

$$y\,(1.2) = 1.10681 + (0.1)\,(1.13785) + \frac{0.01}{2}(1.42433) + \frac{.001)}{6}(.929787)$$

$$y\,(1.2) = 1.22772$$

Example 3: Using Taylor series method, find y at $x = 0.1, 0.2, 0.3, 0.4$ given $\frac{dy}{dx} = x^2 - y, y\,(0) = 1$ (correct to 4 decimal places).



Solution:

Given $y\,(0) = 1$

$\Rightarrow$

$x_0 = 0, y_0 = 1, h = 0.1, x_1 = 0.1, x_2 = 0.2, \ldots x_3 = 0.3, x_4 = 0.4$

$y' = x^2 - y$	$y_0' = x_0^2 - y_0 = -1$
$y'' = 2x - y'$	$y_0'' = 0 - (-1) = 1$
$y''' = 2 - y''$	$y_0''' = 2 - 1 = 1$
$y^{iv} = -y'''$	$y_0^{iv} = -1$ etc.,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^4}{24}y_0^{iv} + \ldots$$

$$y_1 = y\,(0.1) = 1 + (0.1)\,(-1) + \frac{(.01)}{2}(1) + \frac{(.001)}{6}(1) + \frac{(.0001)}{24}(-1) + \ldots$$

$$= 1 - 0.1 + .005 + .0001666 - 0.0000416 + \ldots$$

$$\boxed{\mathbf{y_1 = y\ (0.1)\ = 0.905125}}$$

$$y_1' = x_1^2 - y_1 = 0.01 - 0.905125 = -0.895125$$

$$y_1'' = 2x_1 - y_1' = 0.2 + 0.895125 = 1.095125$$

$$y_1''' = 2 - y_1'' = 2 - 1.095125 = 0.904875$$

$$\therefore \quad y_2 = y_1 + \frac{h}{1} y_1' + \frac{h^2}{2!} y_1'' + \ldots$$

$$y\ (0.2) = y_2 = 0.905125 + (0.1)\ (-0.895125) + \frac{.01}{2}\ (1.095125) + \frac{.001}{6}\ (0.904875) + \ldots$$

$$\boxed{\mathbf{y_2 = y\ (0.2) = 0.128268}}$$

Similarly $y\ (0.3) = 0.7492$ (4 decimals)

$y\ (0.4) = 0.6897$ (4 decimals)

Example 4: Solve $y' = y^2 + x$; $y\ (0) = 1$ using Taylor series method and compute $y\ (0.1)$ and $y\ (0.2)$.

Solution:

Given $y\ (0) = 1 \Rightarrow x_0 = 0,\ y_0 = 1$

$y' = y^2 + x$	$y_0' = 1$
$y'' = 2yy' + 1$	$y_0'' = 3$
$y''' = 2yy'' + 2y'^2$	$y_0''' = 8$
$y^{iv} = 6y'' + 2yy'''$	$y_0^{iv} = 34$

Here $h = 0.1$

To find $y\,(0.1)$

By Taylor algorithm,

$$y_1 = y_0 + \frac{h}{1!}\,y_0' + \frac{h^2}{2!}\,y_0'' + \frac{h^3}{3!}\,y_0''' + \dots$$

$$y\,(0.1) = y_1 = 1 + (0.1)\,(1) + \frac{0.01}{2}\,(3) + \frac{0.001}{6}\,(8) + \frac{0.0001}{24}\,(34)$$

$$y_1 = 1 + 0.1 + 0.015 + 0.0013 + 0.00014$$

$$y\,(0.1) = 1.1164$$

Example 5: Using Taylor's series method, find the value of $y\,(0.1)$, given $\frac{dy}{dx} = x^2 + y^2,\ y\,(0) = 1$

Solution:

Given

$$y\,(0) = 1 \Rightarrow \quad h = x - x_0 = 0.01$$

$$x_0 = 0,\ \ y_0 = 1,\ \ x_1 = 0.1$$

Now, $y' = x^2 + y^2$	$y_0' = x_0^2 + y_0^2 = 0 + 1^2 = 1$
$y'' = 2x + 2yy'$	$y_0'' = 2x_0 + 2y_0\,y_0'$
$y''' = 2 + 2\,(yy'' + y'^2)$	$= 2\,(0) + 2\,(1)\,(1) = 2$
$= 2 + 2yy'' + 2y'^2$	$y_0''' = 2 + 2y_0\,y_0'' + 2y_0'^2$
$y^{iv} = 2\,(yy''' + y'\,y'') + 4y'\,y''$	$= 2 + (2)\,(1)\,(2) + 2\,(1^2) = 8$
$= 2yy''' + 6y'\,y''$	$y_0^{iv} = 2y_0\,y_0''' + 6y_0'\,y_0''$
	$= 2\,(1)\,(8) + 6\,(1)\,(2) = 28$

$\therefore$ By Taylor's series,

$$y_1 = y\,(0.1) = y_0 + \frac{(x - x_0)}{1!}\,y_0' + \frac{(x - x_0)^2}{2!}\,y_0'' + \frac{(x - x_0)^3}{3!}\,y_0''' + \frac{(x - x_0)^4}{4!}\,y_0^{iv}$$

$$= 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(28)$$

$$= 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3 \times 8}{6} + \frac{(0.1)^4 \times 28}{24}$$

$$= 1 + 0.1 + 0.01 + \frac{1.3333}{1000} + \frac{1.1667}{10^4}$$

$$= 1.11145$$

$$\boxed{y\,(0.1) = y_1 = 1.11145}$$

Example 6: Evaluate $y\,(0.1)$ and $y\,(0.2)$, correct to four decimal places by Taylor series method, if $y\,(x)$ satisfies $y' = xy + 1$, $y\,(0) = 1$.

Solution:

Given

$$y\,(0) = 1 \Rightarrow x_0 = 0,\ y_0 = 1,\ h = 0.1$$

Now

$y' = xy + 1$	$y_0' = 1$
$y'' = xy' + y$	$y_0'' = 1$
$y''' = xy'' + 2y'$	$y_0''' = 2$
$y^{iv} = xy''' + 3y''$	$y_0^{iv} = 3$

By Taylor's algorithm,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

Let $x_1 = x_0 + h$ where $h = 0.1$

$$\therefore \quad x_1 = 0.1$$

$$y\,(0.1) = y_1 = 1 + (0.1) + \frac{0.01}{2}\,(1) + \frac{0.001}{3}\,(2) + \frac{0.0001}{24}\,(3)$$

$$= 1 + 0.1 + 0.005 + 0.0007 + 0.0000125$$

$$\boxed{y\,(0.1) = 1.1057}$$

Example 7: Using Taylor's series method find y at $x = 0.1$ if $\frac{dy}{dx} = x^2 y - 1,\ y\,(0) = 1.$ *[A.U. N/D 2004, N/D 2010, N/D 2014 M]*

Solution:

Given

$y' = x^2\, y - 1$ and $x_0 = 0, y_0 = 1, h = 0.1$

By Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}\,y_0' + \frac{h^2}{2!}\,y_0'' + \frac{h^3}{3!}\,y_0''' + \ldots \qquad \ldots (1)$$

$y' = x^2\, y - 1$	$y_0' = x_0^2\, y_0 - 1 = 0 - 1 = -1$
$y'' = 2xy + x^2\, y'$	$y_0'' = 2x_0\, y_0 + x_0^2\, y_0' = 0 + 0 = 0$
$y''' = 2\,[xy' + y] + x^2\, y'' + 2xy'$ $= 2xy' + 2y + x^2\, y'' + 2xy'$ $= 2y + 4xy' + x^2\, y''$	$y_0''' = 2y_0 + 4x_0\, y_0' + x_0^2\, y_0''$ $= 2\,(1) + 4\,(0)\,(-1) + (0)^2\,(0)$ $= 2$
$y^{iv} = 2y' + 4\,[xy'' + y'] + x^2\, y''' + y''\, 2x$ $= 2y' + 4xy'' + 4y' + x^2\, y''' + y''\, 2x$ $= 6y' + 6xy'' + x^2\, y'''$	$y_0^{iv} = 6y_0' + 6x_0\, y_0'' + x_0^2\, y_0'''$ $= 6\,(-1) + 6\,(0)\,(0) + (0)\,(2)$ $= -6$

$$\therefore \text{(1)} \Rightarrow y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(-6) + \ldots$$

$$\text{(i.e.,)} \ y\,(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \ldots$$

$$= 1 - 0.1 + 0.00033 - 0.000025 = 0.900305$$

$\boldsymbol{y\,(0.1) = 0.9003}$ [Correct to four decimal]

Example 8: Solve the differential equation $\frac{dy}{dx} = x + y + xy$, $y\,(0) = 1$ by Taylor series method to get the value of y at $x = h$.

Solution: Given

$$y' = x + y + xy$$

$$x_0 = 0, \ y_0 = 1$$

$y' = x + y + xy$	$y'_0 = 0 + 1 + 0 = 0$
$y' = 1 + y' + xy' + y$	$y''_0 = 1 + 1 + 0 + 1 = 3$
$y'' = 1 + xy' + y' + y'$	$y''_0 = 3 + 1 + 0 + 1 = 5$
$y'' = y'' + 2y' + xy'' + y'$	$y''''_0 = 5 + 6 + 0 + 3 = 14$

Taylor's series is

$$y\,(x) = y_0 + (x - x_0)\,y'_0 + \frac{(x - x_0)^2}{2!}\,y'_0 + \frac{(x - x_0)^3}{3!}\,y''_0 + \frac{(x - x_0)^4}{4!}\,y^{iv} + \ldots$$

$$y(h) = 1 + (h)(1) + \frac{(h)^2}{2}(3) + \frac{(h)^3}{6}(5) + \frac{(h)^4}{24}(14) + \ldots$$

$$y\,(h) = 1 + h + \frac{3}{2}h^2 + \frac{5}{6}h^3 + \frac{7}{12}h^4 + \ldots$$

Example 9: By Taylor series expansion, find y at $x = 0.1$ and $x = 0.2$ correct to three decimal places given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

Solution:

Given $x_0 = 0,\ y_0 = 0$

$y' = 2y + 3e^x$	$y'_0 + 3e^{x0} = 0 + 3(1) = 3$
$y'' = 2y' + 3e^x$	$y'' = 2y' + 3e^{x0} = 2(3) + 3(1) = 9$
$y'' = 2y' + 3e^x$	$y''_0 = 2y''_0 + 3e^{x0} = 2(9) + 3(1) = 21$
$y^{iv} = 2y'' + 3e^x$	$y_0^{iv} = 2y'''_0 + 3e^{x0} = 2(21) + 3(1) = 45$

Taylor's series about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)\, y'_0 + \frac{(x - x_0)^2}{2!}\, y'_0 + \frac{(x - x_0)^3}{3!}\, y''_0 + \frac{(x - x_0)^4}{4!}\, y_0^{iv} + \ldots$$

$$y(x) = 0 + (x - 0)(3) + \frac{(x - 0)^2}{2!}\,(9) + \frac{(x - 0)^3}{3!}\,(21) + \frac{(x - 0)^4}{4!}\,(45) + \ldots$$

$$y(x) = 3x + \frac{9x^2}{2} + \frac{7x^3}{2} + \frac{15x^4}{8} + \ldots$$

$$y\,(0.1) = 3(0.1) + \frac{9\,(0.1)^2}{2} + \frac{7\,(0.1)^3}{2} + \frac{15\,(0.1)^4}{8} + \ldots$$

$$= 0.3 + 0.045 + 0.0035 + 0.0001875$$

$$\boxed{\boldsymbol{y\,(0.1) = 0.3487}}$$

$$y\,(0.2) = 3\,(0.2) + \frac{9\,(0.2)^2}{2} + \frac{7\,(0.2)^3}{2} + \frac{15\,(0.2)^4}{8} + \ldots$$

$$= 0.6 + 0.18 + 0.028 + 0.003$$

$$\boxed{\boldsymbol{y\,(0.2) = 0.811}}$$

5.2 EULER'S METHOD

Itis the simplest one-step method and has a limited application because of its low accuracy. This method yields solution of an ordinary diff. eqn. in the form of a set of tabulated values.

In this method, we determine the change Δy is y corresponding to small increase in the argument x. Consider the differential equation.

$$\frac{dy}{dx} = f(x, y), \; y(x_{0)} = y_0 \qquad \dots (1)$$

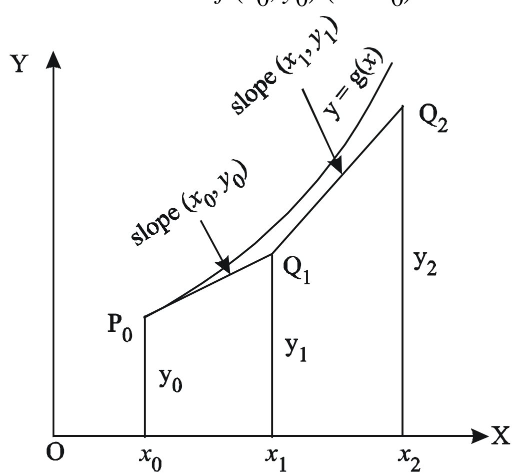
Let $y = g(x)$ be the solution of (1). Let $x_0, x_1, x_2 \dots$ be equidistant values of x.

In this method, we use the property that in a small interval, a curve is nearly a straight line. Thus at the point (x_0, y_0), we approximate the curve by the tangent at the point (x_0, y_0).

The eqn. of the tangent at $P_0\,(x_0, y_0)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{y_0} (x - x_0)$$

$$= f(x_0, y_0)\,(x - x_0)$$



$$\Rightarrow \qquad y = y_n + (x - x_0)\, f(x_0, y_0) \qquad \text{... (2)}$$

This gives the y-coordinate of any point on the tangent. Since the curve is approximated by the tangent in the interval (x_0, x_1), the value of y on the curve corresponding to $x = x_1$ is given by the above value of y in eqn. (2) approximately.

Putting $x = x_1\ (= x_0 + h)$ in eqn.(2), we get

$$y_1 = y_0 + hf(x_0, y_0)$$

Thus Q_1 is (x_1, y_1)

Similarly, approximating the curve in the next interval (x_1, x_2) by a line through $Q_1\ (x_1, y_1)$ with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + hf(x_1, y_1)$$

In general, it can be shown that,

$$\boxed{\boldsymbol{y_{n+1} = y_n + hf\,(x_n, y_n)}}$$

This is called Euler's formula

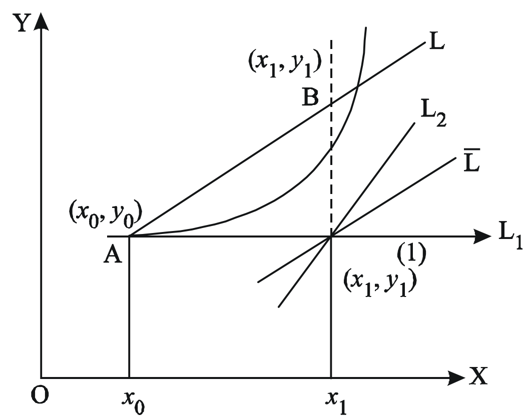
A great disadvantage of this method lies in the fact that if $\frac{dy}{dx}$ changes rapidly over an interval, its value at the beginning of the interval may give a poor approximation as compared to its average value over the interval and thus the value of y calculated from Euler's method may be in much error from its true value. These errors accumulate in the succeeding intervals and the value of y becomes much erroroneous ultimately.

Note: In Euler's method, the curve of actual solution $y = g\,(x)$ is approximated by a sequence of short lines. The process is very slow. If h is not properly chosen, the curve $P_0\, Q_1\, Q_2 \ldots\ldots$ of short lines representing numerical solution deviates significantly from the curve of actual solution.

5.2.1 Improved Euler's Method

The modified Euler's method gives greater improvement in accuracy over the original Euler's method. Here the core idea is that we use a line through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$ where $y_1^{(1)} = y_0 + hf(x_0, y_0)$. This line approximates the curve in the interval (x_0, x_1).

Geometrically, if L_1 is the tangent at $(x_0, y_0), L_2$ is a line through $(x_1, y_1^{(1)})$ but with a slope equal to the average of $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$ then the line L through (x_0, y_0) and parallel to $\overline{L}$ is used to approximate the curve in the interval (x_0, x_1). Thus the ordinate of the point B will give the value of y_1. Now, the eqn. of the AL is give by



$$y_1 = y_0 + (x_1 - x_0)\left[\frac{f(x_0, y_0) + f(x_1, y_1)^{(1)}}{2}\right]$$

$$= y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right] \quad \text{... (1)}$$

A generalized form of improved Euler's formula is

$$\boxed{\boldsymbol{y_1^{(n+1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f[x_1, y_1^{(n)}]; \; n = 0, 1, 2, \ldots\ldots}} \quad \text{... (2)}$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

The above iteration formula can be started by choosing $y_1^{(1)}$ from Euler's formula

$$y_1^{(1)} = y_0 + hf(x_0, y_0) \qquad \text{... (3)}$$

Since this formula attempts to correct the values of y_{n+1} using the predicted value of (by Euler's method), it is classified as a *one-step predictor-corrector method.*

5.2.2 Modified Euler's Method

In this method, the curve in the interval (x_0, x_1), where $x_1 = x_0 + h$ is approximated by the line through (x_0, y_0) with slope $f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$, which is the slope at the middle point whose abscissa is the average of x_0 and x_1.

In the above figure, line AL through $A(x_0, y_0)$ which is parallel to the line $P\bar{L}$ with slope $f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$ approximates the curve in the interval (x_0, x_1). The ordinate at $x = x_1$, meeting the L at B, will give the value of y_1.

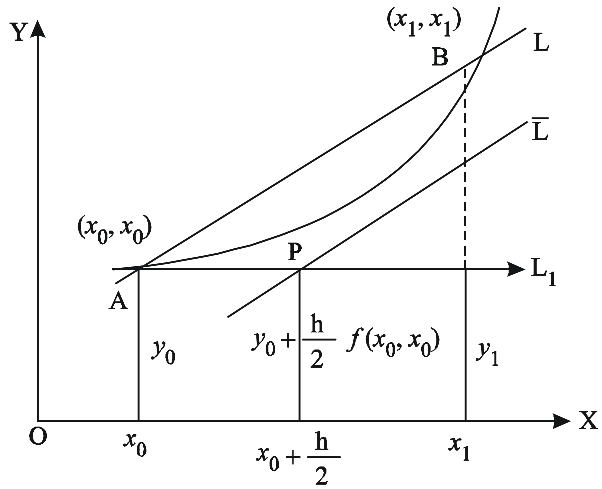
The equation for line AL is

$$y - y_0 = (x - x_0)\left[f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}\right] \qquad \text{... (1)}$$

Putting $x = x_1$ in (1), we get

$$y_1 = y_0 + (x_1 - x_0)\left[f\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}\right]$$

$$y_1 = y_0 + hf\left\{x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right\}$$



Proceeding in the same way, we obtain

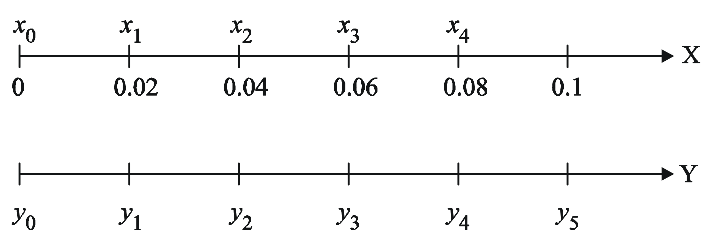
$$\boxed{y_{n+1} = y_n + hf\left\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right\}} \quad \ldots (2)$$

which is the generalized form of **modified Euler's formula**.

WORKED EXAMPLES

Example 1: Use Euler's methods to approximate y when $x = 0.1$ given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y = 1$ and $x = 0$, taking $h = 0.02$.

Solution:



Given $f(x, y) = \frac{y-x}{y+x}$

Also given $x_0 = 0$, $y_0 = 1$ and $h = 0.02$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y_0 + h\left[\frac{y_0 - x_0}{y_0 + x_0}\right]$$

$$y_1 = 1 + 0.02\left[\frac{1 - 0}{1 + 0}\right]$$

$$\boxed{\mathbf{y\ (0.02) = 1.02}}$$

$$x_1 = x_0 + h$$

$$= 0 + 0.02$$

$$\boxed{\mathbf{x_1 = 0.02}}$$

$$y_2 = y_1 + hf(x_1\,, y_1)$$

$$y_2 = y_1 + h\left[\frac{y_1 - x_1}{y_1 + x_1}\right] = 1.02 + (0.02)\left[\frac{1.02 - 0.02}{1.02 + 0.02}\right]$$

$$\boxed{\mathbf{y\ (0.04) = 1.0392}}$$

$$x_2 = x_1 + h$$

$$= 0.02 + 0.02$$

$$\boxed{\mathbf{x_2 = 0.04}}$$

$$y_3 = y_2 + hf(x_2,\ y_2)$$

$$y_3 = y_2 + h\left[\frac{y_2 - x_2}{y_2 + x_2}\right] = 1.0392 + (0.02)\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$

$$= 1.0577$$

$$\boxed{\mathbf{y\ (0.06) = 1.0577}}$$

$x_3 = x_2 + h$

$= 0.04 + 0.02$

$\boxed{x_3 = 0.06}$

$y_4 = y_3 + hf(x_3, \ y_3)$

$$y_4 = y_3 + h\left[\frac{y_3 - x_3}{y_3 + x_3}\right] = 1.0577 + (0.02)\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right]$$

$$= 1.0756$$

$\boxed{y\,(0.08) = 1.0756}$

$x_4 = x_3 + h$

$= 0.06 + 0.02$

$\boxed{x_4 = 0.08}$

$y_5 = y_4 + hf(x_4, y_4)$

$$y_5 = y_4 + h\left[\frac{y_4 - x_4}{y_4 + x_4}\right] = 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right]$$

$$= 1.0928$$

$\boxed{y_5 = y\,(0.1) = 1.0928}$

Example 2: Solve $\frac{dy}{dx} = 1 - y$ with the initial condition $x = 0$, $y = 0$. Using Euler's algorithm, tabulate the solution at $x = 0.1, 0.2, 0.3, 0.4$.

Solution:

Given $f(x, y) = 1 - y$.

Also given $x_0 = 0, y_0 = 0$ and $h = 0.1$

Euler's method

$y_1 = y_0 + hf(x_0, y_0)$

$y_1 = y_0 + h\,[1 - y_0] = 0 + (0.1)\,[1 - 0]$

$= 0.1$

$\boxed{\mathbf{y_1 = y\,(0.1) = 0.1}}$

$x_1 = x_0 + h$

$= 0 + 0.1$

$\boxed{\mathbf{x_1 = 0.1}}$

$y_2 = y_1 + hf(x_{1,}\ y_1)$

$y_2 = y_1 + h\,[1 - y_1] = 0.1 + (0.1)\,[1 - 0.1]$

$= 0.19$

$\boxed{\mathbf{y_2 = y\,(0.2) = 0.19}}$

$x_2 = x_1 + h$

$= 0.1 + 0.1$

$\boxed{\mathbf{x_2 = 0.2}}$

$y_3 = y_2 + hf(x_2 \cdot y_2)$

$y_1 = y_2 + h\,[1 - y_2] = 0.19 + (0.1)\,[1 - 0.19]$

$= 0.271$

$\boxed{\mathbf{y_3 = y\,(0.3) = 0.271}}$

$x_3 = x_2 + h$

$= 0.2 + 0.1$

$\boxed{\mathbf{x_3 = 0.3}}$

$y_4 = y_3 + hf(x_3, y_3)$

$y_4 = y_3 + h[1 - y_3] = 0.271 + (0.1)[1 - 0.271]$

$= 0.3439$

$\boxed{\mathbf{y_4 = y(0.4) = 0.3439}}$

Example 3: Use Euler's method to obtain an approximate value of $y(0.4)$ for the equation $y = x + y, y(0) = 1$ with $h = 0.1$.

Solution:

Here, $f(x, y) = x + y$

Taking $h = 0.1, x_0 = 0, y_0 = 1$, we obtain

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (.1)(0 + 1) = 1.1$$

$\therefore$ $\boxed{\mathbf{y(0.1) = 1.1}}$

Again, $y_2 = y_1 + hf(x_1, y_1) = 1.1 + (.1)(.1 + 1.1) = .122$

$\therefore$ $\boxed{\mathbf{y(0.2) = 1.22}}$

Again, $y_3 = y_2 + hf(x_2, y_2) = 1.22 + (.1)(.2 + 1.22) = 1.362$

$\therefore$ $\boxed{\mathbf{y(0.3) = 1.362}}$

Again, $y_4 = y_3 + hf(x_3, y_3) = 1.362 + (.1)(.3 + 1.362) = 1.5282$

$\therefore$ $\boxed{\mathbf{y(0.4) = 1.5282}}$

Example 4: Using Euler's method, compute y in the range $0 \le x \le 0.5$, if y satisfies $\frac{dy}{dx} = 3x + y^2 \cdot y(0) = 1$.

Solution:

Here $f(x, y) = 3x + y^2, x_0 = 0, y_0 = 1$

By Euler's method

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, \ldots\ldots \qquad \ldots (1)$$

Choosing $h = 0.1$, we compute the values of y using (1)

$$y(0.1) = y_1 = y_0 + hf(x_0, y_0) = 1 + (0.1)[3(0) + (1)^2]$$
$$= 1.1$$
$$y(0.2) = y_2 = y_1 + hf(x_1, y_1) = 1.1 + (0.1)[0.3 + (1.1)^2]$$
$$= 1.251$$
$$y(0.3) = y_3 = y_2 + hf(x_2, y_2) = 1.251 + (0.1)[0.6 + (1.251)^2]$$
$$= 1.4675$$
$$y(0.4) = y_4 = y_3 + hf(x_3, y_3) = 1.4675 + (0.1)[0.9 + (1.4675)^2]$$
$$= 1.7728$$
$$y(0.5) = y_5 = y_4 + hf(x_4, y_4) = 1.7728 + (0.1)[1.2 + (1.7728)^2]$$
$$= 2.2071$$

Example 5: Find $y(1)$ given $y' = xy, y(0) = 1$, taking $h = 0.25$, using Euler's method.

Solution:

Given

$y' = xy, \quad f(x, y) = xy$

$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1,\ y_0 = 1,\ h = 0.25$

$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$y(0.25) = y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.25 \times (0.25 \times 1)$$
$$= 1$$

$$y(0.5) = y_2 = y_1 + hf(x_1, y_1)$$
$$= 1 + 0.25 \times (0.25 \times 1)$$
$$= 1.0625$$
$$y(0.75) = y_3 = y_2 + hf(x_2, y_2)$$
$$= 1.0625 + 0.25\,(0.5 \times 1.0625)$$
$$= 1.1953$$
$$y(1) = y_4 = y_3 + hf(x_3, y_3)$$
$$= 1.1953 + 0.25 \times (0.75 \times 1.1953)$$
$$= 1.4194$$

Example 6: Using Euler's method, solve $y' = y\,(e^x - 1)$, $y(0) = 1$ at $x = (0.1), (0.2), (0.3), (0.4), (0.5)$

Solution:

Given

$$f(x, y) = y' = y\,(e^x - 1)$$
$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4, x_5 = 0.5$$
$$y_0 = 1,\ h = 0.1$$

Euler's formula is $y_{n+1} = y_n + hf(x_0, y_0)$

$$= 1 + 0.1f(0, 1)$$
$$= 1 + 0.1 \times 1\,(e^0 - 1)$$

$$\boxed{y_1 = y(0.1) = 1}$$

$$y(0.2) = y_2 = y_1 + hf(x_1, y_1)$$
$$= 1 + 0.1f(0.1, 1)$$
$$= 1 + 0.1 \times 1\,(e^{0.1} - 1)$$

$$\boxed{y_2 = y(0.2) = 1.0105}$$

$$y\,(0.3) = y_3 = y_2 + hf\,(x_2, y_2)$$

$$= 1.0105 + 0.1f\,(0.2, 1.0105)$$

$$= 1.0105 + 0.1 \times 1.0105\,(e^{0.2} - 1)$$

$$\boxed{\mathbf{y_3 = y\,(0.3) = 1.0329}}$$

$$y\,(0.4) = y_4 = y_3 + hf\,(x_3, y_3)$$

$$= 1.0329 + 0.1f\,(0.3, 1.0329)$$

$$= 1.0329 + 0.1 \times 1.0329\,(e^{0.3} - 1)$$

$$\boxed{\mathbf{y_4 = y\,(0.4) = 1.0690}}$$

$$y\,(0.5) = y_5 = y_4 + hf\,(x_4, y_4)$$

$$= 1.0690 + 0.1f\,(0.4, 1.0690)$$

$$= 1.0690 + 0.1 \times 1.0690\,(e^{0.4} - 1)$$

$$\boxed{\mathbf{y_5 = y\,(0.5) = 1.1216}}$$

Example 7: Solve the initial value problem $\frac{dy}{dx} = x^2 + y^2$ with $y\,(0) = 1$ find $y\,(0.1)$ and $y\,(0.2)$ by Euler method taking the step size $h = 0.1$

Solution:

Given

$$\frac{dy}{dx} = x^2 + y^2$$

$$y' = x^2 + y^2$$

$$f\,(x, y) = x^2 + y^2$$

Initial condition $y\,(0) = 1$

$$x_0 = 0 \quad y_0 = 1$$

Step size $h = 0.1$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_1 + 0.1\ (1)$$

$$= 1.1$$

$$\boxed{\mathbf{y_1 = y\ (0.1) = 1.1}}$$

$$x_1 = 0.1 \quad y_1 = 1.1$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = 1.1 + 0.1\ (1.02)$$

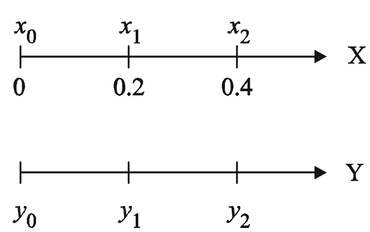
$$y_2 = 1.222$$

$$\boxed{\mathbf{y_2 = y\ (0.2) = 1.222}}$$

Example 8: Solve $\frac{dy}{dx} = x^2 - y$ with $y\ (0) = 1$, find $y\ (0.2)$ and $y\ (0.4)$ by Euler method.

Solution:

Given



$$\frac{dy}{dx} = x^2 - y$$

$$y' = x^2 - y$$

$$f(x, y) = x^2 - y^2$$

Initial condition $y\ (0) = 1$, $x_0 = 0$, $y_0 = 1$, $\mathbf{h = 0.2}$

By Euler method

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.2\ (-\ 1)$$

$$y_1 = 0.8$$

$$\boxed{\mathbf{y_1 = y\ (0.2) = 0.8}}$$

$$x_1 = 0.2, \quad y_1 = 0.8$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 0.82 + 0.2\,(-\,0.76)$$

$$y_2 = 0.648$$

$$\boxed{\mathbf{y_2 = y\,(0.4) = 0.648}}$$

Example 9: Apply modified Euler method, find $y\,(0.1)$ and $y\,(0.2)$ correct to 3 decimal places if $\frac{dy}{dx} = y^2 + x$ and $y\,(0) = 1$

Solution:

Given

$$\frac{dy}{dx} = y^2 + x$$

$$y' = y^2 + x$$

$$f(x, y) = y^2 + x$$

Initial condition $y\,(0) = 1$

$$x_0 = 0, \; y_0 = 1$$

$$h = 0.1$$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

$$= 1 + 0.1f\left[0 + \frac{0.1}{2}, \; 1 + \frac{0.1}{2}(1) \right]$$

$$= 1 + 0.1f\,[0 + 0.05, 1.05]$$

$$= 1 + 0.1f\,[0.05, 1.05]$$

$$= 1 + 0.1\,(1.1525)$$

$$y_1 = 1.115$$

$$y_1 = y\,(0.1) = 1.115$$

$$x_1 = 0.1 \quad y_1 = 1.115$$

$$y_2 = y_1 + hf\left[\, x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f\,(x_1, y_1) \,\right]$$

$$= 1.115 + 0.1f\left[\, 0.1 + \frac{0.1}{2},\; 1.115 + \frac{0.1}{2}\,(1.343) \,\right]$$

$$= 1.115 + 0.1f\,[0.15, 1.182]$$

$$= 1.115 + 0.1\,(1.547)$$

$$= 1.270$$

$$\boxed{\mathbf{y_2 = y\,(0.2) = 1.270}}$$

Example 10: Compute $y\,(0.2)$ and $y\,(0.4)$ given $\frac{dy}{dx} = y - x$ with $y\,(0) = 2$ by Modified Euler method.

Solution:

Given

$$\frac{dy}{dx} = y - x$$

$$y' = y - x$$

$$f\,(x, y) = y - x$$

Initial condition $y\,(0) = 2$

$$x_0 = 0, y_0 = 2$$

$$h = 0.2$$

$$y_1 = y_0 + hf\left[\, x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f\,(x_0, y_0) \,\right]$$

$$= 2 + 0.2f\left[\, 0 + \frac{0.2}{2}, 2 + \frac{0.2}{2} f\,(0, 2) \,\right]$$

$$= 2 + 0.2f\,[0 + 0.1,\ 2 + 0.1\ (2)]$$

$$= 2 + 0.2f\,[0.1,\ 2.2]$$

$$= 2 + 0.2\ (2.1)$$

$\boldsymbol{y\ (0.2) = y_1 = 2.42}$

$$y_2 = y_1 + hf\left[\ x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f\,(x_1,\ y_1)\ \right]$$

$$= 2.42 + 0.2f\left[\ 0.2 + \frac{0.2}{2},\ 2.42 + \frac{0.2}{2} f\,(0.2,\ 2.42)\ \right]$$

$$= 2.42 + 0.2f\,[0.3,\ 2.52\ (2.22)]$$

$$= 2.42 + 0.2f\,[0.3,\ 0.537]$$

$$= 2.42 + 0.2\ (5.2944)$$

$$y_2 = 3.478$$

$$y_2 = y\ (0.4) = 3.478$$

Example 1: Using Euler's method compute the values of $y\ (0.1), y\ (0.2), y\ (0.3)$ and $y\ (0.4)$ for $\frac{dy}{dx} = 1 - 2xy$ given that $y\ (0) = 0$

Solution:

Given

$$\frac{dy}{dx} = 1 - 2xy$$

$$y' = 1 - 2xy$$

$$f\,(x,\ y) = 1 - 2xy$$

Initial condition $y\ (0) = 0$

$$h = 0.1$$

$$y_1 = y_0 + hf\,(x_0,\ y_0)$$

$$x_0 = 0, y_0 = 0$$

$$y_1 = 0 + (0.1) f(0, 0)$$

$$y_1 = 0.1$$

$$\boxed{\mathbf{y_1 = y\ (0.1) = 0.1}}$$

$$x_1 = 0.1, y_1 = 0.1$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 0.1 + 0.1f(0.1, 0.1)$$

$$y_2 = 0.198$$

$$\boxed{\mathbf{y_2 = y\ (0.2) = 0.198}}$$

$$y_2 = x_2 = 0.2, y_2 = 0.198$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 0.198 + 0.2f(0.2, 0.198)$$

$$y_3 = 0.290$$

$$\boxed{\mathbf{y_3 = y\ (0.3) = 0.290}}$$

$$x_3 = 0.3, y_3 = 0.290$$

$$y_4 = y_3 + hf(x_3, y_3)$$

$$= 0.290 + 0.1f(0.3, 0.290)$$

$$y_4 = 0.372$$

$$\boxed{\mathbf{y_4 = y\ (0.4) = 0.372}}$$

Example 12: $\frac{dy}{dx} = x^3 + y,\ y(1) = 1$ **find** $y(1.1), y(1.2)$ **by Euler method by taking** $h = 0.1.$

Solution:

Given

$$\frac{dy}{dx} = x^3 + y$$

$$y' = x^3 + y$$

$$f(x, y) = x^3 + y$$

Initial condition $y(1) = 1$

$$x_0 = 1,\ y_0 = 1,\ h = 0.1$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.1 f(1, 1)$$

$$= 1.2$$

$$\boxed{y_1 = y(0.1) = 1.2}$$

$$x_1 = 1.1,\ y_1 = 1.2$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.2 + 0.1 f(1.1, 1.2)$$

$$y_2 = 1.423$$

$$\boxed{y_2 = y(1.2) = 1.453}$$

Example 13: **Use modified Euler method to get the values of $y\ (0.1), y\ (0.2)$ and $y\ (0.3)$ given $y' = xy + y^2$ with $y\ (0) = 1$**

Solution:

Given

$$y' = xy + y^2$$

$$\frac{dy}{dx} = xy + y^2$$

$$f\,(x, y) = xy + y^2$$

Initial condition $y\ (0) = 1$

$h = 0.1$

$x_0 = 0,\ y_0 = 1$

$$y_1 = y_0 + hf\left[\ x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f\,(x_0, y_0)\ \right]$$

$$= 1 + 0.1f\left[\ 0 + \frac{0.1}{2},\ 1 + \frac{0.1}{2} f\,(0, 1)\ \right]$$

$$= 1 + 0.1f\,[0.05,\ 1.05\ (1)]$$

$$= 1 + 0.1\ (1.155)$$

$$\boxed{\mathbf{y_1 = y\ (0.1) = 1.115}}$$

$x_1 = 0.1,\ y_1 = 1.115$

$$y_2 = y_1 + hf\left[\ x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f\,(x_1, y_1)\ \right]$$

$$= 1.115 + 0.1f\left[\ 0.1 + \frac{0.1}{2},\ 1.115 + \frac{0.1}{2} f\,(0.1,\ 1.15)\ \right]$$

$$= 1.115 + 0.1f\,[0.15,\ 1.182]$$

$y_2 = 1.272$

$$\boxed{\mathbf{y_2 = y\ (0.2) = 1.272}}$$

$x_2 = 0.2,\ y_2 = 1.272$

$$y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f(x_2, y_2) \right]$$

$$= 1.272 + 0.1f\left[0.2 + \frac{0.1}{2}, 1.272 + \frac{0.1}{2} f(0.2, 1.272) \right]$$

$$= 1.272 + 0.1f\,[0.25, 1.365]$$

$y_3 = 1.492$

$\boxed{\mathbf{y_3 = y\,(0.3) = 1.492}}$

Example 14: Use Modified Euler method to solve $y' = x + y + xy$ given that $y(0) = 1$ find y at $x = 0.1$

Solution:

Given

$$y' = x + y + xy$$

$$\frac{dy}{dx} = x + y + xy$$

$$f(x, y) = x + y + xy$$

Initial condition $y(0) = 1$

$h = 0.1$

$x_0 = 0,\ y_0 = 1$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

$$= 1 + 0.1f\left[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2} f(0, 1) \right]$$

$$= 1 + 0.1f\,[0.05, 1.05]$$

$$= 1.115$$

$\boxed{\mathbf{y_1 = y\,(0.1) = 1.115}}$

$x_1 = 0.1,\ y_1 = 1.115$

$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$= 1.115 + 0.1f\left[0.1 + \frac{0.1}{2}, 1.115 + \frac{0.1}{2} f(0.1, 1.115)\right]$$

$$= 1.115 + 0.1f\,[0.15, 1.131]$$

$$= 1.265$$

$$\boxed{\mathbf{y_2 = y\,(0.2) = 1.265}}$$

Example 15: Compute y at $x = 0.25$ by Modified Euler method given $y' = 2xy, y(0) = 1$. *(BR. Nov. 1995)*

Solution:

Here, $f(x, y) = 2xy:\ x_0 = 0, y_0 = 1$

Take $h = 0.25, x_1 = 0.25$

By Modified Euler method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right] \quad \ldots (1)$$

$$\therefore \quad y_1 = y_0 + h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right)\right]$$

$$f(x_0, y_0) = f(0, 1) = 2\,(0)\,(1) = 0$$

$$\therefore \quad y_1 = 1 + (0.25)\,[f(0.125, 1)]$$

$$= 1 + (0.25)\,[2 \times 0.125 \times 1]$$

$$\boxed{\mathbf{y\,(0.25) = 1.0625}}$$

Example 16: Using modified Euler's method, compute $y\,(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}, y\,(0) = 1.$

[A.U. May 1999, CBT N/D 2010]

Solution:

Given

$$f\,(x,\,y) = y - \frac{2x}{y},\, x_0 = 0,\, y_0 = 1,\, x_1 = 0.1,\, h = 0.1$$

By modified Euler's method,

$$\boldsymbol{y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n) \right)}$$

$$n = 0 \Rightarrow \quad y_1 = y_0 + h\left[f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f\,(x_0,\,y_0) \right]\right]$$

$$= 1 + (0.1)\left[f\left[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\left[y_0 - \frac{2x_0}{y_0} \right]\right]\right]$$

$$= 1 + (0.1)\,f\,[0.05,\, 1 + (0.05)\,[1 - 0]]$$

$$= 1 + (0.1)\,f\,[0.05,\, 1.05]$$

$$y\,(0.1) = 1 + (0.1)\left[1.05 - \frac{2\,(0.05)}{1.05} \right] = 1.0955$$

Example 17: Using modified Euler's method, find $y\,(0.1), y\,(0.2)$ given that $y\,' = y + e^x$ with $y\,(0) = 0.$ *[A.U. N/D 2019, R-17]*

Solution:

Given

$$f\,(x,\,y) = y + e^x,\;\; x_0 = 0,\, y_0 = 0,\, h = 0.1$$

$$x_1 = 0.1,\, x_2 = 0.2$$

By modified Euler's method,

$$\boxed{y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n) \right)}$$

$$n = 0 \Rightarrow \quad y_1 = y_0 + h\left[f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right] \right]$$

$$y(0.1) = 0 + (0.1)\left[f\left(0 + \frac{0.1}{2}, 0 + \frac{0.1}{2} f(0, 0) \right) \right]$$

$$= (0.1)\, [f(0.05, (0.05)\, [0 + e^0]]$$

$$= (0.1)\, [f(0.05, 0.05)\,] \qquad [\because e^0 = 1]$$

$$= (0.1)\, [0.05 + e^{0.05}] = 0.1101$$

$$\boxed{\Rightarrow y(0.1) = 0.1101}$$

$$n = 1 \Rightarrow y_2 = y_1 + h\left[f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1) \right) \right]$$

$$= 0.1101 + 0.1\left[f\left(0.1 + \frac{0.1}{2}, 0.1101 + \frac{0.1}{2} f(0.1, 0.1101) \right) \right]$$

$$= 0.1101 + (0.1)\, [f(0.15, 0.1101 + (0.05)\, [0.1101 + e^{0.1}]]$$

$$= 0.1101 + (0.1)\, [f(0.15, 0.1709)]$$

$$= 0.1101 + (0.1)\, [0.1709 + e^{0.15}] = 0.2434$$

$$\boxed{\Rightarrow y(0.2) = 0.2434}$$

Example 18: Evaluate $y(1.2)$ correct to three decimal places, by the modified Euler's method, given that $\frac{dy}{dx} = (y - x^2)^3$, $y(1) = 0$ taking $h = 0.2$ *[A.U. May 1996] [A.U. M/J 2014]*

Solution:

Given

$$f(x, y) = (y - x^2)^3, x_0 = 1, y_0 = 0, h = 0.2, x_1 = 1.2$$

By modified Euler's method,

$$\boxed{y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)}$$

$$n = 0 \Rightarrow \quad y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right)\right]$$

$$= 0 + (0.2)f\left[\left[1 + \frac{0.2}{2}\right], 0 + \frac{0.2}{2}[y_0 - x_0^2]^3\right]$$

$$= (0.2)f[(1.1, (0.1)[0 - 1]^3]$$

$$= (0.2)f(1.1, -0.1)$$

$$= (0.2)[-0.1 - (1.1)^2]^3 = -0.45$$

$$\boxed{y(1.2) = -0.45}$$

Example 19: Solve $y' = 1 - y, y(0) = 0$ by modified Euler's method. Find $y(0.1), y(0.2)$ and $y(0.3)$.

[A.U. April, 2005] [A.U CBT A/M 2011]
[A.U N/D 2022 R-21]

Solution:

Given

$$f(x, y) = 1 - y, x_0 = 0, y_0 = 0, x_1 = 0.1,$$

$$x_2 = 0.2, x_3 = 0.3, h = 0.1$$

$$\boxed{y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{1}{2}f(x_n, y_n)\right]} \quad \text{... (1)}$$

$$n = 0 \Rightarrow \quad y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right] \quad \text{... (2)}$$

$$f(x_0, y_0) = 1 - y_0 = 1 - 0 = 1$$

$$(2) \Rightarrow y_1 = 0 + (0.1)\, f\left[0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}(1) \right]$$

$$= (0.1)\, f\,[0.05, 0.05] = (0.1)\,[1 - 0.05] = 0.095$$

$$\boxed{\Rightarrow \boldsymbol{y\,(0.1) = 0.095}}$$

$$n = 1 \Rightarrow \quad y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f\,(x_1, y_1) \right] \qquad \ldots (3)$$

$$f\,(x_1, y_1) = 1 - y_1 = 1 - 0.095 = 0.905$$

$$y_2 = (0.095) + (0.1)\, f\left[(0.1) + \frac{0.1}{2}, (0.095) + \frac{0.1}{2}(0.905) \right]$$

$$= (0.095) + (0.1)\, f\,[0.15, 0.14025]$$

$$= 0.095 + (0.1)\,[1 - 0.14025]$$

$$= 0.095 + (0.1)\,(0.85975) = 0.18098$$

$$\boxed{\Rightarrow \boldsymbol{y\,(0.2) = 0.18098}}$$

$$n = 3 \Rightarrow \quad y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f\,(x_2, y_2) \right] \qquad \ldots (4)$$

$$f\,(x_2, y_2) = 1 - y_2 = 1 - 0.18098 = 0.81902$$

$$= 0.18098 + (0.1)\, f\left[0.2 + \frac{0.1}{2}, 0.18098 + \frac{0.1}{2}(0.81902) \right]$$

$$= 0.18098 + (0.1)\, f\,[0.25, 0.18098 + 0.040951]$$

$$= 0.18098 + (0.1)\, f\,(0.25, 0.221931)$$

$$= (0.18098) + (0.1)\,[1 - 0.221931] = 0.258787$$

$$\boxed{\Rightarrow \boldsymbol{y\,(0.3) = 0.258787}}$$

Example 20: Using modified Euler's method, find $y(0.1)$ if $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$.

[A.U. N/D 2004] [A.U N/D 2020 R-17 (NM), A/M 2021 R-17 (NM) [A.U A/M 2022 R-21]

Solution:

Given:

$f(x, y) = x^2 + y^2, x_0 = 0, y_0 = 1, h = 0.1, x_1 = 0.1$

By modified Euler's method,

$$\boxed{y_{n+1} = y_n + hf\left[x_0 + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]} \quad \text{... (1)}$$

$$n = 0 \Rightarrow \quad y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right] \quad \text{... (2)}$$

$$f(x_0, y_0) = x_0^2 + y_0^2 = 0 + 1 = 1$$

$$y_1 = 1 + (0.1) f\left[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}(1)\right]$$

$$= 1 + (0.1) f[0.05, 1.05]$$

$$= 1 + (0.1)[(0.05)^2 + (1.05)^2] = 1.1105$$

$\boxed{\textbf{(i.e.,) } y(0.1) = 1.1105}$

5.3 FOURTH ORDER RUNGE - KUTTA METHOD FOR SOLVING FIRST ORDER ODE'S

The Runge - Kutta formula of fourth order for solving the first order ODE $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ is given by the equations

$$\boxed{\Delta y_n = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$y_{n+1} = y_n + \Delta y_n$$

Note

1. This is the most widely used of all the Runge-Kutta formulae, that it is simply referred to as "Runge-Kutta formula". (R-K method of 4^{th} order)

2. This method agrees with Taylor's series solution upto the term in h^4.

WORKED EXAMPLES

Example 1: Solve by Runge-Kutta method, $y\,(0.1), y\,(0.2), y\,(0.3), y\,(0.4)$ for $\frac{dy}{dx} = 1 - 2xy$ with $y\,(0) = 0$.

Solution:

Given

$$\frac{dy}{dx} = 1 - 2xy$$

$$y' = 1 - 2xy$$

$$f\,(x, y) = 1 - 2xy$$

$$y\,(0) = 0$$

$$x_0 = 0, y_0 = 0$$

$$h = 0.1$$

Formula:

$$k_1 = hf\,(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf\,(x_n + h, y_n + k_3)$$

$$k_1 = hf\,(x_0, y_0)$$

$$= 0.1 f\,(0, 0)$$

$$\boxed{k_1 = 0.1}$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}\right)$$

$$= 0.1f\,(0.05, 0.05)$$

$$\boxed{\mathbf{k_2 = 0.0995}}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 0 + \frac{0.0995}{2}\right)$$

$$= 0.1f\,(0.05, 0.049)$$

$$\boxed{\mathbf{k_3 = 0.09951}}$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

$$= 0.1f\,(0 + 0.1, 0 + 0.09951)$$

$$\boxed{\mathbf{k_4 = 0.09800}}$$

$$y\,(0.2) = y_1 = y_0 + \Delta\, y_0$$

$$\Delta\, y_0 = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.1 + 2\,(0.0995) + 2\,(0.9951) + 0.0980)}{6}$$

$$\Delta\, y_0 = 0.099$$

$$y\,(0.1) = y_1 = 0 + 0.099$$

$$y_1 = 0.099$$

$$y\,(0.1) = 0.199$$

$$x_1 = 0.2,\; y_1 = 0.099$$

$$k_1 = hf(x_1, y_1)$$

$$= 0.1f(0.1, 0.099)$$

$$\boxed{\boldsymbol{k_1 = 0.098}}$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0.1 + \frac{0.1}{2}, 0.099 + \frac{0.04}{2}\right)$$

$$= 0.1f(0.15, 0.148)$$

$$\boxed{\boldsymbol{k_2 = 0.09556}}$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1f\left(0.1 + \frac{0.1}{2}, 0.099 + \frac{0.959}{2}\right)$$

$$= 0.1f(0.15, 0.146)$$

$$\boxed{\boldsymbol{k_3 = 0.09562}}$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$$

$$= 0.1f(0.1 + 0.1, 0.099 + 0.09562)$$

$$= 0.1f(0.2, 0.194)$$

$$\boxed{\boldsymbol{k_4 = 0.09224}}$$

$$\Delta y_1 = \frac{k_1 + 2k_2 + 2k_3 + 2k_4}{6}$$

$$= \frac{0.098 + 2\,(0.09556) + 2\,(0.09562) + 0.09224)}{6}$$

$$\Delta y_1 = 0.095$$

$$y_2 = y_1 + \Delta y_1$$

$$= 0.099 + 0.095$$

$$\boxed{\mathbf{y\ (0.2) = y_2 = 0.194}}$$

$$x_2 = 0.2, \ \ y_2 = 0.194$$

$$k_1 = hf\,(x_2, y_2)$$

$$= 0.1f\,(0.2, 0.194)$$

$$\boxed{\mathbf{k_1 = 0.09224}}$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2} \right)$$

$$= 0.1f\left(0.2 + \frac{0.1}{2}, 0.194 + \frac{0.09224}{2} \right)$$

$$= 0.1f\,(0.25, 0.240)$$

$$\boxed{\mathbf{k_2 = 0.088}}$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2} \right)$$

$$= 0.1f\left(0.2 + \frac{0.1}{2}, 0.194 + \frac{0.088}{2} \right)$$

$$= 0.1f\,(0.25, 0.238)$$

$$\boxed{\mathbf{k_3 = 0.0881}}$$

$$k_4 = hf\,(x_2 + h, y_2 + k_3)$$

$$= 0.1f\,(0.2 + 0.1, 0.194 + 0.0881)$$

$$= 0.1f\,(0.3, 0.2824)$$

$$\boxed{\mathbf{k_4 = 0.083074}}$$

$$\Delta y_3 = y_2 + \Delta y_2$$

$$\Delta y_3 = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.09224 + 2\,(0.088) + 2\,(0.0881) + 0.083074}{6}$$

$$\Delta y_3 = 0.0879$$

$$y_3 = 0.194 + 0.087$$

$$y_3 = y_2 = 0.281$$

Example 2: Apply Runge-Kutta method of fourth order to determine $y\,(0.1)$ with $h = 0.1$ from $\frac{dy}{dx} = x^2 + y^2, y\,(0) = 1$

Solution:

Given

$$\frac{dy}{dx} = x^2 + y^2$$

$$y' = x^2 + y^2$$

$$f\,(x, y) = x^2 + y^2$$

$$x_0 = 0, y_0 = 1, h = 0.1$$

Formula:

$$k_1 = hf\,(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf\,(x_n + h, y_n + k_3)$$

$$k_1 = hf\,(x_0, y_0)$$

$$= 0.1 f\,(0, 1)$$

$$\boxed{k_1 = 0.1}$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$\boxed{k_2 = 0.1105}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}\ 1 + \frac{0.1105}{2}\right)$$

$$= 0.1f\,(0.05,\ 1.05525)$$

$$\boxed{\boldsymbol{k_3 = 0.1116}}$$

$$k_4 = hf\,(x_0 + h,\ y_0 + k_3)$$

$$= 0.1f\,(0 + 0.1,\ 1 + 0.1116)$$

$$= 0.1f\,(0.1,\ 1.1116)$$

$$\boxed{\boldsymbol{k_4 = 0.1245}}$$

$$\Delta\, y_0 = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.1 + 2\,(0.1105) + 2\,(0.116) + 0.1245}{6}$$

$$\boxed{\boldsymbol{\Delta\, y_0 = 0.11148}}$$

$$y_1 = y_0 + \Delta\, y_0 = 1 + 0.11148 = 1.1114$$

$$\boxed{\boldsymbol{y\,(0.1) = 1.1114}}$$

Example 3: Solve the $\frac{dy}{dx} = x^3 + x^3, y\,(0) = 1$ find $y\,(0.1)$ by Runge-Kutta method.

Solution:

Given:

$$\frac{dy}{dx} = x^3 + y^3$$

$$y' = x^3 + y^3$$

$$f\,(x, y) = x^3 + y^3$$

Initial condition $= y\,(0) = 1$

Formula:

$$k_1 = hf\,(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf\,(x_n + h,\ y_n + k_3)$$

$$x_0 = 0,\ y_0 = 1$$

$$h = 0.1$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1f(0, 1)$$

$$\boxed{\mathbf{k_1 = 0.1}}$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= 0.1f(0.05, 1.05)$$

$$\boxed{\mathbf{k_2 = 0.1157}}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1157}{2}\right)$$

$$= 0.1f(0.05, 1.0578)$$

$$\boxed{\mathbf{k_3 = 0.1183}}$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(0 + 0.1, 1 + 0.1183)$$

$$= 0.1f(0.1, 1.1183)$$

$$\boxed{\mathbf{k_4 = 0.1399}}$$

$$\Delta y_0 = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.1 + 2\,(0.1157) + 2\,(0.1183) + 0.1399}{6}$$

$$\boxed{\mathbf{\Delta y_0 = 0.1179}}$$

$$y_1 = y_0 + \Delta y_0 = 1 + 0.1179$$

$$= 1.1179$$

$$\boxed{\mathbf{y_1 = y\,(0.1) = 1.1179}}$$

Example 4: Using Runge-Kutta method of fourth order, solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 - x^2}$ with $y\,(0) = 1$ at $x = 0.2$.

[A.U. N/D 2004, 2006, 2007, 2015 (R-13), 2017 (R-18)]
[A.U A/M 2005, 2010, 2015 (R-8), 2013, 2017 (R-8)]
[A.U N/D 2022 R-21]

Solution:

Given:

$$y' = f\,(x, y) = \frac{y^2 - x^2}{y^2 + x^2},\ x_0 = 0,\ y_0 = 1,\ x_1 = 0.2$$

$$h = 0.2$$

$$k_1 = hf\,(x_0, y_0) = (0.2)\left[\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right] = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= (0.2)\,f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$$

$$= (0.2)\,f\,[0.1, 1.1] = (0.2)\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right] = 0.1967$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$$

$$= (0.2)\,f\left[0 + \frac{0.2}{2}, 1 + \frac{0.1967}{2}\right]$$

$$= (0.2)\,f\,[0.1, 1.0984] = (0.2)\left[\frac{(1.0984)^2 - (0.1)^2}{(1.0984)^2 + (0.1)^2}\right] = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.2)\, f(0.2, 1.1967) = (0.2)\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right] = 0.1891$$

$$\Delta y_0 = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.2 + 2\,(0.1967) + 2\,(0.1967) + 0.1891] = 0.1960$$

$$y\,(0.2) = y_1 = y_0 + \Delta y = 1 + 0.1960 = 1.1960$$

Example 5: Using Runge-Kutta method of fourth order find $y\,(0.1)$ and $y\,(0.2)$ for the initial value problem $\frac{dy}{dx} = x + y^2$, $y\,(0) = 1$. *[Anna, Oct. 1996] [A.U N/D 2010] [A.U N/D 2017 R-13, A/M 2023 R-21]*

Solution:

Given:

$$y' = f(x, y) = x + y^2,\ x_0 = 0,\ y_0 = 1,\ h = 0.1$$

To find $y\,(0.1)$:

$$k_1 = hf(x_0, y_0) = (0.1)\, f[0, 1] = (0.1)\,[0 + 1^2] = 0.1$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = (0.1)\, f\left[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right]$$

$$= (0.1)\, f(0.05, 1.05) = (0.1)\,[0.05 + (1.05)^2] = 0.11525$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = (0.1)\, f\left[0 + \frac{0.1}{2}, 1 + \frac{0.11525}{2}\right]$$

$$= (0.1)\, f[0.05, 1.057625]$$

$$= (0.1)\,[0.05 + (1.057625)^2] = 0.116857$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\, f[0 + 0.1,\ 1 + 0.116857]$$

$$= (0.1)\, f[0.1,\ 1.116857] = (0.1)\ [0.1 + (1.116857)^2] = 0.1347$$

$$\Delta\, y = \frac{1}{6}\, [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\, [0.1 + 2\ (0.11525) + 2\ (0.116857) + 0.1347]$$

$$= \frac{1}{6}\, [0.698914] = 0.11649$$

$$y\ (0.1) = y_0 + \Delta\, y = 1 + 0.11649 = 1.1165$$

$$\boxed{\Rightarrow\ \boldsymbol{y\ (0.1) = 1.1165}}$$

(ii) To find y (0.2)

$$\therefore\ x_1 = 0.1,\ y_1 = 1.1165$$

$$k_1 = hf\, [x_1, y_1] = (0.1)\, f\, [0.1,\ 1.1165]$$

$$= (0.1)\ [0.1 + (1.1165)^2] = 0.1347$$

$$k_2 = hf\left[\, x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\,\right]$$

$$= (0.1)\, f\left[\, 0.1 + \frac{0.1}{2},\ 1.1165 + \frac{0.1347}{2}\,\right]$$

$$= (0.1)\, f\, [0.15,\ 1.18385]$$

$$= (0.1)\ [0.15 + (1.18385)^2] = 0.1552$$

$$k_3 = hf\left[\, x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\,\right]$$

$$= (0.1)\, f\left[\, 0.15,\ 1.1165 + \frac{0.1552}{2}\,\right]$$

$$= (0.1)\, f\, [0.15,\ 1.1941] = (0.1)\ [0.15 + (1.1941)^2] = 0.1576$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3)$$

$$k_4 = (0.1) f(0.1 + 0.1, 1.1165 + 0.1576]$$

$$= (0.1) f(0.2, 1.2741) = (0.1)\ [0.2 + (1.2741)^2] = 0.1823$$

$$\Delta y_1 = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\Delta y_1 = \frac{1}{6}[0.1347 + 2(0.1552) + 2(0.1576) + 0.1823] = 0.1571$$

(i.e.,) $y(0.2) = y_1 + \Delta y_1 = 1.1165 + 0.1571 = 1.2736$

$$\boxed{\Rightarrow y(0.2) = 1.2736}$$

Example 6: Use the fourth order Runge-Kutta method to compute y for $x = 0.1$, given $y = \frac{xy}{1+x^2}$, $y(0) = 1$, take $h = 0.1$.

[Anna, May 1999]

Solution:

Given:

$$y' = f(x, y) = \frac{xy}{1+x^2},\ x_0 = 0,\ y_0 = 1,\ h = 0.1,\ x_1 = 0.1$$

To find $y(0.1)$

$$k_1 = hf(x_0, y_0) = (0.1) f[0, 1] = (0.1)\left[\frac{(0)(1)}{1+0^2}\right] = 0$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$= (0.1) f\left[0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right] = (0.1) f[0.05, 1]$$

$$= (0.1)\left[\frac{(0.05)(1)}{1+(0.05)^2}\right] = (0.1)\left[\frac{0.05}{1.0025}\right] = 0.0050$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$$

$$= (0.1) f\left[0.05, 1 + \frac{0.005}{2}\right] = (0.1) f[0.05, 1.0025]$$

$$= (0.1)\left[\frac{(0.05)(1.0025)}{1+(0.05)^2}\right] = 0.005$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.1) f[0 + 0.1, 1 + 0.005] = (0.1) f(0.1, 1.005)$$

$$= (0.1)\left[\frac{(0.1)(1.005)}{1 + (0.1)^2}\right] = 0.00995$$

$$\Delta y_0 = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0 + 2(0.005) + 2(0.005) + 0.00995] = 0.004992$$

$$y_1 = y(0.1) = y_0 + \Delta y_0 = 1 + 0.004992 = 1.004992$$

Example 7: Find $y(0.7)$ given that $y' = y - x^2$, $y(0.6) = 1.7379$ by using Runge-Kutta method of fourth order, Take $h = 0.1$.

[Anna, April, 2000]
[A.U M/J 2012] [A.U N/D 2016 R-13]

Solution:

Given:

$$y' = f(x, y) = y - x^2, x_0 = 0.6, y_0 = 1.7379,$$

$$x_1 = 0.7, h = 0.1$$

To find $y(0.7)$

$$k_1 = hf(x_0, y_0) = (0.1)[y_0 - x_0^2]$$

$$= (0.1)[1.7379 - (0.6)^2] = 0.13779$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = (0.1) f\left[0.6 + \frac{0.1}{2}, 1.7379 + \frac{0.13779}{2}\right]$$

$$= (0.1) f[0.65, 1.806795]$$

$$= (0.1)[1.806795 - (0.65)^2] = 0.13843$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$$

$$= (0.1)\, f\,[0.65,\ 1.807115]$$

$$= (0.1)\,[1.807115 - (0.65)^2] = 0.13846$$

$$k_4 = hf\,[x_0 + h,\ y_0 + k_3]$$

$$= (0.1)\, f\,[0.6 + 0.1,\ 1.7379 + 0.13846]$$

$$= (0.1)\, f\,[0.7,\ 1.87636] = (0.1)\,[1.87636 - (0.7)^2] = 0.13864$$

$$\Delta\, y_0 = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.13779 + 2\,(0.13843) + 2\,(0.13846) + 0.13864]$$

$$\Delta\, y_0 = 0.13837$$

$$y_1 = y\,(0.7) = y_0 + \Delta\, y_0 = 1.7379 + 0.13837 = 1.87627$$

$$= 1.876\ \text{(app)}$$

$$\boxed{\Rightarrow\ y\,(0.7) = 1.876}$$

Example 8: Solve the equation $\frac{dy}{dx} = x + y$ with initial condition $y\,(0) = 1$ by Runge-Kutta rule, from $x = 0$ to $x = 0.2$ with $h = 0.1$.

Solution:

Here $f\,(x, y) = x + y$, $h = 0.1$, $x_0 = 0$, $y_0 = 1$

We have,

$$k_1 = hf\,(x_0, y_0) = 0.1\,(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\,(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.11\,f\left(0.05,\ 1 + \frac{0.11}{2}\right)$$

$$\boxed{\mathbf{k_3 = 0.1105}}$$

$$k_4 = hf\,(x_0 + h,\ y_0 + k_3) = 0.1\,f\,(0 + 0.1,\ 1 + 0.1105)$$

$$\boxed{\mathbf{k_4 = 0.12105}}$$

$$\Delta\, y_0 = \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4) = 0.11034$$

Thus, $x_1 = x_0 + h = 0.1$ and $y_1 = y_0 + \Delta\, y_0 = 1.11034$

Now for the second interval, we have

$$k_1 = hf\,(x_1, y_1) = 0.1\,f\,(0.1, 1.11034) = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1\,f\left(0.1 + \frac{0.2}{2}, \frac{1.11034 + 0.121034}{2}\right)$$

$$\boxed{\mathbf{k_2 = 0.13208}}$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_3 = 0.1\,f\left(0.1 + \frac{0.1}{2}, \frac{1.11034 + 0.13208}{2}\right)$$

$$\boxed{\mathbf{k_3 = 0.13263}}$$

$$k_4 = hf\,(x_1 + h,\ y_1 + k_3)$$

$$= 0.1\,f\,(0.1 + 0.1,\ 1.11034 + 0.13263)$$

$$\boxed{\mathbf{k_4 = 0.14429}}$$

$$\Delta\, y_1 = \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4) = 0.132460$$

Hence $x_2 = 0.2$ and $y_2 = y_1 + \Delta\, y_1 = 1.11034 + 0.13246 = 1.24280$

Example 9: Given $\frac{dy}{dx} = y - x, y(0) = 2$. Find $y(0.1)$ and $y(0.2)$ correct to four decimal places, by R-K method

Solution:

By IV order method

$$k_1 = 0.2, k_2 = 0.205,$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.20525$$

and $$k_4 = hf(x_0 + h, y_0 + k_3) = 0.210525$$

$\therefore$ $$\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2052$$

Thus, $$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y_0 + \Delta y_0 = 2 + 0.2052 = 2.2052$$

Now to determine $y_2 = y(0.2)$, we note that

$$x_1 = x_0 + h = 0.1, y_1 = 2.2052, h = 0.1$$

For II interval,

$$k_1 = hf(x_1, y_1) = 0.21052$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.21605$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.216323$$

and $$k_4 = hf(x_1 + h, y_1 + k_3) = 0.221523$$

$\therefore$ $$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.21613$$

Thus, $$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

and $$y_2 = y_1 + \Delta y_1 = 2.2052 + 0.21613 = 2.4213$$

Hence $y(0.1) = 2.2052$, $y(0.2) = 2.4213$

Example 10: Given the initial value problem:

$$y' = 1 + y^2, y(0) = 0$$

Find $y(0.6)$ by Runge-Kutta fourth order method taking $h = 0.2$.

Solution:

Here, $f(x, y) = 1 + y^2$, $h = 0.2$, $x_0 = 0$, $y_0 = 0$

We have, $k_1 = hf(x_0, y_0) = 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.202$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2020402$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2081640$$

$$\therefore \quad \Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2027074$$

Thus, $y_1 = y(0.2) = y_0 + \Delta y_0 = 0.2027074$

$$x_1 = x_0 + h = 0.2$$

Again, $f(x, y) = 1 + y^2$, $h = 0.2$, $x_1 = 0.2$, $y_1 = 0.2027074$

We have, $k_1 = hf(x_1, y_1) = 0.208218$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.218827$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.219484$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.235649$$

$$\therefore \quad \Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.22008$$

Thus, $y_2 = (0.4) = y_1 + \Delta y_1 = 0.4227874$

$x_2 = y_2 + h = 0.4$

Again, $f(x, y) = 1 + y^2, h = 0.2, x_2 = 0.4, y_2 = 0.4227874$

We have, $k_1 = hf(x_2, y_2) = 0.235749$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.258463$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.260945$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = 0.293498$$

$\therefore$ $$\Delta y_2 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.261344$$

Thus, $y_3 = y(0.6) = y_2 + \Delta y_2 = 0.6841314$

Example 11: Use the Runge-Kutta fourth order method to find the value of y when $x = 1$ given that $y = 1$ when $x = 0$ (taking $h = 0.5$) and $\frac{dy}{dx} = \frac{y - x}{y + x}$.

Solution:

Here, $f(x, y) = \frac{y - x}{y + x}, x_0 = 0, y_0 = 1, h = 0.5$

We have, $k_1 = hf(x_0, y_0) = 0.5 f(0, 1) = 0.5$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.5 f\left(\frac{0.5}{2}, 1 + \frac{0.5}{2}\right)$$

$$\boxed{k_2 = 0.333}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.5 f\left(0 + \frac{0.5}{2}, 1 + \frac{0.333}{2}\right)$$

$$\boxed{k_3 = 0.3235}$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.5 f(0 + 0.5, 1 + 0.3235)$$

$$\boxed{k_4 = 0.2258}$$

$$\therefore \quad \Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.3398$$

Thus, $y_1 = y(0.5) = y_0 + \Delta y_0 = 1.3398$

$$x_1 = x_0 + h = 0.5$$

Again, $k_1 = hf(x_1, y_1) = 0.22823$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.15969$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.15432$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.09906$$

$$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.159218$$

Thus, $y_2 = y(1.0) = y_1 + \Delta y_1 = 1.499018$

5.4 MULTI-STEP METHODS (PREDICTOR – CORRECTOR METHODS)

Introduction

Predictor-corrector for methods are methods which require function values at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$, for the computation of the function value at x_{n+1}. A predictor is used to find the value of y at x_{n+1} and then a corrector formula is used to improve the value of y_{n+1}.

The following two methods are discussed in this section.

(1) Milne's method

(2) Adam's method

5.4.1 Milne's predictor-corrector method

Consider the initial value problem, $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. Assume that $y_0 = y(x_0), y_1 = y(x_1), y_2 = y(x_2)$ and $y_3 = y(x_3)$ where $x_{i+1} = x_i + h, i = 0, 1, 2, 3$ are known. These are the starting values.

$$\frac{dy}{dx} = f(x, y) \Rightarrow y = \int y' \, dx$$

By Newton's forward interpolation formula,

$$y(x_0 + ph) = y_0 + p \, \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \dots$$

$$\Rightarrow y' = y_0' + p \, \Delta y_0' + \frac{p(p-1)}{2!} \Delta^2 y_0' + \frac{p(p-1)(p-2)}{6} \Delta^3 y_0' + \dots$$

Integrating this between x_0 and x_4

$$y_4 = y_0 + \int_{x_0}^{x_4} \left(y_0' + \frac{p(p-1)}{2!} \Delta^2 y_0' + \dots \right) dx$$

where $p = \frac{x - x_0}{h} \Rightarrow ph = x - x_0 \therefore dx = hdp$

$$\therefore y_4 = y_0 + h \int_0^4 \left(y_0' + p \, \Delta y_0' + \frac{p^2 - p}{2} \Delta^2 y_0' \right.$$

$$\left. + \frac{p^3 - 3p^2 + 2p}{6} \Delta^3 y_0' + \dots \right) dp$$

$$\Rightarrow y_4 = y_0 + h \left[4y_0' + 8 \, \Delta y_0' + \frac{20}{3} \Delta^2 y_0' + \frac{8}{3} \Delta^3 y_0' + \dots \right]$$

Neglecting fourth and higher order differences, and expressing

$$\Delta y_0' = y_1' - y_0', \Delta^2 y_0' = y_2' - 2y_1' + y_0'$$

and $\qquad \Delta^3 y_0' = y_3' - 3y_2' + 3y_1' - y_0'$

We have

$$y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3'] + \frac{14h^5}{45} y^{(v)}(\xi) \text{ where } x_0 < \xi < x_4$$

In general, $y_{n+1} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n'$ ... (1)

and the error $= \frac{14h^5}{45} y^{(v)}(\xi)$ where $x_{n-3} < \xi < x_{n+1}$

Expression (1) is called *Milne's predictor formula*.

To obtain a correct formula, consider

$$\int_{x_0}^{x_2} y' \, dx = \int_{x_0}^{x_2} \left(y_0' + p \, \Delta y_0' + \frac{p(p-1)}{2} \Delta^2 y_0' + \ldots\ldots \right) dx$$

$$y_2 = y_0 + h \int_0^2 \left(y_0' + p \, \Delta y_0' + \frac{p(p-1)}{2} \Delta^2 y_0' + \ldots \right) dp$$

$$= y_0 + \frac{h}{3}[(y_0' + 4y_1' + y_2')] - \frac{h^3}{90} \Delta^4 y_0'$$

It can be proved to be error $= -\frac{h^5}{90} y^{(v)}(\xi)$ where $x_0 < \xi < x_2$.

$$y_2 = y_0 + \frac{h}{3}[(y_0' + 4y_1' + y_2')] - \frac{h^5}{90} y^{(v)}(\xi)$$

In general,

$$y_{n+1} = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}'] \qquad \text{... (2)}$$

and the error $= \frac{h^5}{90} y^{(v)}(\xi)$ where $x_{n-1} < \xi < x_{n+1}$.

Expression (2) is called *Milne's corrector formula*.

In particular, to compute y_4 corresponding to $x_4 = x_0 + 4h$, where h is the step-size,

Milne's predictor formula

$$y_{4,\,p} = y_0 + \frac{4h}{3}\,[2y_1' - y_2' + 2y_3'] \text{ and}$$

Milne's corrector formula

$$y_{4,\,c} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4'] \text{ where } y_4' = f(x_4, y_{4,\,p})$$

5.4.2 Adam-Bashforth (or Adam's) Predictor-corrector method

We state below another *predictor-corrector method*, called **Adam's method** or Adam-Bashforth method. We given below predictor and corrector formula without proof. Here also, we require four continuous values of y to find the value of y at the fifth point similar to Milne's method.

Predictor: $y_{n+1,\,p} = y_n + \dfrac{h}{24}$

$$[55y_n' - 59y_{n-1} + 37y_{n-2}' - 9y_{n-3}']$$

Corrector: $y_{n+1,\,c} = y_n + \dfrac{h}{24}$

$$[9y_{n+1}' + 19y_n' - 5y_{n-1}' + 9y_{n-2}']$$

WORKED EXAMPLES

Example 1: Using Milne's method, compute $y\,(0.8)$ given that $\dfrac{dy}{dx} = 1 + y^2$, $y\,(0) = 0$, $y\,(0.2) = 0.2027$, $y\,(0.4) = 0.4228$ and $y\,(0.6) = 0.6841$

Solution:

We have the following table of values

$$\therefore\ x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$$

$y_0 = 0, y_1 = 0.2027, y_2 = 0.4228, y_3 = 0.6841$

$y_0' = 1 + 0^2 = 1$

$y_1' = 1 + (0.2027)^2 = 1.0411$

$y_2' = 1 + (0.4228)^2 = 1.1787$

$y_3' = 1 + (0.6841)^2 = 1.4681$

To find *y* (0.8)

$$x_4 = 0.8. \text{ Here } h = 0.2$$

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 0 + \frac{0.8}{3}[2(1.0411) - 1.1787 + 2(1.4681)]$$

$$\therefore y_{4,p} = 1.0239$$

$$y_4' = f(x_4, y_4) = 1 + (1.0239)^2$$

$$\boxed{y_4' = 2.0480}$$

By Milne's corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 0.4228 + \frac{0.2}{3}[1.1787 + 4(1.4681) + 2.0480]$$

$$= 1.0294$$

$\therefore y(0.8) = 1.0294$

Example 2: Using Milne's method compute $y\,(0.4)$ given that $\dfrac{dv'}{dx} = x^2 + y$, with $y\,(0) = 1, y\,(0.1) = 1.1055, y\,(0.2) = 1.2241$ $y\,(0.3) = 1.3594$ find $y\,(0.4)$.

Solution:

Given

$$y' = x^2 + y$$

$$\frac{dy}{dx} = x^2 + y$$

$$f\,(x, y) = x^2 + y$$

Initial condition $y\,(0) = 1$

x_0	x_1	x_2	x_3	x_4
0	0.1	0.2	0.3	0.4

y_0	y_1	y_2	y_3	y_4
1	1.1055	1.2241	1.3594	?

Predictor formula, $y_{4,\,p} = y_0 + \dfrac{4h}{3}\,[2y_1' - y_2' + 2y_3']$

Corrector formula, $y_{4,\,c} = y_2 + \dfrac{h}{3}\,[y_2' + 4y_3' + y_4']$

x	y	$y^2 = x^2 + y$
$x_0 = 0$	$y_0 = 1$	$y_0' = 1$
$x_0 = 0.1$	$y_1 = 1.1055$	$y_1' = 1.1155$
$x_2 = 0.2$	$y_2 = 0.2241$	$y_2' = 1.2641$
$x_3 = 0.3$	$y_3 = 1.3594$	$y_3' = 1.4494$
$x_4 = 0.4$	$y_4 = 1.5154$	$y_4' = 1.6754$

$$y_{4,\,p} = 1 + \frac{4\,(0.1)}{3}\,[2\,(1.1155) - 1.2641 + 2\,(1.4494)]$$

$$= 1 + \frac{0.4}{3}\,(3.3657)$$

$$\boxed{\mathbf{y_{4,\,p} = 1.5154}}$$

$$y_{4,\,c} = 1.2241 + \frac{0.1}{3}\,[1.2641 + 4\,(1.4494) + 1.6754]$$

$$= 1.2241 + \frac{0.1}{3}\,[0.7271]$$

$$= 1.2241 - 0.9912$$

$$\boxed{\mathbf{y_{4,\,c} = 1.5153}}$$

Adam's method

Predictor formula

$$y_{4,\,p} = y_3 + \frac{h}{24}\,[55y_1' - 59y_2' + 37y_1' - 9y_0']$$

Corrector formula

$$y_{4,\,c} = y_3 + \frac{h}{24}\,[9y_4' + 19y_3' - 5y_2' + y_1']$$

Example 3: Adam's method compute $y\,(0.8)$ given that $\frac{dy'}{dx} = y - x^2$ with $y\,(0) = 1, y\,(0.2) = 1.2186,$ $y\,(0.4) = 1.4682,$ $y\,(0.8) = 1.7379$

Solution:

Given

$$y' = y - x^2$$

$$\frac{dy}{dx} = y - x^2$$

$$f\,(x, y) = y - x^2$$

x_0	x_1	x_2	x_3	x_4
0	0.2	0.4	0.6	0.8

y_0	y_1	y_2	y_3	y_4
1	1.2136	1.4682	1.7379	?

Formula:

$$y_{4,\,p} = y_3 + \frac{h}{24}\,[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$y_{4,\,c} = y_3 + \frac{h}{24}\,[9y_4' + 1.19y_3' - 5y_2' + y_1']$$

x	y	$y' = y - x^2$
$x_0 = 0$	$y_0 = 1$	$y_0' = 1$
$x_1 = 0.2$	$y_1 = 1.2186$	$y_1' = 1.1786$
$x_2 = 0.4$	$y_2 = 1.4682$	$y_2' = 1.3082$
$x_3 = 0.6$	$y_3 = 1.7379$	$y_3' = 1.3779$
$x_4 = 0.8$	$y_4 = 2.0146$	$y_4' = 1.3746$

$$y_{4,\,p} = 1.7379 + \frac{0.2}{24}\,[55\,(1.3779) - 59\,(1.3032) + 37\,(1.1786) - 9\,(1)]$$

$$= 1.7379 + 0.2767$$

$$= 2.0146$$

$$y_{4,\,c} = 1.7399 + \frac{0.2}{24}\,[9\,(1.3146) + 19\,(1.3779) - 5\,(1.3087)\,[1.1736]$$

$$= 2.0144$$

Example 4: Given $\frac{dy}{dx} = xy + y^2, y(0) = 1, y(0.1) = 1.1169,$ $y(0.2) = 1.2773$ find

(i) $y(0.3)$ by Runge-Kutta method of fourth order and

(ii) $y(0.4)$ by Milne's method

Solution:

(i) Runge-Kutta Method

Given

$$\frac{dy}{dx} = xy + y^2$$

$$y' = xy + y^2$$

$$f(x, y) = xy + y^2$$

$$x_0 = 0, y_0 = 1, h = 0.1$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1 f(0.1)$$

$$\boxed{k_1 = 0.1}$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= 0.1 f(0.05, 1.05)$$

$$\boxed{k_2 = 0.1155}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1155}{2}\right)$$

$$= 0.1 f(0.05, 1.0577)$$

$$\boxed{k_3 = 0.1172}$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(0 + 0.1, 1 + 0.1171)$$

$$= 0.1f(0.1, 1.1171)$$

$$\boxed{\mathbf{k_4 = 0.1359}}$$

$$\Delta y_0 = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.1 + 2\,(0.1155) + 2\,(0.1171) + 0.1359}{6}$$

$$\Delta y_0 = 0.1167$$

$$y_1 = y_0 + \Delta y_0 = 1 + 0.1168 = 1.1168$$

$$\boxed{\mathbf{y_1 = y\,(0.3) = 1.1163}}$$

(ii) Milne's method

Given

$$\frac{dy}{dx} = xy + y^2$$

$$y' = xy + y^2$$

$$f(x, y) = xy + y^2$$

x_0	x_1	x_2	x_3	x_4
0	0.1	0.2	0.3	0.4

y_0	y_1	y_2	y_3	y_4
1	1.1169	1.2773	1.1168	?

$$y_{4,\,p} = y_0 + \frac{4h}{3}\,[2y_1' - y_2' + 2y_3']$$

$$y_{4,\,c} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4']$$

x	y	$y' = xy + y^2$
$x_1 = 0.1$	$y_0 = 1$	$y_0' = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$y_1' = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$y_2' = 1.3869$
$x_3 = 0.3$	$y_3 = 1.1168$	$y_3' = 1.5822$
$x_4 = 0.4$	$y_4 = 1.5327$	$y_4' = 2.9622$

$$y_{4,\,p} = 1 + \frac{4\,(0.1)}{3}\,[2\,(1.3591) - 1.8869 + 2\,(1.5822)]$$

$$= 1 + \frac{0.4}{3}\,[3.9957]$$

$$= 1.5327$$

$$y_{4,\,c} = 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(1.5822) + 2.9622]$$

$$= 1.6498$$

Example 5: Solve numerically by Milne's method $y' = x^3 + y$ with the initial values $y\,(0) = 2, y\,(0.2) = 2.073), y\,(0.4) = 2.452, y\,(0.6) = 3.023$ find $y\,(0.8)$

Given

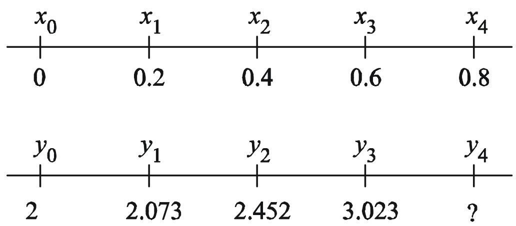
$$\frac{dy}{dx} = x^3 + y$$

$$y' = x^3 + y$$

$$f\,(x, y) = x^3 + y,\ h = 0.2$$

$$y_{4,\,p} = y_0 + \frac{4h}{3}\,[2y_1' - y_2' + 2y_3']$$

$$y_{4,\,c} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4']$$



x	y	$y' = x^3 + y$
$x_0 = 0$	$y_0 = 2$	$y_0{}' = 2$
$x_1 = 0.2$	$y_1 = 2.073$	$y_1{}' = 2.081$
$x_2 = 0.4$	$y_2 = 2.452$	$y_2{}' = 2.516$
$x_3 = 0.6$	$y_3 = 3.023$	$y_3{}' = 3.239$
$x_4 = 0.8$	$y_4 = 4.1644$	$y_4{}' = 4.678$

$$y_{4,\,p} = 2 + \frac{4\,(0.2)}{3}\,[2\,(2.081) - 2.516 + 2\,(3.239)]$$

$$= 2 + \frac{0.8}{3}$$

$$\boxed{\mathbf{y_{4,\,p} = 4.1664}}$$

$$y_{4,\,c} = 2.452 + \frac{0.2}{3}\,[2.516 + 4\,(3.239) + 4.678]$$

$$\boxed{\mathbf{y_{4,\,c} = 3.795}}$$

Example 6: Solve numerically by Milne's method $y' = \dfrac{1}{x+y}$; $y\,(0) = 2$ with the initial values $y\,(0.2) = 2.0933$, $y\,(0.4) = 2.1755$, $y\,(0.6) = 2.2493$

Solution:

Given

$$y' = \frac{1}{x+y}$$

$$\frac{dy}{dx} = \frac{1}{x+y}$$

$$f(x, y) = \frac{1}{x + y}$$

$$x_0 = 0,\ y_0 = 2,\ h = 0.2$$

$$y_{4,\,p} = y_0 + \frac{4h}{3}\,[2y_1' - y_2' + 2y_3']$$

x_0	x_1	x_2	x_3	x_4
0	0.2	0.4	0.6	0.8

y_0	y_1	y_2	y_3	y_4
2	2.0933	2.1755	2.2493	?

$$y_{4,\,c} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4']$$

x	y	$y' = \frac{1}{x + y}$
$x_0 = 0$	$y_0 = 2$	$y_0' = 0.5$
$x_1 = 0.2$	$y_1 = 2.0933$	$y_1' = 0.4360$
$x_2 = 0.4$	$y_2 = 2.1755$	$y_2' = 0.3382$
$x_3 = 0.6$	$y_3 = 2.2493$	$y_2' = 0.3509$

$$y_{4,\,p} = 2 + \frac{4\,(0.2)}{3}\,[2\,(0.4360) - 0.2882 + 2\,(0.3509)]$$

$$\boxed{y_{4,\,p} = 2.3161}$$

$$y_{4,\,c} = 2.1755 + \frac{0.2}{3}\,[0.3882 + 4\,(0.3509) + 0.3209]$$

$$\boxed{y_{4,\,c} = 2.3163}$$

Example 7: Find $y\,(0.4)$ given $y' = y - \frac{2x}{y}$, $y\,(0) = 1$, $y\,(0.1) = 1.0959$, $y\,(0.2) = 1.1841$, $y\,(0.3) = 1.2662$, using Milne's method.

Solution:

Given

$y_0 = 1, y_1 = 1.0959$, $y_2 = 1.1841$ and $y_3 = 1.2662$ for $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$ and $x_3 = 0.3$

Here $h = 0.1$

$$y_0' = 1 - \frac{2\,(0)}{1} = 1$$

$$y_1' = 1.0959 - \frac{2\,(0.1)}{1.0959} = 0.9134$$

$$y_2' = 1.1841 - \frac{0.4}{1.1841} = 0.8463$$

$$y_3' = 1.2662 - \frac{0.6}{1.2662} = 0.7923$$

By predictor formula,

$$y_{4,\,p} = y_0 + \frac{4h}{3}\,[2y_1' - y_2' + 2y_3']$$

$$= 1 + \frac{0.4}{3}\,[2\,(0.9134) - 0.8463 + 2\,(0.7923)]$$

$$= 1.3420$$

$$y_4' = 1.3420 - \frac{0.8}{1.3420} = 0.7459$$

By corrector formula,

$$y_{4,\,c} = y_2 + \frac{h}{3}\,[y_2' + 4y_3' + y_4']$$

$$= 1.1841 + \frac{0.1}{3}\,[0.8463 + 4\,(0.7923) + 0.7459] = 1.3428$$

$$\boxed{\therefore\ y\,(0.4) = 1.3428}$$

Example 8: Given that $\frac{dy}{dx} = x^2 (1 + y), y(1) = 1, \; y(1.1) = 1.233,$ $y(1.2) = 1.5485, \; y(1.3) = 1.9789,$ find $y(1.4)$ by Adam's method.

Solution:

Given the values

		$y' = x^2(1+y)$
$x_0 = 1,$	$y_0 = 1,$	$y_0' = 2$
$x_1 = 1.1,$	$y_1 = 1.233,$	$y_1' = 2.702$
$x_2 = 1.2,$	$y_2 = 1.5485,$	$y_2' = 3.669$
$x_3 = 1.3$	$y_3 = 1.9789$	$y_3' = 5.035$

To find y_4 for $x_4 = 1.4.$ Here $h = 0.1$

By Adam's predictor formula,

$$y_{4,p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$= 1.9789 + \frac{0.1}{24}[55 \times 5.035 - 59 \times 3.966 + 37 \times 2.702 - 9 \times 2]$$

$$= 1.9789 + \frac{0.1}{24}[277 - 216.5 + 99.97 - 18]$$

$$\boxed{y_{4,p} = 2.5726}$$

$$y_4' = 1.96 \times 3.5726 = 7.004$$

Adam's corrector formula is

$$y_{4,c} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

$$= 1.9789 + \frac{0.1}{24}[9 \times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702]$$

$$= 1.9789 + \frac{0.1}{24}[63.036 + 95.665 - 18.345 + 2.702]$$

$$y_{4,c} = 2.5751$$

$$\boxed{y(1.4) = 2.5751}$$

Example 9: Solve $\frac{dy}{dx} = 1 - y$ with the initial condition $x = 0$, $y = 0$ using Euler's algorithm and tabulate the solutions at $x = 0.1, 0.2, 0.3, 0.4$. Using these results, find $y(0.5)$ using Adams-Bashforth predictor and corrector method.

(AU, May/June 2006, MA 038)

Solution:

Given

$$f(x, y) = \frac{dy}{dx} = 1 - y$$

$$h = 0.1$$

$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4, x_5 = 0.5$$

$$y_0 = 0$$

Euler's algorithm is $y_{n+1} = y_n + hf(x_n, y_n)$

$$\therefore y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1\ (1 - 0) = 0.1$$

$$y_2 = y_1 + hf(x_1, y_1) = 0.1 + 0.1\ (1 - 0.1) = 0.19$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.19 + 0.1\ (1 - 0.19) = 0.271$$

$$y_4 = y_3 + hf(x_3, y_3) = 0.271 + 0.1\ (1 - 0.271) = 0.3439$$

$$y_1' = 1 - y_1 - 0.1 = 0.9$$

$$y_2' = 1 - y_2 = 1 - 0.19 = 0.81$$

$$y_3' = 1 - y_3 = 1 - 0.271 = 0.729$$

$$y_4' = 1 - y_4 = 1 - 0.3439 = 0.6561$$

By Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$$

$$\therefore\ y_{5,\,p} = y_4 + \frac{h}{24}\,[55y_4' - 59y_3' + 37y_2' - 9y_1']$$

$$= 0.3439 + \frac{0.1}{24}\,[55\,(0.6561) - 59\,(0.729)$$

$$+ 37\,(0.81) - 9\,(0.9)]$$

$$= 0.4062$$

$$y_5' = y_5 = 1 - 0.4062 = 0.5938$$

Example 10: Solve $2y' - x - y = 0$ given $y\,(0) = 2$, $y\,(0.5) = 2.636$, $y\,(1) = 3.595$, $y\,(1.5) = 4.968$ to get $y\,(2)$ by Adam's method.

(AU, Nov/Dec 2008, MA 1251)

Solution:

Given

$$2y' - x - y = 0,\ h = 0.5$$

$$\therefore\ y' = \frac{1}{2}\,(x + y)$$

$$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2$$

$$y_0 = 2,\ y_1 = 2.636,\ y_2 = 3.595,\ y_3 = 4.968$$

Now

$$y_0' = \frac{1}{2}\,(x_0 + y_0) = \frac{1}{2}\,(0 + 2) = 1$$

$$y_1' = \frac{1}{2}\,(x_1 + y_1) = \frac{1}{2}\,(0.5 + 2.636) = 1.568$$

$$y_2' = \frac{1}{2}\,(x_2 + y_2) = \frac{1}{2}\,(1 + 3.595) = 2.2975$$

$$y_3' = \frac{1}{2}\,(x_3 + y_3) = \frac{1}{2}\,(1.5 + 4.968) = 3.234$$

By Adam's predictor formula,

$$y_{4,\,p} = y_3 + \frac{h}{24}\,(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

$$= 4.968 + \frac{0.5}{24}\,(55\,(3.234) - 59\,(2.2975) + 37\,(1.568) - 9\,(1))$$

$$\boxed{\mathbf{y_{4,\,p} = 6.8708}}$$

$$y_{4,\,p} = \frac{1}{2}\,(x_4 + y_{4,\,p}) = \frac{1}{2}\,(2 + 6.8708) = 4.4354$$

By Adam's corrector formula,

$$y_{4,\,c} = y_3 + \frac{h}{24}\,(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$= 4.968 + \frac{0.5}{24}\,9\,(4.4354) + 19\,(3.234) - 5\,(2.2975) + 1.568)$$

$$\boxed{\mathbf{y_{4,\,p} = 6.8731}}$$

$$y\,(2) = 6.8731$$

Example 11: Given the initial value problem $y' = x^2 - y$, $y\,(0) = 1$, find the value of y at $x = 0.1$ by Taylor series method, at $x = 0.2$ by modified Euler method, at $x = 0.3$ by fourth order Runge - Kutta method and at $x = 0.4$, 0.5 by Adams - Bashforth method.

Solution:

Given

$$y' = x^2 - y,\ h = 0.1$$

$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,$$

$$x_4 = 0.4,\ x_5 = 0.5$$

$$y_0 = 1$$

$$y' = x^2 - y$$

$$y'' = 2x - y'$$

$$y''' = 2 - y''$$

$$y^{iv} = -\, y'''$$

$$y^{v} = -\, y^{iv} = y'''$$

To find *y* (0.1)

Take $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$$y_0{}' = x_0^2 - y_0 = 0 - 1 = -1$$

$$y_0{}'' = 2x_0 - y_0{}' = 0 + 1 = 1$$

$$y_0{}''' = 2 - y_0{}'' = 2 - 1 = 1$$

$$y_0^{iv} = -\, y''' = -1$$

$$y_0^{v} = -\, y^{iv} = 1$$

By Taylor series method,

$$y_1 = y_0 + \frac{h}{1!}\, y_0{}' + \frac{h^2}{2!}\, y_0{}'' + \frac{h^3}{3!}\, y_0{}''' + \dots$$

$$= 1 + \frac{0.1}{1}\,(-1) + \frac{(0.1)^2}{2}\,(1) + \frac{(0.1)^3}{6}\,(1) + \frac{(0.4)^4}{24}\,(-1)$$

$$+ \frac{(0.1)^5}{120}\,(1) + \dots$$

$$= 0.9052$$

$\therefore\ y\,(0.1) = 0.9052$

To find *y* (0.2)

By modified Euler's formula,

$$y_{n+1} = y_n + \frac{1}{2}\, h\,[f\,(x_n, y_n) + f\,(x_n + h,\ y_n + hf\,(x_n, y_n))]$$

$$f(x, y) = x^2 - y, \; x_1 = 0.1, \; y_1 = 0.9052, \; h = 0.1$$

$$f(x_1, y_1) = x_1^2 - y_1$$
$$= (0.1)^2 - 0.9052$$
$$= -0.8952$$

$$y_2 = y_1 + \frac{1}{2} h \, [f(x_1, y_1) + f(x_1 + h, y_1 + hf(x_1, y_1))]$$

$$= 0.9052 + \frac{1}{2} \times 0.1 \times [-0.8952 + f(0.2, 0.9052 + (0.1 \times -0.8952))$$

$$= 0.9052 + \frac{1}{2} \times 0.1 \times [-0.8952 + f(0.2, 0.8952)]$$

$$= 0.9052 + \frac{1}{2} \times 0.1 \times [-0.8952 + ((0.2)^2 - 0.8952)]$$

$$= 0.8217$$

$$y(0.2) = 0.8217$$

To find y (0.3)

$$x_2 = 0.2, \; y_2 = 0.8217, \; h = 0.1$$

$$f(x, y) = x^2 - y$$

$$k_1 = hf(x_2, y_2) = 0.1 \times [(0.2)^2 - 0.8217] = 0.07817$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$$= 0.1 \times f\left(0.2 + \frac{0.1}{2}, 0.8217 - \frac{0.07817}{2}\right)$$

$$= 0.1 \times f(0.25, 0.7826)$$

$$= 0.1 \times ((0.25)^2 - 0.7826)$$

$$= -0.0720$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$$= 0.1 \times f\left(0.2 + \frac{0.1}{2}, 0.8217 - \frac{0.07187}{2}\right)$$

$$= 0.1 \times f(0.25, 0.7826)$$

$$= 0.1 \times ((0.25)^2 - 0.7826)$$

$$= -0.0720$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$$= 0.1 \times f\left(0.2 + \frac{0.1}{2}, 0.8217 - \frac{0.0720}{2}\right)$$

$$= 0.1 \times f(0.25, 0.7857)$$

$$= -0.0723$$

$$k_4 = hf(x_2 + h, y_2 + k_3)$$

$$= 0.1 \times f(0.3, 0.8217 - 0.0723)$$

$$= 0.1 \times f(0.3, 0.7494)$$

$$= 0.1 \times ((0.3)^2 - 0.7494)$$

$$= -0.0659$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.07817 + 2(-0.0720) + 2(-0.0723) - 0.0659)$$

$$= -0.0721$$

$$y_3 = y_2 + \Delta y = 0.8217 - 0.0721 = 0.7496$$

$$\therefore \quad y(0.3) = 0.7496$$

To find y (0.4)

$$y' = x^2 - y,\ h = 0.1$$

$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4$$

$$y_0 = x_1^2 - y_1 = (0.1)^2 - 0.9052 = -0.8952$$

$$y_2' = x_2^2 - y_2 = (0.2)^2 - 0.8217 = -0.7817$$

$$y_3' = x_3^2 - y_3 = (0.3)^2 - 0.7496 = -0.6596$$

By Adams Bashforth predictor formula,

$$y_{4,\,p} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0')$$

$$= 0.7496 + \frac{0.1}{24}[(55 \times -0.6596) - (59 \times -0.7817)$$

$$+ (37 \times -0.8952) - (9 \times -1)]$$

$$\boxed{\mathbf{y_{4,\,p} = 0.6901}}$$

Now $y_{4,\,p} = x_4^2 - y_4 = (0.4)^2 - 0.6901 = -0.5301$

By corrector formula,

$$y_{4,\,c} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$= 0.7496 + \frac{0.1}{24}[(9 \times -0.5301) + (19 \times -0.6596)$$

$$-(5 \times -0.7817) - 0.8952]$$

$$\boxed{\mathbf{y_{4,\,c} = 0.6901}}$$

To find y (0.5)

$$y_{5,\,p} = y_4 + \frac{h}{24}(55y_4' - 59y_3' + 37y_2' - 9y_1')$$

$$= 0.6901 + \frac{0.1}{24}[(55 \times -0.5301) - (59 \times -0.6596)$$

$$+ (37 \times -0.7817) - (9 \times -0.8952)]$$

$$\boxed{\mathbf{y_{5,\,p} = 0.6438}}$$

$$y_{5,p}' = x_5^2 - y_5 = (0.5)^2 - 0.6438 = -0.3938$$

$$y_{5,c} = y_4 + \frac{h}{24}(9y_5' + 19y_4' - 5y_3' + y_2')$$

$$= 0.6901 + \frac{0.1}{24}((9 \times -0.3938) + (19 \times -0.5301)$$

$$-(5 \times -0.6596) - 0.7817)$$

$$\mathbf{y_{5,c} = 0.6439}$$

Example 12: Given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$, evaluate $y(0.4)$ by using Milne's method. *[AU, April/May 2008, MA 1251]*

Solution:

Given

$$y' = xy + y^2, h = 0.1$$

$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4$$

$$y_0 = 1,\ y_1 = 1.1169,\ y_2 = 1.2773,\ y_3 = 1.5049$$

Now

$$y_1' = x_1 y_1 + y_1^2 = (0.1 \times 1.1169) + (1.1169)^2 = 1.3592$$

$$y_2' = x_2 y_2 + y_2^2 = (0.2 \times 1.2773) + (1.2773)^2 = 1.8870$$

$$y_3' = x_3 y_3 + y_3^2 = (0.3 \times 1.5049) + (1.5049)^2 = 2.7162$$

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.1)}{3}((2 \times 1.3592) - 1.8879 + (2 \times 2.7162))$$

$$\boxed{\mathbf{y_{4,p} = 1.8352}}$$

Now

$$y_{4,p}' = x_4 y_4 + y_4^2 = (0.4 \times 1.8352) + (1.8352)^2 = 4.1020$$

By Milne's corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1.2773 + \frac{0.1}{3}(1.8870 + (4 \times 2.7162) + 4.1020)$$

$$= 1.8391$$

$$y(0.4) = 1.8391$$

Example 13: Use Milne's method to find $y(4.4)$ given that $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$. *[AU, Nov/Dec 2004, MA 038 AU, April/May 2008, MA 1251 AU CBE, May/June 2010, 070030010 AU CBE, Nov/Dec 2010, 070230054]*

Solution:

Given

$$5xy' + y^2 - 2 = 0$$

$$\therefore \quad y' = \frac{2 - y^2}{5x}, \; h = 0.1$$

$$x_0 = 4, x_1 = 4.1, x_2 = 4.2, x_3 = 4.3, x_4 = 4.4$$

$$y_0 = 1, y_1 = 1.0049, y_2 = 1.0097, y_3 = 1.0143$$

$$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.0483$$

$$y_2' = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.0467$$

$$y_3' = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.0452$$

By Milne's predictor formula

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4\,(0.1)}{3}((2 \times 0.0483) - 0.0467 + (2 \times 0.0452))$$

$$\boxed{\boldsymbol{y_{4,p} = 1.01871}}$$

Now

$$y_{4,p}' = \frac{2 - y_4^2}{5x_4} = \frac{2 - (1.01871)^2}{5\,(4.4)} = 0.0437$$

By Milne's corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1.0097 + \frac{(0.1)}{3}(0.0467 + (4 \times 0.0452) + 0.0437)$$

$$\boxed{\boldsymbol{y_{4,c} = 1.01874}}$$

$\therefore \quad y\,(4, 4) = 1.01874$

Example 14: Solve and get $y\,(2)$ given $\frac{dy}{dx} = \frac{1}{2}(x + y)$, $y\,(0) = 2$, $y\,(0.5) = 2.636$, $y\,(1) = 3.595$, $y\,(1.5) = 4.968$ by Adam's method.

Solution:

From Example 9 under Milne's method,

We have $y_0' = \frac{1}{2}(0 + 2) = 1$

$$y_1' = 1.5680,\ y_2' = 2.2975,\ y_3' = 3.2340$$

By Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}[55y_n' - 59y_{n-1} + 37y_{n-2}' - 9y_{n-3}']$$

$$\therefore \quad y_{4,p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0'] \qquad \text{... (1)}$$

$$= 4.968 + \frac{0.5}{24}[55\,(3.2340) - 59\,(2.2975) + 37\,(1.5689) - 9\,(1)]$$

$$= 6.8708$$

$$y_{4,c} = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}(2 + 6.8708) = 4.4354$$

By corrector,

$$y_{4,c} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + 9y_1'] \qquad \text{... (2)}$$

$$= 4.968 + \frac{0.5}{24}[9\,(4.4354) + 19\,(3.234) - 5\,(2.2975) + 1.5680]$$

$$= 6.8731$$

Note: We can further improve using this latest $\boldsymbol{y_{4,c}}$ again in (2).

Example 15: Using Adam's method find $y\,(0.4)$ given $\frac{dy}{dx} = \frac{1}{2}xy$, $y\,(0) = 1$, $y\,(0.1) = 1.01$, $y\,(0.2) = 1.022$, $y\,(0.3) = 1.023$.

(MS. 1992)

Solution:

$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4$

$y_0 = 1, y_1 = 1.01, y_2 = 1.022, y_3 = 1.023, y_4 = ?$

By Adam's method,

Predictor: $y_{n+1,p} = y + \frac{h}{24}[55y_n' - 59y_{n-1} + 37y_{n-2}' - 9y_{n-3}']$

$$\therefore\ y_{4,p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0'] \quad \text{... (1)}$$

Here $y_0' = \frac{1}{2} x_0\, y_0 = 0$

$$y_1' = \frac{1}{2} x_1\, y_1 = \frac{(0.1)\,(1.01)}{2} = 0.0505$$

$$y_2' = \frac{1}{2} x_2\, y_2 = \frac{(0.2)\,(1.022)}{2} = 0.1022$$

$$y_3' = \frac{1}{2} x_3\, y_3 = \frac{(0.3)\,(1.023)}{2} = 0.1535$$

Using in (1),

$$y_{4,p} = 1.023 + \frac{0.1}{24}[55\,(0.1535) - 59\,(0.1022) + 37\,(0.0505) - 9\,(0)]$$

$$\boxed{\mathbf{y_{4,p} = 1.0408}}$$

$$y_{4,p}' = \frac{1}{2} x_4\, y_4 = \frac{1}{2}(0.4)\,(1.0408) = 0.20816$$

By Adam's corrector formula

$$y_{n+1,c} = y_n + \frac{h}{24}[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']$$

$$y_{4,c} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

$$= 1.023 + \frac{0.1}{24}[9\,(0.2082) + 19\,(0.1535) - 5\,(0.1022) + 0.0505]$$

$$\boxed{\mathbf{y_{4,c} = 1.0410}}$$

$$y\,(0.4) = y_{4,c} = 1.0410$$

5.5 MISCELLANEOUS PROBLEMS

Fourth Order Runge-Kutta Method

This method is most commonly used in practice. Unless and otherwise state, Runge-Kutta method means only Fourth Order Runge-Kutta method. Let $\frac{dy}{dx} = f(x, y)$ be a given differential equation to be solved under the condition $y(x_0) = y_0$. If h be the length of the interval between equidistant values, then the first increment in y is computed from the formulae.

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Now $x_1 = x_0 + h, y_1 = y_0 + \Delta y_0$

The increment in y for the second interval is computed in a similar manner by means of the formulae.

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and so on for the succeeding intervals.

Example 1: By applying the fourth order Runge-Kutta method find y (0.2) from $y' = y - x, y(0) = 2$ taking $h = 0.1$.

Solution:

Given

$$y' = y - x$$

$$\text{i.e., } f(x, y) = y - x$$

and $y(0) = 2$ *i.e.,* $x_0 = 0$, $y_0 = 2$ and $h = 0.1$. We know that the fourth order Runge-Kutta formula for finding the first increment in y viz Δy is given by

$$\boxed{\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\therefore\ k_1 = (0.1)(y_0 - x_0) = 0.1(2 - 0) = 0.2$$

$$k_2 = (0.1)\left[\left(y_0 + \frac{k_1}{2}\right) - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2 + \frac{0.2}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)[2.1 - 0.05] = 0.205$$

$$k_3 = (0.1)\left[\left(y_0 + \frac{k_2}{2}\right) - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2 + \frac{0.205}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)\,[2.1025 - 0.05]$$

$$= 0.20525$$

$$k_4 = (0.1)\,[(y_0 + k_3) - (x_0 + h)]$$

$$= (0.1)\,[2 + 0.20525 - 0 - 0.1]$$

$$= 0.210525$$

$$\mathbf{\Delta\, y = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]}$$

$$= \frac{1}{6}\,[0.2 + 2\,(0.205) + 2\,(0.20525) + 0.210525]$$

$$= \frac{1}{6}\,[0.2 + 0.41 + 0.4105 + 0.210525]$$

$$= 0.20517$$

$$\therefore\; y\,(0.1) = y_1 = y_0 + \Delta\, y$$

$$= 2 + 0.20517$$

$$\boxed{\therefore\; \mathbf{y\,(0.1) = 2.20517}}$$

Next we have to find $y\,(0.2) = y_2 = y_1 + \Delta\, y$

where $\mathbf{\Delta\, y = \frac{1}{6}\, k_1 + 2k_2 + 2k_3 + k_4}$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

Now $\mathbf{k_1 = hf(x_1, y_1)}$ $= h\,[y_1 - x_1]$

$$= (0.1)\,[2.20517 - 0.1] = 0.210517$$

$$\mathbf{k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)}$$

$$= h\left[\left(y_1 + \frac{k_1}{2}\right) - \left(x_1 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(2.20517 + \frac{0.2105}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)\ [2.31042 - 0.15] = 0.21604$$

$$\boldsymbol{k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)}$$

$$= h\left[\left(y_1 + \frac{k_2}{2}\right) - \left(x_1 + \frac{h}{2}\right)\right]$$

$$= (0.1)\left[\left(0.20517 + \frac{0.21604}{2}\right) - \left(0.1 + \frac{0.1}{2}\right)\right]$$

$$= (0.1)\ [2.31319 - 0.15] = 0.21632$$

$$\boldsymbol{k_4 = hf(x_1 + h, y_1 + k_3)}$$

$$= h\ [y_1 + k_3) - (x_1 + h)$$

$$= (0.1)\ [(2.20517 + 0.21632) - (0.1 + 0.1)]$$

$$= (0.1)\ [2.142149 - 0.2] = 0.22214$$

$$\boxed{\therefore\ \Delta y = \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4]}$$

$$= \frac{1}{6}\ [0.2105 + 2\ (0.21604) + 2\ (0.21632) + 0.22214]$$

$$= \frac{1}{6}\ [0.2105 + 0.43208 + 0.43264 + 0.22214]$$

$$= 0.21622$$

$$\therefore\ y_2 = y_1 + \Delta y$$

$$y_2 = 2.20517 + 0.21622$$

$$y_2 = 2.42139$$

Example 2: Using Runge-Kutta method of order 4 find y for $x = 0.1, 0.2, 0.5$ given that $\frac{dy}{dx} = xy + y^2, y(0) = 1$. Continue the solution at $x = 0.4$ using Milne's method. *[Apr.90]*

Solution:

Given

$$\frac{dy}{dx} = xy + y^2$$

i.e., $f(x, y) = xy + y^2, x_0 = 0, y_0 = 1$ and $h = 0.1$

To find y (0.1)

Here $x_0 = 0, y_0 = 1$

$$k_1 = hf(x_0, y_0)$$

$$= h[x_0 y_0 + y_0^2] = (0.1)(1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)^2\right]$$

$$= (0.1)\left[\left(0 + \frac{0.1}{2}\right)\left(1 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right]$$

$$= (0.1)[0.0525 + 1.1025]$$

$$= 0.1155$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_2}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)^2\right]$$

$$= (0.1)\left[\left(0+\frac{0.1}{2}\right)\left(1+\frac{0.1155}{2}\right)+\left(1+\frac{0.1155}{2}\right)^2\right]$$

$$= (0.1)\ [0.0528 + 1.1188]$$

$$= 0.11717$$

$$\boldsymbol{k_4 = hf(x_0 + h, y_0 + k_3)}$$

$$= h\ [(x_0 + h)\ (y_0 + k_3) + (y_0 + k_3)^2]$$

$$= (0.1)\ [(0 + 0.1)\ (1 + 0.11717) + (1 + 0.11717)^2]$$

$$= (0.1)\ (0.11171 + 1.2480) = 0.13597$$

$$\boldsymbol{\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4]}$$

$$= \frac{1}{6}\ [0.1 + 2\ (0.1155) + 2\ (0.11717) + 0.1359]$$

$$= \frac{1}{6}\ [0.1 + 0.231 + 0.2343 + 0.1359]$$

$$= 0.11687$$

$$\boldsymbol{\therefore y\ (0.1) = y_1 = y_0 + \Delta y}$$

$$= 1 + 0.11686 = 1.11686$$

$$\boldsymbol{\therefore y\ (0.1) = 1.1169}$$

To find *y* (0.2)

Here $x_1 = 0.1, y_1 = 1.1169$

$$\boldsymbol{k_1 = hf(x_1, y_1)}$$

$$= h\ [x_1\ y_1 + y_1^2]$$

$$= (0.1)\ [(0.1)\ (1.1169) + (1.1169)^2]$$

$$= (0.1)\ [0.1117 + 1.2475]$$

$$= 0.1359$$

$$\boldsymbol{k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)}$$

$$= h\left[\left(x_1 + \frac{h}{2}\right)\left(y_1 + \frac{k_1}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right]$$

$$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right) \times \left(1.1169 + \frac{0.1359}{2}\right) + \left(1.1169 + \frac{0.1359}{2}\right)^2\right]$$

$$= (0.1)\ [(0.15)\ (1.18485) + 1.4038]$$

$$= 0.1581$$

$$\boldsymbol{k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)}$$

$$= h\left[\left(x_1 + \frac{h}{2}\right)\left(y_1 + \frac{k_2}{2}\right) + \left(y_1 + \frac{k_2}{2}\right)^2\right]$$

$$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right)\left(1.1169 + \frac{0.1581}{2}\right) + \left(1.1169 + \frac{0.1581}{2}\right)^2\right]$$

$$= (0.1)\ [(0.15)\ (1.19595) + 1.43029]$$

$$= 0.1609$$

$$\boldsymbol{k_4 = hf(x_1 + h, y_1 + k_3)}$$

$$= h\ [(x_1 + h)\ (y_1 + k_3) + (y_1 + k_3)^2]$$

$$= (0.1)\ [(0.1 + 0.1)\ (1.1169 + 0.1609) + (1.1169 + 0.1609)^2]$$

$$= (0.1)\ [(0.2)\ (1.2778) + 1.6327]$$

$$= 0.1888$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1359 + 2\,(0.1581) + 2\,(0.1609) + 0.1888]$$

$$= 0.16045$$

$$\therefore y\,(0.2) = y_2 = y_1 + \Delta y$$

$$= 1.1169 + 0.16045$$

$$\therefore y\,(0.2) = 1.2773$$

To find y (0.3)

Here $x_2 = 0.2$, $y_2 = 1.2773$

Now $k_1 = hf(x_2, y_2)$

$$= h\,[x_2\,y_2 + y_2^2]$$

$$= (0.1)\,[(0.2)\,(1.2773) + (1.2773)^2]$$

$$= (0.1)\,(0.25546 + 1.63145) = 0.1886$$

$$k_2 = hf\left(x_2 \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$$= h\left[\left(x_2 + \frac{h}{2}\right)\left(y_2 + \frac{k_1}{2}\right) + \left(y_2 + \frac{k_1}{2}\right)^2\right]$$

$$= (0.1)\left[\left(0.2 + \frac{0.1}{2}\right)\left(1.2773 + \frac{0.1886}{2}\right)\right.$$

$$\left.\left(1.2773 + \frac{0.1886}{2}\right)^2\right]$$

$$= (0.1)\,[(0.25)\,(1.3716) + 1.8812] = 0.2224$$

$$\boldsymbol{k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)}$$

$$= h\left[\left(x_2 + \frac{h}{2}\right)\left(y_2 + \frac{k_2}{2}\right) + \left(y_2 + \frac{k_2}{2}\right)^2\right]$$

$$= (0.1)\left[\left(0.2 + \frac{0.1}{2}\right)\left(1.2773 + \frac{0.2224}{2}\right)\right.$$

$$\left.\left(1.2773 + \frac{0.2224}{2}\right)^2\right]$$

$$= (0.1)\ [(0.25)\ (1.3885) + 1.9279]$$

$$= 0.2275$$

$$\boldsymbol{k_4 = hf(x_2 + h, y_2 + k_3)}$$

$$= h\ [(x_2 + h)\ (y_2 + k_3) + (y_2 + k_3)^2]$$

$$= (0.1)\ [(0.2 + 0.1)\ (1.2773 + 0.2275) + (1.2273 + 0.2775)^2]$$

$$= (0.1)\ [(0.3)\ (1.5048) + 2.2644]$$

$$= 0.2715$$

$$\boldsymbol{\therefore\ \Delta y = \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4]}$$

$$= \frac{1}{6}\ [0.1886 + 2\ (0.2224) + 2\ (0.2275) + 0.2715]$$

$$= \frac{1}{6}\ [0.1886 + 0.4448 + 0.455 + 0.2715]$$

$$= 0.22665$$

$$\therefore\ y\ (0.3) = y_3 = y_2 + \Delta\ y$$

$$= 1.2773 + 0.22665 = 1.5039$$

$$\boxed{\boldsymbol{\therefore\ y\ (0.3) = 1.5039}}$$

Now we have the following values

x	0	0.1	0.2	0.3
y	1	1.1169	1.2773	1.5039

To find y (0.4) use Milne's predictor and corrector formula

By Milne's predictor formula we have

$$\boxed{y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y_{n-2}' - y_{n-1}' + 2y_n']} \qquad \text{... (A)}$$

$$y_1' = (xy + y^2)_1 = x_1\, y_1 + y_1^2$$

$$= [(0.1)\,(1.1169) + (1.1169)^2]$$

$$= 0.1116 + 1.2474$$

$$= 1.359$$

$$y_2' = (xy + y^2)_2 = x_2\, y_2 + y_2^2$$

$$= [(0.2)\,(1.2773) + (1.2773)^2]$$

$$= 1.8869$$

$$y_3' = (xy + y^2)_3 = x_3\, y_3 + y_3^2$$

$$= [(0.3)\,(1.5039) + (1.5039)^2]$$

$$= 2.7129$$

Putting $n = 3$ in (A) we get

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 1 + \frac{4\,(0.1)}{3}[2\,(1.359) - 1.8869 + 5.4258]$$

$y_{4,p} = 1.8340$ (By Milne's Predictor method)

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}'] \qquad \text{... (B)}$$

$$y_4' = (xy + y^2)_4 = x_4\, y_4 + y_4^2$$

$$= [(0.4)\,(1.834) + (1.834)^2]$$

$$= 0.7336 + 3.3365 = 4.0971$$

Putting $n = 3$ in (B) we get

$$y_{4,\,c} = y_2 + \frac{0.1}{3}\,[y_2' + 4y_3' + y_4']$$

$$= 1.2773 + \frac{0.1}{3}\,[1.8869 + 4\,(2.7129) + 4.0971]$$

$\therefore$ $\boldsymbol{y\,(0.4) = 1.838}$ (By Milne's corrector method)

Example 3: Given $\frac{dy}{dx} = x^3 + y$, $y\,(0) = 2$, $y\,(0.2) = 2.443214$, $y\,(0.4) = 2.990578$, $y\,(0.6) = 3.823516$ find $y\,(0.8)$ by Milne's predictor – corrector method taking $h = 0.2$.

Solution:

Given

$$\frac{dy}{dx} = y' = x^3 + y \text{ and } h = 0.2$$

$x_0 = 0$,	$y_0 = 2$
$x_1 = 0.2,$	$y_1 = 2.443214$
$x_2 = 0.4,$	$y_2 = 2.990578$
$x_3 = 0.6,$	$y_3 = 3.823516$
$x_4 = 0.8,$	$y_4 = ?$

By Milne's Predictor formula we have

$$\boxed{y_{n+1,\,p} = y_{n-3} + \frac{4h}{3}\, 2y_{n-2}' - y_{n-1}' + 2y_n']} \quad \ldots (1)$$

To get y_4, put $n = 3$ in (1) we get

$$\boxed{\boldsymbol{y_{4,p} = y_0 + \frac{4h}{3} [2y_1' - y_2' + 2y_3'}} \quad \text{... (2)}$$

Now $y_1' = (x^3 + y)_1 = x_1^3 + y_1$

$$= (0.2)^3 + 2.443214 = 2.451214 \quad \text{... (3)}$$

$$y_2' = (x^3 + y)_2 = x_2^3 + y_2$$

$$= (0.4)^3 + 2.990578 = 3.054578 \quad \text{... (4)}$$

$$y_3' = (x^3 + y)_3 = x_3^3 + y_3$$

$$= (0.6)^3 + 3.823516 = 4.039516 \quad \text{... (5)}$$

Substituting (3), (4) and (5) in (2) we get

$$y_{4,p} = 2 + \frac{4\,(0.2)}{3} [2\,(2.451214) - 3.04578 + 2\,(4.039516)]$$

$$= 2 + 0.26666\,[9.9363386]$$

$$= 4.649624$$

$\therefore$ **y (0.8) = 4.649624** (By Milne's predictor formula)

By Milne's corrector formula we have

$$\boxed{\boldsymbol{y_{n+1,c} = y_{n-1} + \frac{h}{3} (y_{n-1}' + 4y_n' + y_{n+1}')}}$$

To get y_4, put $n = 3$, we get

$$y_{4,c} = y_2 + \frac{h}{3} (y_2' + 4y_3' + y_4') \quad \text{... (6)}$$

Now $\quad y_4' = (x^3 + y)_4 = x_4^3 + y_4$

$$= (0.8)^3 + 4.649624 = 4.713624 \quad \text{... (7)}$$

Substituting (4), (5), (7) in (6) we get

$$y_{4,\,c} = 2.990578 + \frac{0.2}{3}$$

$$[3.045748 + 4\ (4.039516) + 4.713624]$$

$$= 4.584914284$$

$\boldsymbol{y}$ **(0.8) = 4.584914** (By Milne's corrector formula

$\boldsymbol{y}$ **(0.8) = 4.649624** (By Milne's predictor formula)

Example 4: Using R-K method of fourth order, solve $\boldsymbol{y' = 3x + \frac{1}{2} y}$ with $\boldsymbol{y}$ (0) = 1 at $\boldsymbol{x}$ = 0.2 taking $\boldsymbol{h}$ = 0.1

Solution: Given $f(x, y) = 3x + \frac{1}{2} y$

Also given $x_0 = 0, y_0 = 1$. Take $h = 0.1$

To find $\boldsymbol{y}$ (0.1)

$$k_1 = hf(x_0, y_0) = (0.1)\left(3x_0 + \frac{y_0}{2}\right) = (0.1)\left(3\ (0) + \frac{1}{2}\right)$$

$$= 0.05$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.05}{2}\right)$$

$$= (0.1)\, f(0.05, 1.025) = 0.1\left(3(0.05) + \frac{1.025}{2}\right)$$

$$= 0.0663$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_n + \frac{k_2}{2}\right) = (0.1)\, f\left(0 + \frac{0.1}{2} \cdot 1 + \frac{0.0663}{2}\right)$$

$$= (0.1)\, f(0.05, 1.0332)$$

$$= 0.1\left(3\ (0.05) + \frac{1.0332}{2}\right) = 0.0667$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\, f(0 + 0.1,\ 1 + 0.0667)$$

$$= (0.1)\left(3\,(0.1) + \frac{1.0667}{2}\right)$$

$$= 0.0833$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.05 + 2(0.0663) + 2\,(0.0667) + 0.0833]$$

$$= 0.0666$$

$$y_1 = y_0 + \Delta y$$

$$= 1 + 0.0666$$

$$= 1.0666$$

(i.e.,) $y\,(0.1) = 1.0666$

$$x_1 = x_0 + h$$

$$= 0 + 0.1$$

$$= 0.1$$

To find *y* (0.2)

$$k_1 = hf(x_1, y_1) = (0.1)\left(3x + \frac{y_1}{2}\right) = (0.1)\left(3\,(0.1) + \frac{1.0666}{2}\right)$$

$$= 0.0833$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)\, f\left(0.1 + \frac{0.1}{2}, 1.0666 + \frac{0.0833}{2}\right)$$

$$= (0.1)\, f(0.15,\ 1.1083) = 0.1\left(3\,(0.15) + \frac{1.1083}{2}\right)$$

$$= 0.1004$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)\, f\left(0.1 + \frac{0.1}{2} \cdot 1.0666 + \frac{0.1004}{2}\right)$$

$$= (0.1)\, f(0.15, 1.1168) = 0.1\left(3\,(0.15) + \frac{1.1168}{2}\right)$$

$$= 0.1008$$

$$k_4 = hf(x_1 + h, y_1 + k_2) = (0.1)\, f(0.1 + 0.1,\ 1.0666 + 0.1008)$$

$$= (0.1)\left(3\,(0.2) + \frac{1.1674}{2}\right)$$

$$= 0.1184$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.0833 + 2\,(0.1004) + 2\,(0.1008) + 0.1184]$$

$$= 0.1007$$

$$y_2 = y_1 + \Delta y$$

$$= 1.0666 + 0.1007$$

$$= 1.1673$$

(i.e.,) $y\,(0.2) = 1.1673$

Example 5: Use 4th order R-K method to solve $y' = xy$ for $x = 1.2, 1.4, 1.6$. Initially $x = 1, y = 2$. (take $h = 0.2$)

Solution: Given $f(x, y) = xy$

Also given $x_0 = 1, y_0 = 2$. Take $h = 0.2$

To find y (1.2)

$$k_1 = hf(x_0, y_0) = (0.2)\,(x_0\, y_0) = (0.2)\,[(1)(2)]$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\, f\left(1 + \frac{0.2}{2}.2 + \frac{0.4}{2}\right)$$

$$= (0.2)\, f\,(1.1, 2.2) = (0.2)\,[(1.1)\,(2.2)]$$

$$= 0.484$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)\, f\left(1 + \frac{0.2}{2}.2 + \frac{0.484}{2}\right)$$

$$= (0.2)\, f\,(1.1, 2.242) = (0.2)\,[(1.1)\,(2.242)]$$

$$= 0.4932$$

$$k_4 = hf\,(x_0 + h, y_0 + k_1) = (0.2)\, f\,(1 + 0.2, 2 + 0.4932$$

$$= (0.2)\, f\,(1.2, 2.4932) = (0.2)\,[(1.2)\,(2.4932)]$$

$$= 0.5984$$

$$\Delta y = \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 ++ k_4]$$

$$= \frac{1}{6}\,[0.4 + 2\,(0.484) + 2\,(0.4932) + 0.5984)]$$

$$= 0.4921$$

$$y_1 = y_0 + \Delta y$$

$$= 2 + 0.4921$$

$$= 2.4921$$

(i.e.,) $y\,(1.2) = 2.4921$

$$x_1 = x_0 + h$$

$$= 1 + 0.2$$

$$= 1.2$$

To find y (1.4)

$k_1 = hf(x_1, y_1) = (0.2)(x_1 y_1) = (0.2)[(1.2)(2.4921)]$

$= 0.5981$

$k_2 = hf\left(x_1 + \frac{h}{2} \cdot y_1 + \frac{k_1}{2}\right)$

$= (0.2) f\left(1.2 + \frac{0.2}{2}, 2.4921 + \frac{0.5981}{2}\right)$

$= (0.2) f(1.3, 2.7912) = (0.2)[(1.3)(2.7912) = 0.7257$

$k_3 = hf\left(x_1 + \frac{h}{2} \cdot y_1 + \frac{k_2}{2}\right) = (0.2) f\left(1.2 + \frac{0.2}{2} \cdot 24921 + \frac{0.7257}{2}\right)$

$= (0.2) f(1.3, 2.8550)$

$= (0.2)[(1.3)(2.8550)] = 0.7423$

$k_4 = hf(x_1 + h, y_1 + k_3$

$= (0.2) f(1.2 + 0.2,\ 2.4921 + 0.7423)$

$= (0.2) f(1.4, 3.2344) = (0.2)[1.4(3.2344)]$

$= 0.9056$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4$

$= \frac{1}{6}[0.5981 + 2(0.7257) + 2(0.7423) + 0.9056] = 0.74$

$y_2 = y_1 + \Delta y$

$= 2.4921 + 0.74 = 3.2321$

(i.e.,) y (1.4) = 3.2321

$x_2 = x_1 + h$

$= 1.2 + 0.2$

$= 1.4$

To find y (1.6)

$$k_1 = hf(x_2, y_2) = (0.2)\ (x_2 y_2)$$

$$= (0.2)\ [(1.4)\ (3.2321)]$$

$$= 0.9050$$

$$k_2 = hf\left(x_2 + \frac{h}{2},\ y_2 + \frac{k_1}{2} \right)$$

$$= (0.2)\, f\left(1.4 + \frac{0.2}{2} \cdot 3.2321 + \frac{0.9050}{2} \right)$$

$$= (0.2)\, f(1.5,\ 3.6846)$$

$$= (0.2)\ [(1.5)\ (3.6846)] = 1.1054$$

$$k_3 = hf\left(x_2 + \frac{h}{2},\ y_2 + \frac{k_2}{2} \right)$$

$$= (0.2)\, f\left(1.4 + \frac{0.2}{2},\ 3.2321 + \frac{1.1054}{2} \right)$$

$$= (0.2)\, f(1.5,\ 3.7848)] = 1.1354$$

$$k_4 = hf(x_2 + h,\ y_2 + k_3$$

$$= (0.2)\, f(1.4 + 0.2,\ \ 3.232 + 1.1354$$

$$= (0.2)\, f(1.6,\ 4.3675)$$

$$= (0.2)\ [(1.6)\ (4.3675)] = 1.3976$$

$$\Delta y = \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}\ [0.9050 + 2\ (1.1054) + 2\ (1.1354) + 1.3976$$

$$= 1.1307$$

$$y_2 = y_2 + \Delta y$$

$$= 3.2321 + 1.1307 = 4.3628$$

(i.e.,) y (1.6) = 4.3628

Example 6: Solve $y' = x - y^2, 0 \le x \le 1,$ $y(0) = 0, y(0.2) = 0.02,$ $y(0.4) = 0.0795,$ $y(0.6) = 0.1762$ by Milne's method to find $y(0.8)$ and 1).

Solution:

Given $y' = x - y^2$ and $h = 0.2$.

$x_0 = 0$	$y_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$
$x_2 = 0.4$	$y_2 = 0.0795$
$x_3 = 0.6$	$y_3 = 0.1762$
$x_4 = 0.8$	$y_4 = ?$
$x_5 = 1$	$y_5 = ?$

By Milne's predictor formula, we have

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

To get y_4, put $n = 3$ we get

$$y_{4,p} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3$$

Now, $y'_1 = x_1 - y_1^2$

$= 0.2 - (0.02)^2$

$= 0.1996$

$y'_2 = x_2 - y_2^2$

$= 0.4 - (0.0795)^2$

$= 0.3937$

$$y'_3 = x_3 - y_3^2$$

$$= 0.6 - (0.1762)^2$$

$$= 0.5690$$

$$y_{4,\,p} = 0 + \frac{4\,(0.2)}{3}\,[2(0.1996) - (0.3937) + 2\,(0.5690)]$$

$y\,(0.8)_n =$ **0.3049**

By **Milne's corrector formula,** we have

$$y_{n+1,\,c} = y_{n-1} + \frac{h}{3}\,[y'_{n-1} + 4y'_n + y'_{n+1}$$

To get y_4, put $n = 3$ we get

$$y_{4,\,c} = y_2 + \frac{h}{3}\,[y'_2 + 4y'_3 + y'_4]$$

Now, $y'_4 = x_4 - y_4^2$

$$= 0.8 - (0.3049)^2$$

$$= 0.7070$$

$$y_{4,\,c} = 0.0795 + \frac{0.2}{3}\,[0.3937 + 4\,(0.5690) + 0.7070]$$

$$y\,(0.8)_c = 0.3046$$

Again, $y'_4 = x_4 - y_4^2 = 0.8 - (0.3046)^2$

$$= 0.7072$$

$$y_{4,\,x} = 0.0795 + \frac{0.2}{3}\,[0.3937 + 4\,(0.5690) + 0.7072]$$

$y\,(0.8)_c^{(2)} =$ **0.3046**

To find y_5 (or) $y\,(1)$, put $n = 4$ in the Milne's formula.

To get y_5, put $n = 4$ in Milne's predictor formula, we get

$$y_{5,\,p} = y_1 + \frac{4h}{3}\,[2y'_2 - y'_3 + 2y'_4]$$

$$= 0.02 + \frac{4\,(0.2)}{3}\,[2\,(0.3937) - 0.5690 + 2\,(0.7070)]$$

$y\,(1) = \mathbf{0.4553}$

Now put $n = 4$ in Milne's corrector formula, we get

$$y5,\,c = y_3 + \frac{h}{3}\,[y'_3 + 4y'_4 + y'_5$$

$$y'_5 = x_5 - y_5^2 = 1 - (0.4553)^2 = 0.7927$$

$$y_{5,\,c} = 0.1762 + \frac{0.2}{3}\,[0.5690 + 4\,(0.7070) + 0.7927]$$

$y\,(1)_c = 0.4555$

Again, $y'_5 = x_5 - y_5^2 = 1 - (0.4555)^2 = 0.7925$

$$y_{5,\,c} = 0.172 + \frac{0.2}{3}\,[0.5690 + 4\,(0.7070) + 0.7925]$$

$y\,(1)_{c\,(2)} = \mathbf{0.4555}$

Example 7: Given $\frac{dy}{dx} = x^3 + y,\, y\,(0) = 2$

(i) Compute $y\,(0.2), y\,(0.4)$ and $y\,(0.6)$ by R-K method of 4^{th} order. (ii) Hence find $y\,(0.8)$ by Milne's predictor corrector method taking $h = 0.2$

Solution:

Given $f\,(x, y) = x^3 + y$

Also given $x_0 = 0,\ y_0 = 2.$ Take $h = 0.2$

To find y (0.2)

$$k_1 = hf(x_0, y_0) = (0.2)\,(x_0^3 + y_0)$$

$$= (0.2)\,[0 + 2] = 0.4$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\left(\left(x_0 + \frac{h}{2}\right)^3 + \left(y_0 + \frac{k_1}{2}\right)\right]$$

$$= (0.2)\left(\left(0 + \frac{0.2}{2}\right)^3 + \left(2 + \frac{0.4}{2}\right)\right]$$

$$= 0.4402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)\left(\left(x_0 + \frac{h}{2}\right)^3 + \left(y_0 + \frac{k_2}{2}\right)\right]$$

$$= (0.2)\left(\left(0 + \frac{0.2}{2}\right)^3 + \left(2 + \frac{0.4402}{2}\right)\right]$$

$$= 0.4442$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)\,[(x_0 + h)^3 + (y_0 + k_3)]$$

$$= (0.2)\,[(0 + 0.2)^3 + (2 + 0.4442)]$$

$$= 0.4904$$

$$\Delta y = \frac{1}{6}\,[k_1 + 2k_2 + + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.4 + 2\,(0.4402) + 2\,(0.4442) + 0.4904]$$

$$= 0.4432$$

$$y_1 = y_0 + \Delta y = 2 + 0.4432$$

$$= 2.4432$$

(i.e.,) $y\,(0.2) = 2.4432$

$$x_1 = x_0 + h$$

$$= 0 + 0.2 = 0.2$$

To find *y* (0.4)

$$k_1 = hf(x_1, y_1) = (0.2)\,(x_1^3 + y_1)$$

$$= (0.2)\,[(0.2)^3 + 2.4432] = 0.4902$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)\left[\left(x_1 + \frac{h}{2}\right)^3 + \left(y_1 + \frac{k_1}{2}\right)\right]$$

$$= (0.2)\left[\left(0.2 + \frac{0.2}{2}\right)^3 + \left(2.4432 + \frac{0.4902}{2}\right)\right]$$

$$= 0.5431$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)\left[\left(x_1 + \frac{h}{2}\right)^3 + \left(y_1 + \frac{k_2}{2}\right)\right]$$

$$= (0.2)\left[\left(0.2 + \frac{0.2}{2}\right)^3 + \left(2.4432 + \frac{0.5431}{2}\right)\right]$$

$$= 0.5484$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)\,[(x_1 + h)^3 + (y_1 + k_3)]$$

$$= (0.2)\,[(0.2 + 0.2)^3 + (2.4432 + 0.5484)]$$

$$= 0.6111$$

$$\Delta y = \frac{1}{6}\,[k_1 + 2k_2 + + 2k_3 + k_4]$$

$$= \frac{1}{6}\,[0.4902 + 2\,(0.5431) + 2\,(0.5484) + 0.6111]$$

$$= 0.5474$$

$$y_2 = y_1 + \Delta y = 2.4432 + 0.5474$$

$$= 2.9906$$

(i.e.,) $y\,(0.4) = 2.9906$

$x_2 = x_1 + h$

$= 0.2 + 0.2 = 0.4$

To find y (0.6)

$k_1 = hf(x_2, y_2) = (0.2)\,(x_2^3 + y_2)$

$= (0.2)\,[(0.4)^3 + 2.9906] = 0.6109$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0.2)\left(\left(x_2 + \frac{h}{2}\right)^3 + \left(y_2 + \frac{k_1}{2}\right)\right]$$

$$= (0.2)\left(\left(0.4 + \frac{0.2}{2}\right)^3 + \left(2.9906 + \frac{0.6109}{2}\right)\right]$$

$= 0.6842$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.2)\left(\left(x_2 + \frac{h}{2}\right)^3 + \left(y_2 + \frac{k_2}{2}\right)\right]$$

$$= (0.2)\left(\left(0.4 + \frac{0.2}{2}\right)^3 + \left(2.9906 + \frac{0.6842}{2}\right)\right]$$

$= 0.6915$

$k_4 = hf(x_2 + h, y_2 + k_3) = (0.2)\,[(x_2 + h)^3 + (y_2 + k_3)]$

$= (0.2)\,[(0.4 + 0.2)^3 + (2.9906 + 0.6915)]$

$= (0.2)\,[3.8981] = 0.7796$

$\Delta y = \frac{1}{6}\,[k_1 + 2k_2 + + 2k_3 + k_4]$

$= \frac{1}{6}\,[0.6109 + 2\,(0.6842) + 2\,(0.6915) + 0.7796]$

$= 0.6903$

$y_3 = y_2 + \Delta y = 2.9906 + 0.6903$

$= 3.6809$

(i.e.,) $y(0.6) = 3.6809$

$$x_3 = x_2 + h$$
$$= 0.4 + 0.2 = 0.6$$

To find y (0.8)

Given $y' = x^3 + y$ and $h = 0.2$

$x_0 = 0$	$y_0 = 2$
$x_1 = 0.2$	$y_1 = 2.4432$
$x_2 = 0.4$	$y_2 = 2.9906$
$x_3 = 0.6$	$y_3 = 3.6809$
$x_4 = 0.8$	$y_4 = ?$

By **Milne's predictor formula,** we have

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

To get y_4, put $n = 3$ we get

$$y_{4,p} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

Now, $y'_1 = x_1^3 + y_1 = (0.2)^3 + 2.4432 = 2.4512$

$$y'_2 = x_2^3 + y_2 = (0.4)^3 + 2.9906 = 3.0546$$
$$y'_3 = x_3^3 + y_3 = (0.6)^3 + 3.6809 = 3.8969$$
$$y_{4,p} = \frac{4\,(0.2)}{3}[2\,(2.4512) - (3.0546) + 2\,(3.8969)]$$

$y\,(0.8)_p = 4.5711$

By **Milne's corrector formula,** we have

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

To get y_4, put $n = 3$ we get

$$y_{4,\,c} = y_2 + \frac{h}{3}\,[y'_2 + 4y'_3 + y'_4$$

Now, $y'_4 = x_4^3 + y_4$

$$= (0.8)^3 + 4.5711 = 5.0831$$

$$y_{4,\,c} = 2.9906 + \frac{0.2}{3}\,[3.0546 + 4\,(3.8969)\,+) + 5.083]$$

$$y\,(0.8)_c = 4.5723$$

Again, $y'_4 = x_4^3 + y_4$

$$= (0.8)^3 + 4.5723$$

$$= 5.0843$$

$$y_{4,\,c} = 2.9906 + \frac{0.2}{3}\,[3.0546 + 4\,(3.8969) + 5.0843]$$

$$y\,(0.8)_c^{(2)} = 4.5724$$

Example 8: Given $y' + xy^2 + y = 0$, $y\,(0) = 1$, find the value of $y\,(0.1)$ and $y\,(0.2)$ by using Runge kutta method of 4^{th} order.

Solution Hint:

$$y\,' = -\,ny^2 - y$$

To find $y\,(0.1)$

$k_1 = -\,0.1$, $k_2 = -\,0.0995$, $k_3 = -\,0.0995$,

$k_4 = -\,0.0982$, $\Delta y = -\,0.0994$

$$\boxed{y\,(0.1) = 0.9006}$$

To find y (0.2)

$k_1 = -0.0982, k_2 = -0.0960,\ k_3 = -0.0962,$

$k_4 = -0.0934,\ \Delta y = -0.0960$

$\boxed{y\,(0.2) = 0.8046}$

Example 9: Apply Runge-kutta method to find approximate value of y for $x = 0.2$ in 2 steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$.

Solution:

Hint:

To find y (0.1)

$k_1 = 0.1, k_2 = 0.1153,\ k_3 = -0.1169,\ k_4 = 0.1347,\ \Delta y = 0.1165$

$\boxed{y\,(0.1) = 1.1165}$

To find y (0.2)

$k_1 = 0.1347, k_2 = 0.1552,\ k_3 = 0.1576,\ k_4 = 0.1823,$

$\Delta y = -0.1571$

$\boxed{y\,(0.2) = 1.2736}$

EXERCISES

Taylor series method

1. Find y (0.2) and y (0.4) given $y' = x - y^2$, y (0) = 1, using Taylor series method. [**Ans:** 0.8457; 0.7752]

2. Solve by Taylor series method, $y' = xy + y^2, y\,(0) = 1$ at $x = 0.1$ and 0.2 correct to four decimal places. [**Ans:** 1.1169; 1.2774]

3. Solve $y' = y^2 + x$; $y\,(0) = 1$ using Taylor series method and compute $y\,(0.1)$ and $y\,(0.2)$.

4. Find $y(0.1)$ and $y(0.2)$ by Taylor series method if $y(x)$ satisfies $\frac{dy}{dx} = y - \frac{2x}{y}, y(0) = 1$ taking $h = 0.1$ correct to four decimal places. [**Ans:** 1.0954; 1.1830]

5. Apply Taylor series method to find the value of $y(1.1), y(1.2)$ and $y(1.3)$ correct to the four decimal places given that $\frac{dy}{dx} = xy^{1/3}, y(1) = 1.$ [**Ans:** 1.1066; 1.2275, 1.3641]

6. Given $y' = -y, y(0) = 1,$ solve for the values of y at $x = (0.01)\,(0.01)\,(0.04)$ by Euler's method. [**Ans:** 0.99, 0.9801, 09703, 09606]

7. Find $y(0.3)$ taking $h = 0.1$, given $\frac{dy}{dx} = 1 + y^2, y(0) = 0,$ using Euler's method. [**Ans:** 0.6841]

8. Using Euler's method, solve $y' = x + y + xy, y(0) = 1$. Find y at $x = (0.2)\,(0.2)\,(1.0)$ [**Ans:** 1.2, 1.528, 2.0358, 2.8072, 3.9778]

9. Given $\frac{dy}{dx} = 3x + y^2,\ y(0) = 1,$ find y at $x = (0.1)\,(0.1)\,(0.5)$ by Taylor series method. [**Ans:** 1.1, 1.251, 1.4675, 1.7728, 2.2071]

10. If $y' = y - 1,\ y(0) = 1.1,$ find $y(1.5)$ taking $h = 0.5.$ [**Ans:** 1.15]

Modified Euler's method

11. Using modified Euler's method, find $y(0.1)$ given $y' = \frac{y - x}{y + x},$ $y(0) = 1$ [**Ans:** 1.0932]

12. Find $y(0.1)$ given $y'' = x^2 + y, y(0) = 1,$ using modified Euler's method. [**Ans:** 1.1055]

13. Given $y' = x^2 - y, y(0) = 1,$ find $y(0.1)$ using modified Euler's method. [**Ans:** 0.9055]

14. Solve $y' = y + e^x$, $y(0) = 0$, for $x = 0.2, 0.4$ by modified Euler's method **[Ans:** 0.24214, 0.59116]

15. Given $y' = \log(x + y)$, $y(0) = 1$, find $y(0.2)$ and $y(0.5)$ by modified Euler's method. **[Ans:** 1.079, 1.0480]

16. Solve $y' = y + \sin x$, $y(0) = 2$ find $y(0.1)$, $y(0.2)$ using modified Euler's method. **[Ans:** 2.215, 2.463]

17. Using Euler's modified method, find $y(1.1), y(1.2), y(1.3)$ given $\frac{dy}{dx} = xy^{1/3}$, $y(0) = 1$ **[Ans:** 1.1068, 1.2369, 1.3679)

18. Solve $y' = -2x - y$, $y(0) = -1$, to find y at $x = 0.1, 0.2, 0.3$ by modified Euler's method. **[Ans:** $-0.9150, -0.8571, -0.8237$]

Runge - Kutta method

19. Solve $\frac{dy}{dx} = \frac{1}{x + y}$, $y(0) = 1$ for $y(0.1)$ and $y(0.2)$ using Runge - Kutta method of fourth order. **[Ans:** 1.0914, 1.1696]

20. Use Runge - Kutta method of the fourth order to find $y(0.2)$ and $y(0.4)$ given that $y\frac{dy}{dx} = y^2 - x$, $y(0) = 2$, taking $h = 0.2$. **[Ans:** 2.4334, 2.9478]

21. Use Runge - Kutta method to find y when $x = 1.2$ in steps of 0.1, if $\frac{dy}{dx} = x^2 + y^2$ and $y(1) = 1.5$ **[Ans:** 1.8955, 2.5043]

22. Obtain the values of y at $x = 0.1$, 0.2 using Runge - Kutta method given $y' = -y$, $y(0) = 1$ **[Ans:** 0.9048, 0.8187[

23. Solve $y' = \frac{y - x}{y + x}$, $y(0) = 1$ to obtain $y(0.2)$ using Runge - Kutta method. **[Ans:** 1.165]

24. Evaluate $y(0.1), y(0.2), y(0.3)$ using fourth order Runge - Kutta method given $y' = \frac{1}{2}(1+x)y^2, y(0) = 1$.

[**Ans:** 1.0552, 1.123, 1.2073]

25. Solve $y' = xy + 1$, at $x = 0.2, 0.4, 0.6$ given $y(0) = 2$, taking $h = 0.2$. [**Ans:** 2.243, 2.589, 3.072]

26. Find $y(0.2)$ given $\frac{dy}{dx} = y - x$, $y(0) = 2$, taking $h = 0.1$

[**Ans:** 2.4214]

27. Find $y(0.1)$ given $y'' = y^3, y(0) = 5$ by Runge - Kutta method. [**Ans:** 17.4148]

28. Using Runge - Kutta method, find $y(0.1)$ given $y'' + 2xy' - 4y = 0$, $y(0) = 0.2$, $y'(0) = 0.5$. [**Ans:** 0.2542]

29. Using Runge - Kutta method of order four, solve $y'' = y + xy', y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$

[**Ans:** 0.9802, $-$ 0.196]

30. Find $y(0.2)$ and $y(0.4)$ by Runge - Kutta method, given method, given that $y''(x) = (1+x^2)\,y$, $y(0) = 1$, $y'(0) = 1$. Take $h = 0.2$. [**Ans:** 1.0202, 1.0833]

Milne's Predictor - Corrector Method

31. Evaluate $y(0.4)$ by Milne's predictor corrector method, given $\frac{dy}{dx} = \frac{1}{2}(1+x^2)\,y^2$, $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$. [**Ans:** 1.2798]

32. Determine the value of $y(0.4)$ using Milne's method given $y' = xy + y^2, y(0) = 1$. Using Taylor series method to get the values of $y(0.1), y(0.2)$ and $y(0.3)$.

[**Ans:** 1.1167, 1.2767, 1.5023, 1.8370]

33. Given $\frac{dy}{dx} = \frac{1}{2}(x + y)$, find $y(2)$ by Milne's method, given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$.

[**Ans:** 6.8732]

34. Solve $\frac{dy}{dx} = x - y^2$ to find $y(0.8)$ and $y(1)$ given $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$

[**Ans:** 0.3046, 0.4556]

35. Given $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$, $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$ find $y(1.4)$ and $y(1.5)$ using Milne's method.

[**Ans:** 0.8552, 0.9378]

Adams-Bashforth method

36. Using Adams-Bashforth method, find $y(0.4)$ given $\frac{dy}{dx} = \frac{1}{2}xy$, $y(0) = 1$, $y(0.1) = 1.01$, $y(0.2) = 1.022$, $y(0.3) = 1.023$.

[**Ans:** 1.0410]

37. Obtain $y(0.6)$ given $\frac{dy}{dx} = x + y$, $y(0) = 1$ with $h = 0.2$ by Adam's method. Obtain $y(-0.2)$, $y(0.2)$, $y(0.4)$ by Taylor's method. [**Ans:** 0.8373, 1.2427, 1.5834, 2.0439]

38. Using Adams' predictor - corrector formula, evaluate $y(1.4)$ if y satisfies $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$. [**Ans:** 0.949]

39. Given the initial value problem $y' = y - \frac{2x}{y}$, $y(0) = 1$, $y(0.1) = 1.0959$, $y(0.2) = 1.1841$, $y(0.3) = 1.2662$, find $y(0.4)$ using Adams method. [**Ans:** 1.3431]

SHORT QUESTIONS AND ANSWERS

1. Write down the Taylor series formula.

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \ldots\ldots$$

$n = 0,$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

2. Write down the Euler's formula.

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$y_1 = y_0 + hf(x_0, y_0)$

$n = 0$

3. Write down the modified Euler's Algorithm.

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

4. Write down the Runga-Kutta method formula.

$y_1 = y_0 + \Delta y_0$ where $\Delta y_0 = \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_1 = hf(x_0, y_0)$ $k_2 = hf\left(x_0 + h_2, y_0 + \dfrac{k_1}{2} \right)$

$k_3 = hf\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2} \right)$ $k_4 = hf(x_0 + h, y_0 + k_3)$

5. Write the single step and multi step methods

- Euler's method
- Modified Euler's method
- Taylor series method
- R-K method

- Milne's predictor corrector method
- Adam's bash forth method

6. Find $y\,(0.1)$ by Euler's method, given that $\frac{dy}{dx} = 1 - y$, $y\,(0) = 0$

$y_1 = y_0 + hf\,(x_0, y_0) = 0 + (0.1)\,[1 - 0] = 0.1$

$y\,(0.1) = 0.1$

7. Using Euler's method, compute for $x = 0.1$ and 0.2 with $h = 0.1$ given $y' = y - \frac{2x}{y}$, $y\,(0) = 1$

$y_1 = 1 + (0.1)\,[1 - 0] = 1.10 = y\,(0.1) = 1.10$

$$y_2 = 1.1 + 0.1\left[\,1.1 - \frac{0.2}{1.1}\,\right] = 1.19$$

$y\,(0.2) = 1.19$

8. Which is better Taylor series method or Runge-Kutta method? Why?

Runge-Kutta method is better since higher order derivatives of 'y' are not required. Taylor series method involves use of higher order derivatives which may be difficult in case of complicated algebraicfunctions.

9. Distinguish single-step and multi-step methods.

Single-step methods	Multi-step methods
To find y_{n+1}, the information at y_n is enough	To find y_{n+1}, the past four valves y_{n-3}, y_{n-2}, y_{n-1} and y_n are needed.

10. Find $y\,(0.1)$ from $\frac{dy}{dx} = x + y$ given that $y\,(0) = 1$ by Taylor's method.

Given: $\frac{dy}{dx} = x + y,\ \ y\,(0) = 1$

$$y_{n+1} = y_n + \frac{h}{1!}\, y_n' + \frac{h^2}{2!}\, y_n'' + \frac{h^3}{3!}\, y_n''' + \ldots$$

$$y_1 = y\,(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2!}\,(2) + \frac{(0.1)^3}{3!}\,(2)$$

$$= 1.1103$$

11. State Adam's predictor and corrector formula.

Adam's predictor formula is

$$y_{n+1,\,p} = y_n + \frac{h}{24}\,[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}']$$

Adam's corrector formula is

$$y_{n+1,\,c} = y_n + \frac{h}{24}\,[9y_{n+1}' + 19y_n' - 5y_{n-2}' + y_{n-3}']$$

12. Write down Milne's Predictor – Corrector formula.

Milne's Predictor formula

$$y_{n+1,\,p} = y_{n-3} + \frac{4h}{3}\,[2y_{n-2}' - y_{n-1}' + 2y_n']$$

Corrector formula

$$y_{n+1,\,c} = y_{n-1} + \frac{h}{3}\,[y_{n-1}' + 4y_n' + y_{n+1}']$$

13. When the method of iteration will be useful?

Method of iteration with be useful if the coefficient matrix of the system of equations is diagonally dominant.

14. What is the order of convergence in Newton - Raphson Method?

Two

15. Gauss-Seidel method is better than Gauss - Jacobi method. Why?

In Gauss-Seidel method the latest values of unknowns at each of iteration are used in proceeding to the next stage of iteration. Hence the convergence in Guss-Seidel method is more rapid than Gauss-Jacobi Method.

16. What is the condition for convergence of Gauss-Jacobi method of iteration?

The coefficient matrix should be diagonally dominant.

17. Write the sufficient condition for convergence of Gauss Seidel Method.

Let the given set of equation of

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_1$$

$$a_3x + b_3y + c_3z = d_1$$

Then the sufficient condition for convergence of Gauss-Siedal method is

$$|a_1| \geq |b_1| + |c_1|$$

$$|b_2| \geq |a_2| + |c_2|$$

$$|c_3| \geq |a_3| + |b_3|$$

18. Establish an iteration formation to find the reciprocal of a positive number N by Newton - Raphson Method?

Solution: Let $x = \frac{1}{N}$; $N = \frac{1}{x}$; $\frac{1}{x} - N = 0$

$$f(x) = \frac{1}{x} - N$$

$$f'(x) = -\frac{1}{x^2}$$

The Newton's formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\left(-\frac{1}{x_n^2}\right)}$$

$$= x_n + \left(\frac{1}{x_n} - N\right) \times x_n^2$$

$$= x_n + x_n - Nx_n^2$$

$$= 2x_n - Nx_n^2$$

$$= x_n (2 - Nx_n)$$

19. What is the condition to apply Jacobi's method to solve a system of equation?

Diagonally dominant.

20. Write the first iteration values of x, y, z when the equations $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 5z = 110$ are solved Gauss Seidel method.

$$x = \frac{1}{27}[85 - 6y + z] = \frac{85}{27} \quad [\because \text{ putting } y = z = 0]$$

$$y = \frac{1}{15}[72 - 6x - 2z] = \frac{1}{15}\left[72 - 6\left(\frac{85}{27}\right)\right] \quad [\text{putting } z = 0]$$

$$z = \frac{1}{5}[110 - x - y]$$

$$= \frac{1}{5}\left[110 - \frac{85}{27} - \frac{1}{15}\right]\left[72 - y\left(\frac{85}{27}\right)\right]$$

21. Give Newton - Raphson iterative formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}$$

22. Explain briefy Gauss-Jordan iteration to solve simultaneous equation.

In this method, the coefficient Matrix is reduced to a diagonal matrix (or even a unit matrix) rather than a triangular matrix as in the Gaussian method. Here the elimination of the unknown is done not only in the equations below, but also in the equations above the leading diagonal. Here we get the solution without using the back substitution method.

23. Show that Newton-Raphson formula to find $\sqrt{a}$ (or) $\sqrt{N}$ can be expressed in the form

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 0, 1, 2, \ldots$$

Let $x = \sqrt{a}$ (or) $x^2 - a = 0$

Let $f(x) = x^2 - a$

$f'(x) = 2x$

$$x_{n+1} = x_n = \frac{f(x_n)}{f'(x_n)}$$

24. Why Gauss-seidel method is a better method than Jacob's iterative method?

In Gauss-Seidel method as soon as a new value for a variable is found by iteration, it is used immediately in the subsequent equation. So the convergence in Gauss-Seidel method is more rapid that Jacobi's method.

25. Give the name of any two iteration method in Numerical methods.

1. Gauss - Seidel method.
2. Jacobi's method.

QUESTION BANK

UNIT I – TESTING OF HYPOTHESIS

PART - A

1. What are the expected frequencies of a 2×2 contingency table

a	b
c	d

(A/M 2015, M/J 2016, A/M 2017)

2. Write down the formula of test statistic t to test the significance of difference between the means of two large samples. *(A/M 2015, N/D 2016)*

3. What is random sampling? *(N/D 2015)*

4. Write about F-test. *(N/D 2015)*

5. What are type I and type II errors? *(M/J 2016)*

6. Give the main use of χ^2 test. *(N/D 2016)*

7. A standard sample of 200 tins of coconut oil gave an average weight of 4.95 kgs with a standard deviation of 0.21 kg. Do we accept that the net weight is 5 kg per tin at 5% level of significance? *(A/M 2017)*

8. What is meant by level of significance and critical region? *(N/D 2017)*

9. State any two applications/uses of Chi-square test. *(N/D 2017)*

10. Define the following terms: Statistic, Parameter, Standard error and random sampling. *(A/M 2018)*

11. Mention the type of sampling.

12. Define the following terms: Population, Sample and sampling.

13. What is mean by null and alternative hypothesis?

14. Write down any two properties of Chi-square distribution.
15. Define Chi-square test of goodness of fit.
16. What are the assumptions of t-test?
17. What are the properties of t distribution.
18. State the properties of F distribution.
19. What is meant by confidential limits?
20. What is meant by one-tail and two-tail test?
21. Write the applications of 't' test.
22. Write the test statistic formula for the significant difference of mean in a single large sample.
23. Write the test statistic formula for difference of means for a single small sample.
24. What are the applications of F-test?
25. Write the test statistic formula for the Chi-square test for goodness of fit.
26. What are the applications of large sample test?
27. Write the test statistic formula for difference of means for two small samples.
28. Write the test statistic formula for difference of variances in F-test
29. What is mean by attributes?
30. What is mean by contingency table?

PART - B
[First Half] (All are 8 marks)

I - Large sample test

1. Examine whether the different in the variability in yields is significant at 5% level of significance for the following.

	Set of 40 plots	Set of 60 plots
Mean yield per plot	1256	1243
S.D per plot	34	28

2. A mathematics test was given to 50 girls and 75 boys. The girls made an average grade of 76 with an S.D of 6 and the boys made an average grade of 82 with an S.D of 2. Test whether there is a difference between the performance of boys and girls.

3. A sample of 900 members has a mean of 3.4 cms and s.d. is 2.61 cms. Is the sample from a large population of mean 3.25 cm and s.d. is 2.61 cms. If the population is normal and its mean is known, find the 95% fiducial limits of the mean.

4. A normal population has a mean of 6.48 and s.d. of 1.5. In a sample of 400 members, mean is 6.75. Is the difference significant?

II - Student *t*-test

5. The IQ's of 10 girls are respectively 120, 110, 70, 88, 101, 100, 83, 98, 95 and 107. Test whether the population mean IQ is 100.

6. Test made on the breaking strength of 10 pieces of a metal gave following results 578, 572, 570, 570, 572, 596 and 584 kg. Test if mean breaking strength of the wire can assumed as 577 kg.

7. A random sample of size 16 values from a normal population showed a mean of 53 and a sum of square of deviation from the mean equals to 150. Can this sample be regarded as taken from the population having 53 as mean obtain 95% confidence limits of the mean of the population?

8. The average numbers of articles produced by two machines per day are 200 and 250 with standard deviation 20 and 25 respectively on the basis of records of 25 days production. Can you regard both the machines equally efficient at 1% level.

[Second Half] (All are 8 marks)

III - ψ^2-test (or) chi-square test

9. The demand for particular spare parts in a factory was found to vary from day-to-day. In a sample study the following information was obtained.

Days	Mon	Tue	Wed	Thu	Fri	Sat
No. of spare parts demanded	1124	1125	1110	1120	1126	1115

Test the hypothesis that the number of parts demanded does not depend on the day of the week.

10. The number of automobile accidents in a certain locality was 12, 8, 20, 2, 14, 10, 15, 6, 9, 4. Are these frequencies in agreement with the belief that accident conditions were the same during this 10 week period?

11. The theory predicts the proportion of beans, in the four groups *A*, *B*, *C*, *D* should be 9:3:3:1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory?

12. Test whether the following attributes are independent at 5% level.

Vaccination			
Small pox	Given	Not given	Total
Attacked	35	333	368
Not attacked	308	806	1114
Total	343	1139	1482

13. The following table gives the classification of 100 workers according to sex and nature of work. Test whether the nature of work is independent of the sex of the worker.

	Stable	Unstable
Males	40	20
Females	10	30

14. Four coins were tossed 160 times and the following results were obtained. Under the assumption that the coins are unbiased and test the goodness of fit.

No. of heads	0	1	2	3	4
Observed frequency	17	52	54	31	6

15. On the basis of information given below about the treatment of 200 patients suffering from a disease, state whether the new treatment is comparatively superior to the conventional treatment.

	Favorable	Not favorable
New	60	30
Conventional	40	70

III - F-Ratio test

16. Two independent samples of sizes 9 and 7 from a normal population had the following values

Sample-I	18	13	12	15	12	14	16	14	15
Sample-II	16	19	13	16	18	13	15	-	-

Do the estimate of population variance difference at 5% significance.

17. A group of 10 rats fed on diet *A* and another group of 8 rats fed on diet *B*, recorded the following increase in weight

Diet *A*	5	6	8	1	12	4	3	9	6	10
Diet *B*	2	3	6	8	10	1	2	8	-	-

Do the estimates of population variance difference from the samples are not significant different.

18. Do the following sample variances vary significantly at 5% level.

Sample-I	39	41	43	41	45	39	-
Sample-II	40	42	40	44	39	38	40

19. Test if the variances are significantly different for

X_1	24	27	26	21	25	-
X_2	27	30	32	36	28	23

20. Two random samples gave the following results

Sample	Size	Sample mean	Sum of square of deviation from mean
1	10	15	90
2	12	14	108

UNIT II – DESIGN OF EXPERIMENTS

PART - A

1. What are the uses of ANOVA? *(A/M 2017)*
2. What are the basic principles in the design of experiment? *(A/M 2017, M/J 2016)*
3. What are the basic elements of an ANOVA table for one way classification? *(A/M 2018)*
4. What are the basic designs of experiment? *(A/M 2018)*
5. Is 2×2 Latin square design possible? Why? *(N/D 2015, M/J 2016)*
6. Write two advantages/uses/applications of completely randomized experimental design. *(N/D 2015)*
7. What is ANOVA? *(N/D 2016)*
8. Define experimental error. *(N/D 2016)*
9. What is the aim of the design of experiment? *(N/D 2017)*
10. What is a completely randomized design? *(N/D 2017)*
11. What is mean by randomization?
12. What is mean by replication?
13. What are the demerits of completely randomized design?
14. What are the demerit of Latin square design?
15. What are the assumptions made to validate the F-test in ANOVA?
16. What are the demerits of randomized block design?
17. What is mean by local control?
18. What are the merits of randomized block design?
19. What are the merits of Latin square design?
20. What is mean by 2^2 factorial design?
21. Write two advantages/uses/applications of randomized block design.

22. Write two advantages/uses/applications of Latin square design.
23. What is mean by one way classification?
24. What is mean by two way classification?
25. What is mean by three way classification?
26. Compare completely randomized design and randomized block design.
27. Compare randomized block design and Latin square design.
28. Compare completely randomized design and Latin square design.
29. What are the merits of 2^2 factorial design?
30. What are the demerits of 2^2 factorial design?

PART - B
[First Half] (All are 16 marks)

I-One way classification (Completely Randomized Design-CRD)

1. The following table gives the yields of 15 samples of plot under three varieties of seed.

A	20	21	23	16	20
B	18	20	17	15	25
C	25	28	22	28	32

Test using analysis of variance whether there is a significant difference in the average of yield of seeds.

2. The following table shows the lies in hours of four electrical lamps brand.

A	1610	1610	1650	1680	1700	1720	1800	-
B	1580	1640	1640	1700	1750	-	-	-
C	1460	1550	1600	1620	1640	1660	1740	1820
D	1510	1520	1530	1570	1600	1680	-	-

Perform an analysis of variance and test the homogeneity of the mean lives of four brands of lamps.

3. The three samples below have been obtained from normal population with equal variances. Test the hypothesis that the sample means are equal.

Samples		
8	7	12
10	5	19
7	10	13
14	9	12
11	9	14

II-Two way classification (Randomized Block Design-RBD)

4. A company appoints 4 salesmen A, B, C and D and observes their sales in 3 seasons, summer, winter and monsoon. The figures are given in the following table.

Season	Salesmen			
	A	*B*	*CüzçzC*	*D*
summer	45	40	28	37
winter	43	41	45	38
monsoon	39	39	43	41

5. Give that:

Detergent	Engine		
	1	2	3
A	45	43	51
B	47	46	52
C	48	50	55
D	42	37	49

Perform ANOVA and test at 0.05 level of significance whether these are difference in the detergents or in the engines.

6. Three varieties of coal were analyzed by 4 chemists and the ash content is tabulated here. Perform an analysis of variance.

	Chemists				
	A	***B***	***C***	***D***	
I	8	5	5	7	
Coal	II	7	6	4	4
	III	3	6	5	4

7. The result of an RBD experiment on 3 blocks with 4 treatments A, B, C and D are tabulated here. Carry out an analysis of variance.

Blocks	Treatment effects			
I	A36	D35	C21	B36
II	D32	B29	A28	C31
III	B28	C29	D29	A26

8. A company appoints 4 salesmen A, B, C and D and observes their sales in 3 seasons, summer, winter and monsoon. The figures are given in the following table.

Season	Salesmen			
	A	*B*	*CüzçzC*	*D*
summer	36	36	21	35
winter	28	29	31	32
monsoon	26	28	29	29

9. Carry out ANOVA table.

Workers		*A*	*B* C	*üzçzC*	*D*
	1	44	38	47	36
	2	46	40	52	43
Workers	3	34	36	44	32
	4	43	38	46	33
	5	38	42	49	39

10. Five doctors each test five treatments for a certain disease and observe the number of days each patient takes to recover.

Doctors	Treatments				
	1	2	3	4	5
1	10	14	23	18	20
2	11	15	24	17	21
3	9	12	20	16	19
4	8	13	17	17	20
5	12	15	19	15	22

[Second Half] (All are 16 marks)

III-Three way classification (Latin Square Design-LSD)

11. Analyze the variance in the following Latin square of yields of paddy where *A*, *B*, *C* and *D* denote the difference methods of calculation.

D 122	A 121	C 123	B 122
B 124	C 123	A 122	D 125
A 120	B 119	D 120	C 121
C 122	D 123	B 121	A 122

Examine whether the different methods of cultivation have given significantly different yields.

12. A variable trail was conducted on wheat with 4 varieties in a Latin square design. The plan of the experiment and the per plot yield are given below.

C 25	B 23	A 20	D 20
A 19	D 19	C 21	B 18
B 19	A 14	D 17	C 20
D 17	C 20	B 21	A 15

13. Given that

Routes	Drivers			
	1	2	3	4
1	18 (C)	12 (D)	16 (A)	20 (B)
2	26 (D)	34 (A)	25 (B)	31 (C)
3	15 (B)	22 (C)	10 (D)	28 (A)
4	30 (A)	20 (B)	15 (C)	9 (D)

14. In a Latin square experiment given below are the yields in quintals per acre on the paddy crop carried out for testing the effect of five fertilizers A, B, C and D. Analyze the data for variations.

B 25	A 18	E 27	D 30	C 27
A 19	D 31	C 29	E 26	B 23
C 28	B 22	D 33	A 18	E 27
E 28	C 26	A 20	B 25	D 33
D 32	E 25	B 23	C 28	A 20

UNIT III – SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

PART - A

1. Write the formula for Newton-Raphson method.
2. Write the order and condition for the convergence in Newton-Raphson method. *(A/M 2015)*
3. What is the sufficient condition for the convergence of fixed point iteration method? *(N/D 2015)*
4. Write the formula for fixed point iteration method.
5. What are the two major types of methods to solve the system of equations?
6. What are the iterative methods available to solve a system of equations?
7. What are the direct methods available to solve a system?
8. Write the sufficient condition for the convergence of iterative or indirect methods. *(A/M 2015)*
9. What is mean by diagonally dominant? *(A/M 2016)*
10. Write the procedure to solve a system of equations by Gauss elimination method.
11. Write the working procedure for the power method.
12. Write the procedure to solve a system of equations by Gauss – Jordan method.
13. Write the principles used in Gauss elimination method and Gauss – Jordan method.
14. Distinguish between Gauss elimination *(N/D 2016)*
15. Write the difference between Gauss elimination method and Gauss – Jordan method. *(N/D 2015)*
16. Explain why the Newton's method is better than fixed point iteration method.
17. Compare Gauss Jacobi and Gauss Seidel method.

18. What are the applications of Newton's method?
19. Explain briefly the power method. *(A/M 2016)*
20. What are the applications of Gauss – Jordan method?
21. The convergence in Gauss Seidel method in more when compared to Gauss Jacobi method. Why? *(N/D 2016)*
22. What are the applications of Gauss elimination method?
23. What are the applications of power method?
24. What are the applications of Jacobi's method to find the eigen value?
25. Write the disadvantage of fixed point iteration method over Newton-Raphson method.
26. Write the disadvantage of Gauss Jacobi method over Gauss Seidel method.
27. Write the procedure to solve a system of equations by Gauss Jacobi method.
28. Write the procedure to solve a system of equations by Gauss Seidel method.
29. Write the procedure to find eigen values by Jacobi's method.
30. Compare power method and Jacobi's method to find eigen values.

PART - B
[First Half] (All are 8 marks)

I - Fixed Point Iteration method

1. Solve by iteration method, $f(x) = x^4 - x - 10$
2. Solve by iteration method, $f(x) = x^4 - x - 9$

II - Newton - Raphson method

3. Solve by Newton - Raphson method, $f(x) = x^3 - 6x + 4$
4. Solve by Newton - Raphson method, $x^2 = -4 \sin x$
5. Evaluate $\sqrt{12}$ (or) $x = \sqrt{12}$ by Newton - Raphson method.

6. Find the root of the equations $e^x = 4x$ by Newton - Raphson method.

7. Find the Newton's Raphson formula to find the value of $\frac{1}{N}$, where N is a real number and hence evaluate $\frac{1}{26}$ correct to 4-decimal places.

III - Gauss-Elimination Method

1. Solve by Gauss elimination method,
$10x - y + 2z = 4,\ x + 10y - z = 3,\ 2x + 3y + 20z = 7$

2. Solve by Gauss elimination method,
$2x + y + 4z = 12,\ 8x - 3y + 2z = 20,\ 4x + 11y - z = 33$

3. Solve by Gauss elimination method,
$5x - 2y + z = 4,\ 7x + y - 5z = 8,\ 3x + 7y + 4z = 10$

IV - Gauss-Jordan Method

4. Solve by Gauss elimination method,
$2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16$

5. Solve by Gauss elimination method,
$x + y + z = 9,\ 2x - 3y + 4z = 13,\ 3x + 4y + 5z = 40$

[Second Half] (All are 8 marks)

V - Gauss-Jacobi's Method

1. Solve by Gauss – Jacobi's method, $5x + 2y + z = 12,$
$x + 4y + 2z = 15,\ x + 2y + 5z = 20$

2. Solve by Gauss – Jacobi's method,
$28x + 4y - z = 32,\ x + 3y + 10z = 24,\ 2x + 17y + 4z = 35$

3. Solve by Gauss – Jacobi's method,
$20x + y - 2z = 17,\ 3x + 20y - z = -18,\ 2x - 3y + 20z = 25$

4. Solve by Gauss – Jacobi's method,
$30x - 2y + 3z = 75,\ x + 17y - 2z = 48,\ x + y + 9z = 15$

VI-Gauss - Seidel Method

1. Solve by Gauss - Seidel Method,
$5x + 2y + z = 12,\ x + 4y + 2z = 15,\ x + 2y + 5z = 20.$

2. Solve by Gauss - Seidel Method,
$x + y + 9z = 15,\ x + 17y - 2z = 48,\ 30x - 2y + 3z = 75.$

3. Solve by Gauss - Seidel Method,
$5x - 2y + z = -4,\ x + 6y - 2z = -1,\ 3x + y + 5z = 13.$

4. Solve by Gauss - Seidel Method,
$8x - y + z = 18,\ 2x + 5y - 2z = 3,\ x + y - 3z = -6.$

VII-Eigen value of a matrix by Power Method

1. Find the dominant eigen value by power method,
$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2. Find the dominant eigen value by power method,
$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

3. Find the dominant eigen value by power method,
$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$

VIII-Eigen value of a matrix by Jacobi's Method

4. Find the eigen value by Jacobi's method,
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

5. Find the eigen value by Jacobi's method, $A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$

UNIT IV – INTERPOLATION, NUMERICAL DIFFERENTIATION AND NUMERICAL INTEGRATION

PART - A

I-Lagrange's Method

1. State Lagrange's interpolation formula.
2. What is the assumption we make when Lagrange's formula is used? *(A/M 2015)*
3. What advantage has Lagrange's formula over Newton?
4. What is the disadvantage in practice in applying Lagrange's interpolation formula?
5. What is 'inverse interpolation'? *(N/D 2017)*
6. Give the inverse of Lagrange's interpolation formula?
7. Use Lagrange's formula, to find the quadratic polynomials that take these values.

X:	0	1	3
Y:	0	1	0

Then find $y(2)$. *(A/M 2016)*

8. Explain briefly interpolation.
9. Find the divided table for the following data *(A/M 2015)*

X:	2	5	10
Y:	5	29	109

10. From the divided difference table for the following table

X:	2	5	10
Y:	5	29	109

11. Give the Newton's divided difference interpolation formula.
12. State any 2 properties of divided differences. *(A/M 2016)*

13. Using Newton's divided difference formula determine $f(3)$ from the data

x :	0	1	2	4	5
$f(x)$:	1	14	15	5	6

14. Using Newton's divided difference interpolation formula find the missing value.

x :	1	2	4	5	6
$f(x)$:	14	15	5	-	9

15. State the Order of Convergence of cubic spline.

16. State the properties of Cubic spline.

17. Derive Newton's backward difference formula by using operator method.

18. Derive Newton's forward difference formula by using operator method.

19. State Newton's backward formula.

20. State Gregory-Newton's for difference formula.

21. When Newton's backward Interpolation formula is used.

22. Obtain the interpolation quadratic polynomial for the given data by using Newton's forward difference formula.

X :	0	2	4	6
Y :	-3	5	21	4

(N/D 2016)

23. Obtain the divided difference table for the following table

X :	2	3	5
Y :	0	14	102

24. Find the polynomial for the following data by Newton's backward difference formula.

x :	0	1	2	3
$f(x)$:	-3	2	9	18

25. Find the divided difference table for the following data *(N/D 2017)*

X :	-1	0	2	3
Y	-8	3	1	12

26. Using Lagrange's interpolation, find the polynomial through (0, 0), (1, 1) and (2, 2)

27. Show that the divided differences are symmetrical in their arguments. *(N/D 2015)*

28. What is the nature of n^{th} divided differences of a polynomial of n^{th} degree?

29. Find the second divided differences with arguments a, b, c if $f(x) = \frac{1}{x}$. *(N/D 2016)*

30. Show that $\underset{bcd}{\overset{3}{\Delta}}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

PART - B

[First Half] (All are 8 marks)

I-Lagrange's Method

1. Using Lagrange formula, fit a polynomial to the data and hence find y at $x = 1.5$ and $x = 1$

X	-1	0	2	3
Y	-8	3	1	12

2. Use Lagrange's formula to fit a polynomial to the data:

x :	-1	0	2	3
y :	-8	3	1	12

(N/D 2016)

3. Find x when $y = 20$ using Lagrange's interpolation formula for the data: (8) *(8) (N/D 2017)*

x :	1	2	3	4
y :	1	8	27	64

4. Find the Lagrange's polynomial of degree 3 to fit the data: $y(0) = -12$, $y(1) = 0$, $y(3) = 6$ and $y(4) = 12$. Hence, find $y(2)$. (8)

II - Newton's Divided Difference Method

1. Find $y(10)$, $y'(6)$ using Newton's divided difference formula

x	0	2	3	4	7	9
y	4	26	58	112	466	922

2. Using Newton's divided difference formula, find the value of $f(8)$ and from the following table: *(8) (N/D 2016)*

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

3. Using Newton's divided difference formula find $f(x)$ and $f(6)$ from the following data: (8)

x :	1	2	7	8
$f(x)$:	1	5	5	4

4. If $f(0) = 0$, $f(2) = -12$, $f(5) = 600$ and $f(7) = 7308$, find a polynomial that satisfies this data using Newton's divided difference interpolation formula. Hence, find $f(6)$. (8)

III-Newton's Forward and Backward Method

1. Find the value of y, when $x = 43$ and $x = 84$. From the following data.

X	40	50	60	70	80	90
y	184	204	226	250	276	304

2. The following table gives the values of a function at equal intervals, Evaluate $f(1.8)$

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.3889	0.3521	0.2420	0.1295	0.0540

3. Interpolate $y(12)$, if

x	10	15	20	25	30
$Y(x)$	35	33	29	27	22

4. The table gives the distances I nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x	100	150	200	250	300	350	400
y	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when $x = 118$ ft and 390 ft

5. Given that the table, and find $y(175)$

x	140	150	160	170	180
y	3.685	4.854	6.302	8.076	10.225

6. Find $y\,(22)$, given that

x	20	25	30	35	40	45
$Y\,(x)$	354	332	291	260	231	204

7. Estimate the premium for policies maturing at age 46

Age (x)	45	50	55	60	65
Premium (y)	114.84	96.16	83.32	74.48	68.48

[Second Half] (All are 8 marks)

IV-Differentiation of Newton's Forward and Backward (or) To find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$

1. Given that:

x	1	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ at $x = 1.1$ and $x = 1.6$

2. Find the $\left(\frac{dy}{dx}\right)$ and $\left(\frac{d^2 y}{dx^2}\right)$ at $x = 51$, from the data given below

x	50	60	70	80	90
y	19.96	36.65	58.81	72.21	94.61

3. For the following table obtain $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ for $x = 1.2$

x	1.0	1.2	1.4	1.6	1.8	20	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

4. Compute $f'\,(0)$ and $f''\,(4)$ from the following table.

V-Trapezoidal and Simpson's $\frac{1}{3}$ rule (Single integral)

x	0	1	2	3
$f(x)$	1	2.718	7.381	20.086

1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using the trapezoidal and Simpson's $\frac{1}{3}$ rule and compare with its exact solution.

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using the trapezoidal and Simpson's $\frac{1}{3}$ rule by taking $h = \frac{1}{4}$ (or) $h = 0.25$

3. Evaluate $\int_{-3}^{3} x^4\, dx$ using the trapezoidal and Simpson's $\frac{1}{3}$ rule, correct the three decimals dividing the range of integration into 8-equal parts. Also compare with exact solution.

4. Evaluate $\int_0^1 \frac{dx}{1+x}$ using the trapezoidal and Simpson's $\frac{1}{3}$ rule by taking $h = \frac{1}{4}$ (or) $h = 0.25$

5. By dividing the range into ten equal parts, Evaluate $\int_0^{\pi} \sin x\, dx$ by using trapezoidal and Simpson's $\frac{1}{3}$ rule

VI-Trapezoidal and Simpson's $\frac{1}{3}$ rule (Double integral)

1. Evaluate the integral $\int_1^2 \int_1^2 \frac{dx\,dy}{x+y}$, using the trapezoidal and Simpson's $\frac{1}{3}$ rule with $h = 0.5$ and $k = 0.25$

2. Evaluate the integral $\int_1^2 \int_1^2 \frac{dx\,dy}{x+y}$, using the trapezoidal and Simpson's $\frac{1}{3}$ rule with $h = 0.25$ and $k = 0.25$

3. Evaluate the integral $\int_1^2 \int_1^2 \frac{dx\,dy}{x^2+y^2}$, using the trapezoidal and Simpson's $\frac{1}{3}$ rule with $h = 0.2$ and $k = 0.25$

4. Evaluate the integral $\int_0^2 \int_1^2 \sin(9x+y)\,dx\,dy$, using the trapezoidal and Simpson's $\frac{1}{3}$ rule with $h = 0.25$ and $k = 0.5$

UNIT V – NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

PART - A

1. Define initial conditions and initial value problem.
2. Write the formula for Taylor's series method.
3. Write the working procedure for Taylor's series method.
4. Write the difference between Euler and Modified Euler method. *(N/D 2017)*
5. Solve $\frac{dy}{dx} = x + y$, given $y = 0$ when $x = 1$ upto $x = 1.1$ with $h = 0.1$ by Taylor's series method. *(A/M 2015)*
6. Name some multi-step method.
7. Write the formula for Euler's method.
8. How many previous values should be given for multi-step methods?
9. Write the formula for Modified Euler's method.
10. Write the working procedure for Modified Euler's method.
11. What is Adam – Bash Forth Predictor – Corrector method?
12. What are the two types of methods of obtaining the solution of initial value problem?
13. What is the use of Runge-Kutta method and what is its working rule for fourth order?
14. Solve $\frac{dy}{dx} = x + y$, given $y = 0$ when $x = 1$ upto $x = 1.2$ with $h = 0.2$ by Taylor's series method. *(N/D 2017)*
15. Solve $y' = \sin x + y$, $y(0) = 2$ by the modified Euler's method to get y at $x = 0.1\ (0.1)\ 0.3$. *(N/D 2015)*
16. Define Predictor – Corrector methods and what are the types of it?

17. Name some single-step methods. *(N/D 2016)*

18. Given $\frac{dy}{dx} = x + y^2$ and $y(0) = 1$. Find an approximate value of y at $x = 0.5$ by modified Euler's method. Taking $h = 0.1$. *(A/M 2015)*

19. Write the working procedure for Adam's method.

20. Write the working procedure for Milne's method.

21. What do you mean by single – step method? *(N/D 2015)*

22. Write the difference between Taylor and Euler method.

23. What is Milne's Predictor - Corrector method?

24. What is the use of Runge-Kutta method and what is its working rule for fourth order?

25. What are the uses of Taylor's series method? *(N/D 2016)*

26. What are the applications of Adam's method?

27. Compare Milne's method and Adam's method.

28. Compare Runge-Kutta method of fourth order and Taylor's series method.

29. Compare Runge-Kutta method of fourth order and Euler's method.

30. Compare Runge-Kutta method of fourth order and Modified Euler's method.

PART - B
[First Half] (All are 8 marks)

I-Taylor's Series Method

1. By means of Taylor series expansion, find y at $x = 0.1$ and $x = 0.2$ correct to three decimal places, given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$. *(N/D 2016)*

2. Find by Taylor series method, the values of y at $x = 0.1$ and $x = 0.2$, to four decimal places $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$.

3. Solve numerically $\frac{dy}{dx} = x + y$ when $y(1) = 0$ using Taylor's series upto $x = 1.2$ with $h = 0.1$. *(N/D 2017)*

4. Using Taylor series method, find correct to four decimal places, the value of $y(0.1)$, given $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$. *(A/M 2015)*

5. Using Taylor series method, find y at $x = 1$, given $\frac{dy}{dx} = x^2 - y,\ y(0) = 1$, correct to 4 decimal places.

6. Solve $\frac{dy}{dx} = xy + y^2,\ y(0) = 1$, using Milne's predictor-corrector formula and find $y(0.4)$. Use Taylor series method to find $y(0.1),\ y(0.2)$ and $y(0.3)$. *(16 m) (N/D 2015)*

II-Euler's Method

7. Using Euler's method, solve numerically the equation $y' = x + y,\ y(0) = 1$, for $x = 0.0\ (0.2)\ (1.0)$. Check your answer with the exact solution.

III-Modified Euler's Method

8. Apply the modified Euler's method to find $y(0.2)$ and $y(0.4)$, given that $\frac{dy}{dx} = x^2 + y^2\ y(0) = 1$. Take $h = 0.2$.

9. Using modified Euler method, find $y(0.1)$ and $y(0.2)$ given $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$ with $h = 0.1$

10. Solve $\frac{dy}{dx} = \log_{10}(x+y)$, $y(0) = 2$ by Euler's modified method and find the values of $y(0.2)$, $y(0.4)$ and $y(0.6)$, taking $h = 0.2$. *(N/D 2016)*

IV-Adams-Bash Forth Predictor Corrector Method

11. By using Adam's predictor method find y when $x = 0.4$, given $\frac{dy}{dx} = \frac{xy}{2}$, $y(0.1) = 1$, $y(0.1) = 1.01$, $y(0.2)$ 1.022, $y(0.3) = 1.023$. *(N/D 2015)*

12. Solve $2y' - x - y = 0$ given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$ to find $y(2)$ by Adam's method.

13. Obtain $y(0.6)$ given $\frac{dy}{dx} = x + y$, $y(0) = 1$ using $h = 0.2$ by Adam's method if $y(-0.2) = 0.8373$ $y(0.2) = 1.2427$ and $y(0.4) = 1.5834$. *(N/D 2016)*

[Second Half] (All are 8 marks)

V-Fourth Order Runge-Kutta Method *(N/D 2015)*

14. Apply Runge-Kutta method to find approximate value of y for $x = 0.2$ in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$ when $x = 0$. *(N/D 2015)*

15. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ given $y(0) = 1$ at $x = 0.2$.

16. Using Runge-Kutta method of fourth order, find $y(0.8)$ correct to 4 decimal places if $y' = y - x^2$, $y(0.6) = 1.7379$ with $h = 0.1$. *(A/M 2015)*

17. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ given $y(0) = 1$ at $x = 0.2$. *(N/D 2016)*

VI-Milne's Predictor Corrector Method

18. Using Milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$; $y(4.1) = 1.0049$; $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$. *(N/D 2015)*

19. Given $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)\, y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3)$ 1.21, evaluate $y(0.4)$ by Milne's predictor-corrector method. *(A/M 2015)*

20. Given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, find $y(0.3)$ by Runge-Kutta method of order four and $y(0.4)$ using Milne's predictor-corrector method.

U23MAT22 - STATISTICS AND NUMERICAL METHODS

MODEL QUESTION PAPERS - I

Time: Three hours — Maximum: 100 Marks

Answer ALL questions

PART – A **($10 \times 2 = 20$ Marks)**

1. Define student's t-test for difference of means of two samples.
2. State the main use of χ^2 test.
3. Write down the ANOVA for completely randomized block design.
4. Compare and contrast LSD and RBD.
5. State Newton Raphson formula for iterative method.
6. Using Gauss elimination method solve $5x + 4y = 15$; $3x + 7y = 12$.
7. State Newton's formula to find $f'(x)$ using forward differences.
8. State Simpson's one-third rule formula.
9. Write the merits and demerits of the Taylor Method of solution.
10. State the algorithm for modified Euler method.

PART – B ($5 \times 16 = 80$ marks)

11. (a) (i) A sample of 26 bulbs gives a mean life of 990 hours with a standard deviation of 20 hours. The manufacturer claims that the mean life of bulbs is 1000 hours. Is the sample not upto the standard? (8)

    (ii) A sample of 900 members has a mean of 3.4 cms and standard deviation is 2.61 cms. Is the sample from a large population of mean 3.25 cm and standard deviation is 2.61 cm? If the population is normal and its mean is unknown find the 95% fiducial limits of true mean. (8)

(OR)

(b) (i) A sample of size 13 gave an estimated population variance of 3.0, while another sample of size 15 give an estimate of 2.5 could both samples be from populations with the same variance. (8)

(ii) Two groups of 100 people each were taken for testing the use of a vaccine 15 persons contracted the disease out to the inoculated persons, while 25 contracted the disease in the other group. Test the efficiency of the vaccine using χ^2 test. (8)

12. (a) Three varieties A, B, C of a crop are test in a randomized block design with four replications. The plot yields in pounds are as follows

A 6	C 5	A 8	B 9
C 8	A 4	B 6	C 9
B 7	B 6	C 10	A 6

Analysis the experimental yield and state your conclusion.

(OR) (16)

(b) A variable trial was conducted on wheat with varieties in a Latin square design. The plan of the experiment and the plot yield are given below

C 25	B 23	A 20	D 20
A 19	D 19	C 21	B 20
B 19	A 14	D 17	C 20
D 17	C 20	B 21	A 15

Analyze the data and interpret the result. (16)

13. (a) (i) Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton-Raphson method. (8)

(ii) Solve the following equations by Gauss-Seidal method
$27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72.$ (8)

(OR)

(b) Find the dominant Eigen value and Eigen vector of
$A = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$ using power method. (16)

14. (a) The following table gives the census population of a town for the years 1931-1971. Estimate the population (i) for the year 1933 (ii) for the year 1965 by using appropriate interpolation formula. (16)

Year	1931	1941	1951	1961	1971
Population in lakhs	36	66	81	93	101

(OR)

(b) Evaluate $\int_0^1 \int_0^1 e^{x+y}\, dx\, dy$ using Trapezoidal and Simpson's rule. (16)

15. (a) (i) Solve $y'(x) = x + y$ given $y(1) = 0$ get $y(1, 1)$ using Taylor series method. (8)

(ii) Given $y'(x) = x^3 + y, y(0) = 2.$ Compute $y(0.2)$ by R-K method of fourth order. (8)

(OR)

(b) Solve $y' = x - y^2,\quad 0 \le x \le 1, y(0) = 0,\quad y(0.2) = 0.02,$ $y(0.4) = 0.0795,$ $y(0.6) = 0.1702.$ Use Milne's method to find $y(0.8)$ and $y(1).$ (16)

MODEL QUESTION PAPERS - II

Time: Three hours Maximum: 100 Marks

Answer ALL questions

PART – A **($10 \times 2 = 20$ Marks)**

1. Define Type-I and Type-II error.
2. For a 2×2 contingency table

a	b
c	d

Write down the corresponding x^2 value.

3. What are the basic principles of the design of experiments?
4. Why a 2×2 Latin square design is not possible?
5. Compare the differences between direct methods and indirect methods.
6. What is the condition for convergence of iterative method?
7. What is interpolation?
8. Write the Newton's divided difference interpolation formula.
9. Define Euler's formula.
10. State the Adams-Bashforth predictor-corrector formula.

PART – B ($5 \times 16 = 80$ marks)

11. (a) (i) The mean height obtained from a random sample of size 100 is 64 inches. The standard deviation of the height of the distribution of the population is known to be 3 inches. Test the statement that the mean height of the population is 67 inches at 5% level of significance. (8)

(ii) The heights of 10 students of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable is believe that the average height is greater than 64 inches?

(OR)

(b) (i) Weighs in kilograms of 10 students are given below 38, 40, 45, 53, 47, 43, 55, 48, 52, 49. Can you say that the variance of the distribution from which the above sample was drawn equal to 20 kgms. (8)

(ii) The melting points of two alloys used in formulating solider were investigated by melting 20 sample of each material. The sample standard deviation for alloy 1 was $s_1 = 4°F$. While for alloy 2 they were $s_2 = 3°F$. Do the sample data support a claim that both alloys have the same variance of melting point? Use $\alpha = 0.05$ in reaching your conclusion. (8)

12. (a) Three varieties A, B, C of a crop are tested in a randomized block design with four replications. The plot yields in pounds are as follows: (16)

A 260	C 250	A 280	B 290
C 280	A 240	B 260	C 290
B 270	B 260	C 300	A 260

Analyze the experimental yield and state your conclusion.

(OR)

(b) The following data resulted from an experiment to compare three burners B_1, B_2, B_3. A Latin square design was used as the tests were made on 3 engines and were spread over 3 days. (16)

	Engine 1	Engine 2	Engine 3
Day 1	16 (B_1)	17 (B_2)	20 (B_3)
Day 2	16 (B_2)	21 (B_3)	15 (B_1)
Day 3	15 (B_3)	16 (B_1)	17 (B_2)

Carry out an analysis of variance. What inference can you draw from the data given?

13. (a) Solve $2x + y + 4z = 12,\ 8x - 3y + 2z = 20,$ $4x + 11y - z = 33$ by
(i) Gauss Elimination method (ii) Gauss Jordan Method. (16)

(OR)

(b) Solve $20x + y - 2z = 17,\ 3x + 20y - z = -18,$ $2x - 3y + 20z = 25$ by
(i) Gauss Jacobi method (ii) Gauss Seidal method. (16)

14. (a) (i) Use Lagrange's interpolation formula to find the value of x when $y = 20$, using the following data: (8)

x :	1	2	3	4
y :	1	8	27	64

(ii) Evaluate $\int_{1/2}^{1} \frac{1}{x}\, dx$ by trapezoidal rule, dividing the range into 4 equal parts. (8)

(OR)

(b) (i) The following are data from the steam table:

Temperature °C :	140	150	160	170	180
Pressure Kg f/cm^2 :	3.685	4.854	6.302	8.076	10.225

Using Newton's formula, find the pressure of the steam for a temperature of 142°C. (8)

(ii) Evaluate $\int_{0}^{\pi} \sin x\, dx$ dividing the interval into 8 equal sub-intervals using Simpson's 1/3 rule. (8)

15. (a) Compute the first 3 steps of the initial value problem $\frac{dy}{dx} = \frac{x - y}{2}$, $y\,(0) = 1$ by Taylor series method and next step by Milne's method with step length $h = 0.1$. (16)

(OR)

(b) Given $\frac{dy}{dx} = xy + y^2$, $y\,(0) = 1$, $y\,(0.1) = 1.1169$, $y\,(0.2) = 1.2773$, Find $y\,(0.3)$ by Runge-Kutta of fourth order method and $y\,(0.4)$ using Milne's predictor-corrector method. (16)

MODEL QUESTION PAPERS - III

Time: Three hours Maximum: 100 Marks

Answer ALL questions

PART – A **($10 \times 2 = 20$ Marks)**

1. Define Type I and Type II errors in taking a decision.
2. Mention various steps involved in testing of hypothesis.
3. Define Mean sum of squares.
4. Discuss the advantages and disadvantages of Randomized block design.
5. State the iterative formula and the error term for Newton Raphson method.
6. Solve by using Gauss elimination method for $x + y = 2$; $2x + 3y = 5$.
7. Write down the Lagrange's interpolating formula.
8. Create a forward difference table for the following data and state the degree of polynomial for the same.

X:	0	1	2	3
Y:	-1	0	3	8

9. Using modified Euler's method, find $y(0.1)$ if $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.
10. Write down the Milne's predictor corrector formula for solving initial value problem in first order differential equation.

PART – B ($5 \times 16 = 80$ marks)

11. (a) (i) A sample of 900 members has a mean 3.4 cm and standard deviation 2.61 cm. Is the sample from a large population of mean 3.25 cms and standard deviation of 2.61 cms? (8)

(ii) A machine produce 16 imperfect articles in a sample of 500. After machine is overhauled, it produces 3 imperfect articles in a batch of 100. Has the machine been improved? (8)

(OR)

(b) (i) A group of 10 rats fed on diet A and another group of 8 rats fed on diet B, recorded the following increase in weight. (8)

Diet A :	5	6	8	1	12	4	3	9	6	10
Diet B :	2	3	6	8	10	1	2	8		

Test the hypothesis that the samples have same from populations with equal variance at 5% level of significance.

(ii) Examine whether the difference in the variability in yields is significant at 5% level of significance, for the following. (8)

	Set of 40 plots	Set of 60 plots
Mean yield per plot	1256	1243
S.D per plot	34	28

12. (a) The following are the numbers of mistakes made in 5 successive days of 4 technicians working for a photographic laboratory:

Technician I (X_1)	Technician II (X_2)	Technician III (X_3)	Technician IV (X_4)
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

Test at the level of significance $\alpha = 0.01$ whether the differences among the 4 sample means can be attributed to chance. (16)

(OR)

(b) Four varieties A, B, C, D of a fertilizer are tested in a RBD with 4 replications. The plot yields in pounds are as follows: (16)

A12	D20	C16	B10
D18	A14	B11	C14
B12	C15	D19	A13
C16	B11	A15	D20

Analyze the experimental yield.

13. (a) Solve the following system of equations by Gauss-Jacobi method and Gauss Seidal method. (16)

$27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$

(OR)

(b) Find all the Eigen values of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ using power method, using $x_1 = [1, 0, 0]^T$ as the initial vector. (16)

14. (a) Using Newton's divided difference formula, find the values of $f(2)$, $f(3)$, $f(8)$ and $f(12)$ given the following data: (16)

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(OR)

(b) Evaluate $\int_1^2 \int_3^4 \frac{1}{(x+y)^2}\,dx\,dy$ by using Trapezoidal, Simpsons rules with $h = k = 0.5$. (16)

15. (a) Using R.K Method of fourth order, Solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ given $y(0) = 1$ find y at $x = 0.2x = 0.4$ with $h = 0.1$. (16)

(OR)

(b) Given $y' = x^2 + y$, $y(0) = 1$, find $y(0.1)$ by Taylors method, $y(0.2)$ by modified Euler's method, $y(0.3)$ by Runge-Kutta method and $y(0.4)$ by Milne's method. (16)

MODEL QUESTION PAPERS - IV

Time: Three hours Maximum: 100 Marks

Answer ALL questions

PART – A **($10 \times 2 = 20$ Marks)**

1. Define testing a hypothesis.

2. What are the applications of ψ^2 – test?

3. Compare one-way classification model with two-way classification model.

4. Define Analysis of variance (ANOVA).

5. What is order of convergence for fixed point iteration?

6. Define the principle used in Gauss Jordan method.

7. What is Lagrange's interpolation formula?

8. State trapezoidal formula for single integral.

9. List the advantages of Taylor's series method.

10. Find $y(0.2)$ from $\frac{dy}{dx} = x + y, y(0) = 1$ with $h = 0.2$. Using Euler's method.

PART – B ($5 \times 16 = 80$ marks)

11. (a) A sample of 900 members has a mean 3.4 cm and standard deviation 2.61 cm. Is the sample from a large population of mean 3.25 cms and standard deviation of 2.61 cms? (Test at 5% level of significance. The value of z at 5% level is $|Z_\alpha| < 1.96$). (16)

(Or)

(b) Apply χ^2 to determine if any distinction is made in appointment on the basis of gender. Out of 8000 graduates in a town 800 are females; out of 600 graduate employees 120 are females. Value of χ^2 at 5% level for one degree of freedom is 3.84. (16)

12. (a) Analyze the experimental yield and state your conclusion. Given three varieties A, B and C of a crop are tested in a randomized block design with four replications. The plot yield in pounds are as follows: (16)

A	6	*C*	5	*A*	8	*B*	9
C	8	*A*	4	*B*	6	*C*	9
B	7	*B*	6	*C*	10	*A*	6

(Or)

(b) Analyze data and interpret the result. Given a variable trial was conducted on wheat with 4 varieties in a Latin Square Design. The plan of the experiment and the per plot yield are given below. (16)

C	25	*B*	23	*A*	20	*D*	20
A	19	*D*	19	*C*	21	*B*	18
B	19	*A*	14	*D*	17	*C*	20
D	17	*C*	20	*B*	21	*A*	15

13. (a) Construct the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to six decimal places. (16)

(Or)

(b) Solve by Power method, find the dominant eigen value and corresponding eigen vector of the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

. (16)

14. (a) Construct by Newton's forward and backward interpolation formulae, from the following data, find θ at $x = 43$ and $x = 84$.

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

(Or)

(b) Apply Trapezoidal rule, Evaluate $\int_1^2 \int_1^2 \frac{dx\,dy}{x^2 + y^2}$ numerically with $h = 0.2$ along x-direction and $k = 0.25$ along y-direction. (16)

15. (a) (i) Develop R.K method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$. (10)

(ii) Apply Taylor's series method find y at $x = 0.1$ correct to four decimal places from $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$, with $h = 0.1$. (6)

(Or)

(b) Apply Adam's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$. (16)

MODEL QUESTION PAPERS - V

Time: Three hours Maximum: 100 Marks

Answer ALL questions

PART – A **($10 \times 2 = 20$ Marks)**

1. What is called a critical region?

2. Find χ^2 for the following data.

O_i :	37	44	19
E :	31	38	31

3. What are the types of ANOVA?

4. What is a Latin square design?

5. Show that the iterative formula for finding the reciprocal of N is $X_{n+1} = X_n (2 - Nx_n)$.

6. Solve the equations by using Gauss Jordan method $5x + 4y = 15,\ 3x + 7y = 12$.

7. Find the second degree polynomial through the points (0,2), (2,1), (1,0) using Lagragen's formula.

8. Find the divided difference of $f(x)$ which takes the values 1, 4, 40, 85 with arguments 0, 1, 3, 4.

9. Find $y(0.1)$ by using Euler's method given that $\frac{dy}{dx} = x + y,\ y(0) = 1$.

10. Compare single step methods and multi-step methods.

PART – B ($5 \times 16 = 80$ marks)

11. (a) (i) A sample of 900 members has a mean 3.4 cm and standard deviation 2.61 cm the sample from a large population of 3.25 cms and standard deviation of 2.61 cm? (Test at 5% level of significance. The value of Z at 5% level is $|z_a| < 1.96|$. (8)

(ii) The sales manager of large company conducted a sample survey in two places *A* and *B* tasking 200 samples in each case. The results were the following table. Test whether the average sales in the same in the 2 areas at 5% level.

	Place *A*	Place *B*
Average sales	Rs.2,000	Rs.1,700
S.D	Rs.200	Rs.450

(OR)

(b) (i) Two independent samples of sizes 9 and 7 from a normal population had the following values

Sample I:	18	13	12	15	12	14	16	14	15
Sample II:	16	19	13	16	18	13	15	–	–

Does the estimation of the population variance differ significantly at 5% level? (8)

(ii) 4 coins were tossed 160 times and the following results were obtained:

No. of heads:	0	1	2	3	4
Observed frequencies:	17	52	54	31	6

Under the assumption that the coins are unbiased, find the expected frequencies of getting 0, 1, 2, 3, 4 heads and test the goodness of fit. (8)

12. (a) The following data represent the number of units production per day turned out by different workers using 4 different types of machines.

Machine type					
		A	*B*	*C*	*D*
Workers	1	44	38	47	36
	2	46	40	52	43
	3	34	36	44	32
	4	43	38	46	33
	5	38	42	49	39

Test whether the five men differ with respect to mean productivity and test whether the mean productivity is the same for the four different machine types. (16)

(OR)

(b) A farmer wishes to test the effects of four different fertilizers A, B, C, D on the yield of wheat. In order to eliminate sources of error due to variability in soil fertility, he uses the fertilizers, in a Latin square arrangements as indicated in the following table, where the numbers indicate yields in bushels per unit area.

A 18	*C* 21	*D* 25	*B* 11
D 22	*B* 12	*A* 15	*C* 19
B 15	*A* 20	*C* 23	*D* 24
C 22	*D* 21	*B* 10	*A* 17

Perform an analysis of variance to determine if there is a significant difference between the fertilizers at $\alpha = 0.05$ levels of significance. (16)

13. (a) Solve the following equations by Gauss-Jacobi and Gauss-Seidel method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110 \qquad (16)$$

(OR)

(b) Using power method, find all the Eigen values of $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$. (16)

14. (a) The population of a town is as follows.

X year:	1941	1951	1961	1971	1981	1991
Y population:	20	24	29	36	36	51

Estimate the population increase during the period 1946 to 1976.

(OR)

(b) Evaluate $\int_1^{1.4} \int_2^{2.4} \frac{1}{xy}\, dx\, dy$ by numerical double integration using Simpson's rule and Trapezoidal rule. (16)

15. (a) Evaluate $y(0.1)$ and $y(0.2)$ by using Runge-Kutta method of order 4, given $\frac{dy}{dx} = y - x^2$, $y(0) = 1$. (16)

(OR)

(b) Evaluate $y(0.4)$ by using Milne's predictor – corrector formula, Given $\frac{dy}{dx} = \frac{(1 + x^2)\, y^2}{2}$, $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$ and $y(0.3) = 1.21$. (16)

Values of $F_{0.05}$

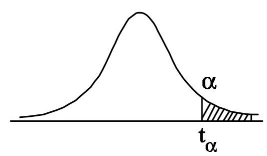
v_2 = degree of freedom for denominator	v_1 = Degree of freedom for numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.46	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.63	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.52	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.00	3.94	3.87	3.83	3.81	3.77	3.74	3.70	3.6	
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.40	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.11	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.89	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.73	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.60	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.50	2.47	2.38	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.41	2.38	2.34	2.30	2.22	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.34	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.28	2.25	2.70	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.23	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.18	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.14	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.07	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.02	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.00	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.97	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.88	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.78	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.69	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.60	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.51	1.46	1.39	1.32	1.22	1.00

Values of $F_{0.05}$

v_2 = degree of freedom for denominator	v_1 = Degree of freedom for numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,982	6,023	6,056	6,106	6.157	6,209	6,240	6,261	6.287	6,313	6,339	6,366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.57	99.47	99.48	99.49	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.58	26.50	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.91	13.84	13.75	13.65	13.56	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.86	9.72	9.55	9.45	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.30	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.06	5.99	5.91	5.82	5.74	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.26	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.71	4.65	4.57	4.48	4.40	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.31	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.01	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.76	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.57	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.41	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.28	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.16	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.07	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	2.98	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.91	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.84	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.79	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.73	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.69	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.64	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.60	2.54	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.45	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.27	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.10	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.93	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.77	1.70	1.59	1.47	1.32	1.00

Values of t_α (one tail)

Note: 5% in one-tail = 10% in two-tail
0.5% in one-tail = 1% in two-tail



Two tail	0.2	0.1	0.05 (5%)	0.02	0.0167	0.0125	0.01 (1%)	
One tail ν	**α = 0.10**	α **= 0.05** (5%)	α **= 0.025**	**α = 0.01** (1%)	α **= 0.00833**	α **= 0.00625**	α = 0.005	ν
1	3.078	6.314	12.706	31.821	38.204	50.923	63.657	1
2	1.886	2.920	4.303	6.965	7.650	8.860	9.925	2
3	1.638	2.353	3.182	4.541	4.857	5.392	5.841	3
4	1.533	2.132	2.776	3.747	3.961	4.315	4.604	4
5	1.476	2.015	2.571	3.365	3.534	3.810	4.032	5
6	1.440	1.943	2.447	3.143	3.288	3.521	3.707	6
7	1.415	1.895	2.365	2.998	3.128	3.335	3.499	7
8	1.397	1.860	2.306	2.896	3.016	3.206	3.355	8
9	1.383	1.833	2.262	2.821	2.934	3.111	3.250	9
10	1.372	1.812	2.228	2.764	2.870	3.038	3.169	10
11	1.363	1.796	2.201	2.718	2.820	2.981	3.106	11
12	1.356	1.782	2.179	2.681	2.780	2.934	3.055	12
13	1.350	1.771	2.160	2.650	2.746	2.896	3.012	13
14	1.345	1.761	2.145	2.624	2.718	2.864	2.977	14
15	1.341	1.753	2.131	2.602	2.694	2.837	2.947	15
16	1.337	1.746	2.120	2.583	2.673	2.813	2.921	16
17	1.333	1.740	2.110	2.567	2.655	2.793	2.898	17
18	1.330	1.734	2.101	2.552	2.639	2.775	2.878	18
19	1.328	1.729	2.093	2.539	2.625	2.759	2.861	19
20	1.325	1.725	2.086	2.528	2.613	2.744	2.845	20
21	1.323	1.721	2.060	2.518	2.602	2.732	2.831	21
22	1.321	1.717	2.074	2.508	2.591	2.720	2.819	22
23	1.319	1.714	2.069	2.500	2.582	2.710	2.807	23
24	1.318	1.711	2.064	2.492	2.574	2.700	2.797	24
25	1.316	1.708	2.060	2.485	2.566	2.692	2.787	25
26	1.315	1.706	2.056	2.479	2.559	2.684	2.779	26
27	1.314	1.703	2.052	2.473	2.553	2.676	2.771	27
28	1.313	1.701	2.048	2.467	2.547	2.669	2.763	28
29	1.311	1.699	2.045	2.462	2.541	2.663	2.756	29
inf	1.282	1.645	1.960	2.326	2.394	2.493	2.576	inf

Values of χ^2_α

ν	$\alpha = 0.99$	$\alpha = 0.975$	$\alpha = 0.95$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$	ν	
1	0.0000393	0.000157	0.000982	0.00393	3.841	5.024	6.635	7.879	1
2	0.100	0.0201	0.0506	0.103	5.991	7.378	9.210	10.597	2
3	0.0717	0.115	0.216	0.352	7.815	9.348	11.345	12.838	3
4	0.207	0.297	0.484	0.711	9.488	11.143	13.277	14.860	4
5	0.412	0.554	0.831	1.145	11.070	12.832	15.056	16.750	5
6	0.676	0.872	1.237	1.635	12.592	14.449	16.812	18.548	6
7	0.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278	7
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955	8
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589	9
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188	10
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757	11
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300	12
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.818	13
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319	14
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801	15
16	5.142	5.812	6,908	7.962	26.296	28.845	32.000	34.267	16
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718	17
18	6.625	7.015	8.231	9.390	28.869	31.526	34.805	37.156	18
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582	19
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997	20
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401	21
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796	22
23	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181	23
24	9.886	10.856	12.401	13.484	36.415	39.364	42.980	45.558	24
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928	25
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290	26
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645	27
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993	28
29	13.121	14.256	16.047	17.708	42.557	45.772	49.588	52.336	29
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672	30
40	20.706	22.164	24.433	26.509	55.758	59.342	63.691	66.766	40
50	27.991	29.707	32.357	34.764	67.505	71.420	76.154	79.490	50
60	35.535	37.485	40.482	43.118	79.082	83.298	88.379	91.952	60
70	43.275	45.442	48.758	51.739	90.531	95.023	100.425	104.215	70
80	51.172	53.540	57.153	60.391	101.879	106.629	112.329	116.321	80
90	59.196	61.754	65.646	69.126	113.145	118.136	124.116	128.299	90
100	67.328	70.065	74.222	77.929	124.342	129.561	135.807	140.169	100